National Taiwan University - Calculus I for Economics and Management
Worksheet 2 : Derivatives in Economics and Point Elasticity

Name: **ID:** **Department:**

Introduction.
In economics, the term *marginal* is used to denote the rate of change of a quantity with respect to its dependent variable. In other words, the *marginal* of a function is simply the *derivative* of a function! For example,

- **Marginal Cost**: Suppose that $C(x)$ is the total cost function for a manufacturer to produce x units of a commodity. Then the *marginal cost of production* is $C'(x)$ which can be used to estimate the extra cost in producing one more unit of the commodity, i.e.
$$\Delta C = C(x+1) - C(x) = \frac{C(x+1) - C(x)}{1} \approx C'(x).$$
- **Marginal Revenue, Marginal Profit**: Suppose that $R(x)$ is the revenue generated when x units of a commodity is produced and sold, and $P(x)$ is the corresponding profit. Then, the *marginal revenue* $R'(x)$ is used to approximate the additional revenue in producing one more unit of the commodity $\Delta R = R(x+1) - R(x)$.
(Similarly the *marginal profit* $P'(x)$ is used to approximate $\Delta P = P(x+1) - P(x)$.)
- **Marginal Utility**: Let $U(x)$ be the utility of consuming x units of a commodity. Then, we use *marginal utility* $U'(x)$ to estimate the additional utility of consuming one more unit of the commodity.

Question 1. Suppose the weekly cost to manufacture x sets of Pulsar 40-in television is given by
$$C(x) = 10^{-4}x^3 - 0.06x^2 + 500x + 10000.$$

(a) Use the marginal cost to estimate the extra cost of producing the 11^{th} Pulsar TV, $C(11) - C(10)$.
Solution.
We use $C'(10)$ to estimate $C(11) - C(10)$.
$$C'(x) = 3 \times 10^{-4}x^2 - 0.12x + 500,\ C'(10) = 498.83$$

(b) When is the marginal cost decreasing and when is it increasing? Explain this phenomena from the perspective of economics.
Solution.
$C'(x) = 3 \times 10^{-4}x^2 - 0.12x + 500$, $C''(x) = 6 \times 10^{-4}x - 0.12$.
$C''(x) < 0$ for $0 < x < 200 \Rightarrow C'(x)$ is decreasing for $0 < x < 200$. (This is called the "economy of scale".)
$C''(x) > 0$ for $x > 200 \Rightarrow C'(x)$ is increasing for $x > 200$.

(c) Let x be the weekly demand for Pulsar TV and p be its unit price. It is known that $0 \le x \le 4800$ and $p = -0.125x + 600$. How many TV sets should be produced weekly in order to maximize the profit?
Solution.
The profit is $\Pi(x) = x \cdot p(x) - C(x) = -0.125x^2 + 600x - 10^{-4}x^3 + 0.06x^2 - 500x - 10000$
To find the maximum value of $\Pi(x)$ on $[0, 4800]$, we find critical number of $\Pi(x)$ in $(0, 4800)$.
Solve $\Pi'(x) = -3 \times 10^{-4}x^2 - 0.13x + 100 = 0 \Rightarrow x = 400$ or $-\frac{2500}{3}$.
$\Pi(400) = 13200$, $\Pi(0) = -10000$, $\Pi(4800) < 0$.
Hence the maximum profit is 13200 which is obtained by producing 400 TV sets weekly.

Introduction.
In economics, various scales of measurement are used in counting production quantity (e.g. in thousands vs. in millions) or pricing (in different currencies). Therefore, the derivative $f'(p)$ is not ideal to describe the sensitivity of the demand $y = f(p)$ of a commodity with respect to its price p.

