23. Show that the equation has exactly one real solution.
$2x + \cos x = 0$

Sol.
Let $f(x) = 2x + \cos x$. Then $f(-\pi) = -2\pi - 1 < 0$ and $f(0) = 1 > 0$.

Since f is the sum of the polynomial $2x$ and the trignometric function $\cos x$, f is continuous and differentiable for all x. By the Intermediate Value Theorem, there is a number c in $(-\pi, 0)$ such that $f(c) = 0$.

Thus, the given equation has at least one real solution.

If the equation has distinct real solutions a and a with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 2 - \sin r > 0$ since $\sin r \leq 1$.

This contradiction shows that the given equation can't have two distinct real solutions, so it has exactly one solution.

24. Show that the equation has exactly one real solution.
$x^3 + e^x = 0$

Sol.
Let $f(x) = x^3 + e^x$. Then $f(-1) = -1 + \frac{1}{e} < 0$ and $f(0) = 1 > 0$.

Since f is the sum of the polynomial and the natural exponential function, f is continuous and differentiable for all x. By the Intermediate Value Theorem, there is a number c in $(-1, 0)$ such that $f(c) = 0$.

Thus, the given equation has at least one real solution.

If the equation has distinct real solutions a and a with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 3r^2 + e^r > 0$.

This contradiction shows that the given equation can't have two distinct real solutions, so it has exactly one solution.

28. (a) Suppose that f is differentiable on $\mathbb{R}$ and has two zeros. Show that f' has at least one zero.
(b) Suppose f is twice differentiable on $\mathbb{R}$ and has three zeros. Show that f'' has at least one real zero.
(c) Can you generalize parts (a) and (b)?

Sol.

29. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

Sol.
(a)
Suppose that $f(a) = f(b) = 0$ where $a < b$.
By Rolle's Theorem applied to f on $[a, b]$, there is a number c such that $a < c < b$ and $f'(c) = 0$.
(b)
Suppose that $f(a) = f(b) = f(c) = 0$ where $a < b < c$.
By Rolle's Theorem applied to f on $[a, b]$ and $[b, c]$, there are numbers d and e such that $a < d < b < e < c$ with $f'(d) = f'(e) = 0$.
By Rolle's Theorem applied to f' on $[d, e]$, there is a number g such that $d < g < e$ and $f''(g) = 0$
(c)
Suppose that f is n times differentiable on $\mathbb{R}$ and has $n + 1$ distinct zeros. Then $f^{(n)}$ has at least

one real zero

30. Suppose that $3 \le f'(x) \le 5$ for all values of x. Show that $18 \le f(8) - f(2) \le 30$.

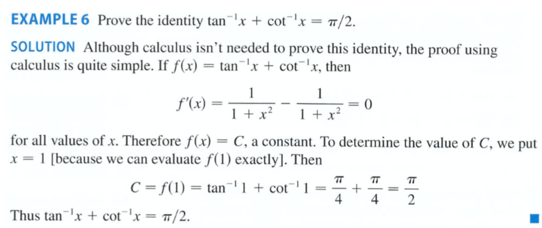
Sol.
By the Mean Value Theorem, $f(8) - f(2) = f'(c)(8-2)$ for some $c \in (2,8)$.
(Note that f is differentiable for all x, so, in particular, f is differentiable on $(2,8)$ and continuous on $[2,8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.)
Since $f(8) - f(2) = 6f'(c)$ and $3 \le f'(c) \le 5$, it follows that $6 \cdot 3 \le 6f'(c) \le 6 \cdot 5 \Rightarrow 18 \le f(8) - f(2) \le 30$

32. Suppose that f and g are continuous on $[a,b]$ and differentiable on (a,b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$.
[Hint: Apply the Mean Value Theorem to the function $h = f - g$]

Sol.
Intuitively, the statement shows that f and g start at the same point $(a, f(a))$. But since f grows faster than g, we have that $f(b) < g(b)$

Let $h = f - g$. Note that since $f(a) = g(a)$, $h(a) = f(a) - g(a) = 0$.
Then since f and g are continuous on $[a,b]$ and differentiable on (a,b), so is h, and thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a < c < b$ such that $h(b) = h(b) - h(a) = h'(c)(b-a)$.
Given $f'(x) < g'(x)$, we have $f' < g' < 0$ or, equivalently, $h' < 0$. Now since $h'(c) < 0$, $h'(c)(b-a) < 0$, so $h(b) = f(b) - g(b) < 0$ and hence $f(b) < g(b)$.

39. Use the method of Example 6 to prove the identity.
$2\sin^{-1}x = \cos^{-1}(1-2x^2), \quad x \ge 0$


EXAMPLE 6 Prove the identity $\tan^{-1}x + \cot^{-1}x = \pi/2$.

SOLUTION Although calculus isn't needed to prove this identity, the proof using calculus is quite simple. If $f(x) = \tan^{-1}x + \cot^{-1}x$, then

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

for all values of x. Therefore $f(x) = C$, a constant. To determine the value of C, we put $x = 1$ [because we can evaluate $f(1)$ exactly]. Then

$$C = f(1) = \tan^{-1}1 + \cot^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Thus $\tan^{-1}x + \cot^{-1}x = \pi/2$.

Sol.
Let $f(x) = 2\sin^{-1}x - \cos^{-1}(1-2x^2)$. Then

$$f'(x) = \frac{2}{\sqrt{1-x^2}} - \left(-\frac{-4x}{\sqrt{1-(1-2x^2)^2}}\right) = \frac{2}{\sqrt{1-x^2}} - \frac{4x}{\sqrt{4x^2-4x^4}}$$
$$= \frac{2}{\sqrt{1-x^2}} - \frac{4x}{2x\sqrt{1-x^2}} \; [\text{since} x \ge 0] = 0$$

Thus, $f'(x) = 0$ for all $x \in (0,1)$, and hence, $f(x) = C$ on $(0,1)$.
To find C, let $x = 0.5$ to get $f(0.5) = 2\sin^{-1}(0.5) - \cos^{-1}(0.5) = 2(\frac{\pi}{6}) - \frac{\pi}{3} = 0 = C$
We conclude that $f(x) = 0$ for $x \in (0,1)$. By continuity of f, $f(x) = 0$ on $[0,1]$.
Therefore, we see that $f(x) = 0 \Rightarrow 2\sin^{-1}x = \cos^{-1}(1-2x^2)$