Calculus 1 MATH 4006-15
Homework 5

(1) (25 pts)Consider the function $f(x) = \arctan(e^x) + e^x$.
(a) Show that f is a one-to-one function.
(b) What is the range of of f?
(c) Write down the linear approximation of $f^{-1}(x)$ at $x = 1 + \frac{\pi}{4}$? Here $f^{-1}(x)$ denotes the inverse of f.
(d) Approximate the value of $f^{-1}(1 + \frac{\pi}{5})$.
(e) Is your estimation in (d) an over- or under-estimation? Explain. (Hint : consider the concavity of f^{-1}. Recall that $(f^{-1})''(x) = \frac{-f''(f^{-1}(x))}{(f'(f^{-1}(x)))^3}$.)

Solution:

(a) From $f(x) = \arctan(e^x) + e^x$, we have $f'(x) = \frac{e^x}{1+e^{2x}} + e^x > 0$. So f is strictly increasing and f is one to one.

(b) The domain of f is $(-\infty, \infty)$. Using $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$, we have $\lim_{x\to-\infty} \arctan(e^x)+e^x = \arctan(0)+0 = 0$ and $\lim_{x\to\infty} \arctan(e^x)+e^x = \frac{\pi}{2} + \infty = \infty$. So the range of f is $(0, \infty)$.

(c) We have $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. So $(f^{-1})'(1 + \frac{\pi}{4}) = \frac{1}{f'(f^{-1}(1+\frac{\pi}{4}))}$. To find $f^{-1}(1 + \frac{\pi}{4})$, we have to solve $f(x) = 1 + \frac{\pi}{4}$, i.e $\arctan(e^x) + e^x = 1 + \frac{\pi}{4}$. Note that $e^0 = 1$ and $\arctan(e^0) = \arctan(1) = \frac{\pi}{4}$. It is obvious that $x = 0$ is a solution. Thus $f^{-1}(1 + \frac{\pi}{4}) = 0$ $(f^{-1})'(1 + \frac{\pi}{4}) = \frac{1}{f'(0)}$. Using $f'(x) = \frac{e^x}{1+e^{2x}} + e^x$, We have $f'(0) = \frac{1}{1+1} + 1 = \frac{3}{2}$ and $(f^{-1})'(1 + \frac{\pi}{4}) = \frac{1}{f'(0)} = \frac{2}{3}$.

(d) The linear approximation of $f^{-1}(x)$ at $x = 1 + \frac{\pi}{4}$ is
$L(x) = f^{-1}(1 + \frac{\pi}{4}) + (f^{-1})'(1 + \frac{\pi}{4})[x - (1 + \frac{\pi}{4})] = \frac{2}{3}[x - (1 + \frac{\pi}{4})]$.
$L(1 + \frac{\pi}{5}) = \frac{2}{3}[1 + \frac{\pi}{5} - (1 + \frac{\pi}{4})] = -\frac{2}{3}\frac{\pi}{20} = -\frac{\pi}{30}$.

(e) $(f^{-1})''(1 + \frac{\pi}{4}) = \frac{-f''(f^{-1}(1+\frac{\pi}{4}))}{(f'(f^{-1}(1+\frac{\pi}{4})))^3} = \frac{-f''(0)}{(f'(0))^3}$.
From $f'(x) = \frac{e^x}{1+e^{2x}} + e^x$. We have $f''(x) = \frac{e^x(1+e^{2x}) - e^x \cdot 2e^{2x}}{(1+e^{2x})^2} + e^x = \frac{e^x(1-e^{2x})}{(1+e^{2x})^2} + e^x$ and $f''(0) = 1$.
Thus $(f^{-1})''(1 + \frac{\pi}{4}) = \frac{-1}{(\frac{2}{3})^3} = -\frac{27}{8} < 0$. So f^{-1} is concave down near $x = 1 + \frac{\pi}{4}$ and the estimate in (d) is an overestimate.