To eliminate the dependence of the scales, economists use the **point elasticity of demand** to describe the price sensitivity of a commodity, which is defined as the percentage change of quantity demanded divided by the percentage change in prices.
In the case when the units are divisible, the **point elasticity of demand** is given mathematically as

$$\varepsilon = \lim_{\Delta p \to 0} \frac{\frac{y(p+\Delta p)-y(p)}{y(p)}}{\frac{\Delta p}{p}} = \frac{p}{y} \cdot \frac{dy}{dp}$$

A commodity is said to be **elastic** if $\varepsilon < -1$ and to be **inelastic** if $-1 < \varepsilon < 0$.

Question 2.

(a) Let $b < 0$. Consider a demand function $y(p) = a + bp$ for $0 \le p \le -\frac{a}{b}$.
For what values of p is the point elasticity (i) between 0 and -1, (ii) smaller than -1 ?
Solution. $\varepsilon = \frac{dy}{dp} \cdot \frac{p}{y} = \frac{b \cdot p}{y} = \frac{b \cdot p}{a+bp}$
(i) $-1 < \varepsilon < 0 \Rightarrow -1 < \frac{bp}{a+bp} < 0 \Rightarrow -a - bp < bp < 0$

$$\left.\begin{aligned} &\because b < 0, -a - bp < bp \Leftrightarrow -a < 2bp \Rightarrow p < -\frac{a}{2b} \\ &\because b < 0, bp < 0 \Rightarrow p > 0 \end{aligned}\right\} \Rightarrow 0 < p < -\frac{a}{2b}$$

(ii) $\varepsilon < -1 \Leftrightarrow \frac{bp}{a+bp} < -1 \Leftrightarrow bp < -a - bp$
$\because b < 0$, $bp < -a - bp \Leftrightarrow 2bp < -a \Rightarrow p > -\frac{a}{2b} \Rightarrow -\frac{a}{2b} < p < -\frac{a}{b}$.

(b) Consider a demand function $y(p) = Kp^{-r}$ where K and r are positive constants.
What is its point elasticity? (Does it depend on the price p?)
Show that if the point elasticity is a constant $-c$, then the demand function is $y(p) = Kp^{-c}$ for some constant $K > 0$. (`Hint.` Show that $y(p) \cdot p^c$ is a constant function.)
Solution.
$\varepsilon = \frac{dy}{dp} \cdot \frac{p}{y} = (k \cdot (-r) \cdot p^{-r-1}) \times \frac{p}{kp^{-r}} = -r$.
The point elasticity is a contant function $\varepsilon(p) = -r$ which doesn't depend on price p.
Suppose that the point elasticity is a constant $-c$.
Then $\frac{dy}{dp} \cdot \frac{p}{y} = \varepsilon = -c \Rightarrow y'(p) = -c\frac{y}{p}$.
Consider a new function $f(p) = y(p) \cdot p^c$.
Then $\frac{d}{dp} f = y'(p) \cdot p^c + y \cdot c \cdot p^{c-1} = -c\frac{y}{p} \cdot p^c + cy \cdot p^{c-1} = 0$
Hence $f(p) = y(p) \cdot p^c$ is a constant function, say $f(p) = k$.
Thus $y(p) = k \cdot p^{-c}$.

(c) Prove that if a commodity is inelastic, then an increase in price p leads to an increase in total revenue $y(p) \cdot p$.
Analogously, show that if a commodity is elastic, then an increase in price p leads to an decrease in total revenue $y(p) \cdot p$.
Solution.
$\frac{d}{dp}(y(p) \cdot p) = y'(p) \cdot p + y(p) = y(p) \left[\frac{y'}{y} \cdot p + 1\right] = y \cdot (\varepsilon(p) + 1)$.
If $-1 < \varepsilon(p) < 0$, then $\frac{d}{dp}(y(p) \cdot p) > 0$ and the revenue $y(p) \cdot p$ is increasing in terms of p.
If $\varepsilon(p) < -1$, then $\frac{d}{dp}(y(p) \cdot p) < 0$ and the revenue $y(p) \cdot p$ is decreasing in terms of p.