(2) (20 pts) Sketch the curve $y = f(x) = \frac{x|x+1|}{x+2}$ for each of the following function $f(x)$. Indicate on your sketch (if any) the local extrema, inflection points and asymptotes of the curve. Hint: Find the domain first and discuss the case $x > -1$ and $x < -1$ separately.

Solution: The domain of f is

$$D = (-\infty, -2) \cup (-2, \infty)$$

where f is well-defined. Rewrite the function

$$f(x) = \begin{cases} -\frac{x(x+1)}{x+2} & \text{if } x \leq -1 \text{ and } x \neq -2 \\ \frac{x(x+1)}{x+2} & \text{if } x > -1 \end{cases}.$$

Note the $f(-1) = 0$.

Local extrema, inflection points

Calculate the f' directly

$$f'(x) = \begin{cases} -(\frac{x(x+1)}{x+2})' = -\frac{x^2+4x+2}{(x+2)^2} & \text{if } x < -1 \text{ and } x \neq -2 \\ (\frac{x(x+1)}{x+2})' = \frac{x^2+4x+2}{(x+2)^2} & \text{if } x > -1 \end{cases}$$

Moreover, f is not differentiable at $x = -1$ by

$$\lim_{x \to -1^-} \frac{f(x) - f(-1)}{x+1} = -\frac{1}{3} \neq \frac{1}{3} = \lim_{x \to -1^+} \frac{f(x) - f(-1)}{x+1}.$$

Now, $f'(x) = 0$ if $x = -2 \pm \sqrt{2}$. We could get $f'(x) < 0$ on $(-\infty, -2-\sqrt{2})$ and $(-1, -2+\sqrt{2})$ and $f'(x) > 0$ on $(-2-\sqrt{2},\ -2)$, $(-2, -1)$, $(-2+\sqrt{2},\ \infty)$. Thus, since f' change sign at $x = -2 \pm \sqrt{2}, -1$, the local maximum is $(-1, 0)$ and local minimum are $(-2-\sqrt{2},\ 2\sqrt{2}+3)$ and $(-2+\sqrt{2},\ 2\sqrt{2}-3)$. Calculate the f'' directly

$$f''(x) = \begin{cases} \frac{-4}{(x+2)^3} & \text{if } x < -1 \text{ and } x \neq -2 \\ \frac{4}{(x+2)^3} & \text{if } x > -1 \end{cases}$$

Now, $f''(x)$ is not defined at $x = -2, -1$. We could get $f''(x) > 0$ on $(-\infty, -2)$, $(-1, -\infty)$ and $f''(x) < 0$ on $(-2, -1)$. Thus, since f'' change sign only at $x = -1$, the inflection point is $(-1, 0)$.

Asymptotes

Since

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} -\frac{x(x+1)}{x+2} = -\infty, \quad \lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} -\frac{x(x+1)}{x+2} = \infty$$

There is an asymptotes $x = -2$. On the other hands,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -\frac{x(x+1)}{x+2} = \infty, \quad \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x(x+1)}{x+2} = \infty$$

There is no horizontal asymptotes. Now, check the slant asymptotes, called $L(x) = mx + b$. Consider $x \to -\infty$,

$$m = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{-(x+1)}{(x+2)} = -1$$

and

$$b = \lim_{x\to-\infty} (f(x) - (-x)) = \lim_{x\to-\infty} \left(\frac{-x(x+1)}{x+2} + x \right) = \lim_{x\to-\infty} \frac{x}{x+2} = 1$$

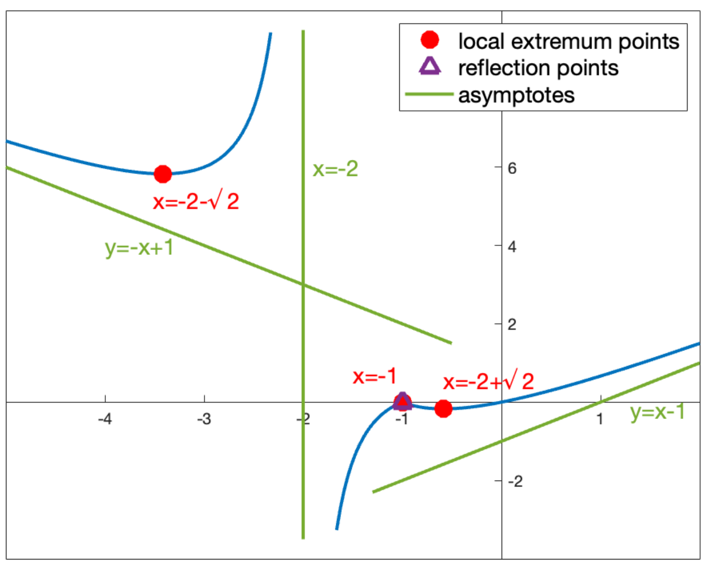
On the other hands, consider $x \to \infty$,

$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{(x+1)}{(x+2)} = 1$$

and

$$b = \lim_{x\to\infty} (f(x) - x) = \lim_{x\to\infty} \left(\frac{x(x+1)}{x+2} + x \right) = \lim_{x\to\infty} \frac{-x}{x+2} = -1$$

Hence, the slant asymptotes are $y = -x + 1$ and $y = x - 1$.



(3) (15 pts) Find the following limit $\lim_{x\to\infty} \cos(\sqrt{e^{2x}-e^x+1})-\cos(\sqrt{e^{2x}-e^x-1})$.
Solution: By mean value theorem, we have

$$\begin{aligned}
&|\cos(\sqrt{e^{2x}-e^x+1})-\cos(\sqrt{e^{2x}-e^x-1})| \\
\leq&|\sqrt{e^{2x}-e^x+1}-\sqrt{e^{2x}-e^x-1}| \\
=&\left|\frac{(\sqrt{e^{2x}-e^x+1}-\sqrt{e^{2x}-e^x-1})(\sqrt{e^{2x}-e^x+1}+\sqrt{e^{2x}-e^x-1})}{\sqrt{e^{2x}-e^x+1}+\sqrt{e^{2x}-e^x-1}}\right| \\
=&\left|\frac{2}{e^x(\sqrt{1-e^{-x}+e^{-2x}}+\sqrt{1-e^{-x}-e^{-2x}})}\right|
\end{aligned}$$

Since $\lim_{x\to\infty} \frac{2}{e^x(\sqrt{1-e^{-x}+e^{-2x}}+\sqrt{1-e^{-x}-e^{-2x}})}=0$,

$$\lim_{x\to\infty} \cos(\sqrt{e^{2x}-e^x+1})-\cos(\sqrt{e^{2x}-e^x-1})=0$$

by the squeeze theorem.

(4) (20 pts) $f(x) = x^{\frac{2}{3}}(x+5)$.

(a) Find the interval where f is increasing or decreasing.

(b) Find the interval where f is concave up or concave down.

(c) Find the local maximum and local minimum values on $(-\infty, \infty)$.

(d) Find the absolute maximum and minimum values on $(-\infty, \infty)$.

(e) Find the absolute maximum and minimum values on $[-8, 1]$.

(f) Find the inflection points.

(g) Sketch the graph of $y = f(x)$.

Solution:

(a) Calculate directly

$$f'(x) = \frac{5}{3}x^{-1/3}(x+2) \quad \text{where } x \neq 0$$

Note that f is not differentiable at $x = 0$. Now, $f'(x) = 0$ if $x = -2$. We could get that $f'(x) > 0$ on $(-\infty, -2)$, $(0, \infty)$ and $f'(x) < 0$ on $(-2, 0)$. Thus, f is increasing on $(-\infty, -2)$, $(0, \infty)$ and f is decreasing on $(-2, 0)$

(b) Calculate directly

$$f''(x) = \frac{10}{9}x^{-4/3}(x-1) \quad \text{where } x \neq 0$$

Now, $f''(x) = 0$ if $x = -1$. We could get that $f''(x) > 0$ on $(1, \infty)$ and $f'(x) < 0$ on $(-\infty, 0)$, $(0, 1)$. Thus, f is concave up on $(1, \infty)$ and f is concave down on $(-\infty, 0)$, $(0, 1)$.

(c) Consider $x = -2, 0$. If $x = 2$, f is concave down near $x = 2$. Hence, f has maximum $f(-2) = 3\sqrt[3]{4}$ at $x = 2$. If $x = 0$, f' change sign near 0. Hence, f has minimum $f(0) = 0$ at $x = 0$.

(d) Since

$$\lim_{x \to -\infty} x^{2/3}(x+5) = -\infty, \quad \lim_{x \to \infty} x^{2/3}(x+5) = \infty\,,$$

there is no absolute maximum and minimum on $(-\infty, \infty)$

(e) Since f is continuous on $[-8, 1]$, there is absolutely maximum and minimum on $[-8, 1]$. Now, consider the critical points -2, 0, we have

$$f(-2) = 3\sqrt[3]{4} \quad f(0) = 0\,.$$

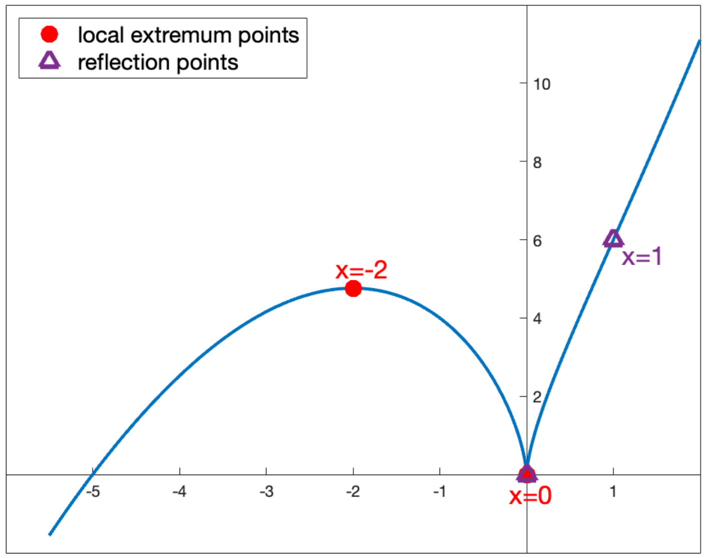
consider the boundary points -8, 1, we have

$$f(-8) = -12 \quad f(1) = 6\,.$$

Thus, compare the four value. The absolutely maximum is $f(1) = 6$ at $x = 1$ and absolutely minimum $f(-8) = -12$ at $x = -12$.

(f) Consider $x = 1, 0$. If $x = 1$, then f'' change sign near $x = 1$. Hence, $x = 1$ is inflection points. If $x = 0$, then f'' doesn't change sign. $x = 0$ is not inflection point.

(g) As mentioned above, we could sketch as following.



(5) (20 pts) Let $f(x) = (x^2 - 4x + 4)e^{2x}$.

(a) Find the absolute maximum and minimum values on $[0, 2]$.

(b) Find the absolute maximum and minimum values on $[0, 5]$.

Solution: Compute $f'(x) = (2x-4)e^{2x} + 2(x^2-4x+4)e^{2x} = 2(x^2-5x+6)e^{2x} = 2(x-2)(x-3)e^{2x}$. $f'(x) = 0$ if $x = 2$ or $x = 3$.

(a) f is continuous and differentiable. There is no critical number in $(0, 2)$. We evaluate $f(0) = 4$ and $f(2) = 0$. On $[0, 2]$, the absolute maximum is 4 and the absolute minimum is 0.

(b) The critical number of f in $(0, 5)$ is $x = 2$ and $x = 3$. Evaluate $f(0) = 4$, $f(2) = 0$ and $f(3) = (3^2 - 4 \cdot 3 + 4)e^{2 \cdot 3} = e^6 > 4$. On $[0, 4]$, the absolute maximum is e^6 and the absolute minimum is 0.