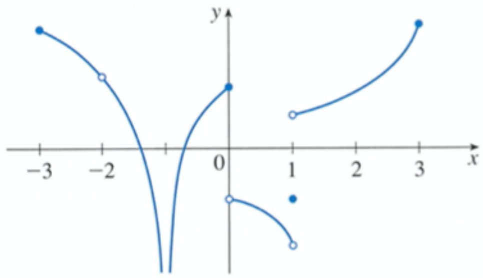
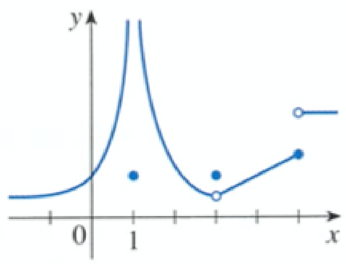
4. From the given graph of g, state the numbers at which g is discontinuous and explain why.



Sol.
g is not continuous at -2 since $g(-2)$ is not defined.
g is not continuous at -1 since the limit at $x = -1$ does not exist ($-\infty$ from both side. Or you can just say $g(-1)$ is not defined.
g is not continuous at 0 and 1 since the limits do not exists. (The left and right limits are not equal)

6. The graph of a function f is given.
(a) At what numbers a does $\lim_{x\to a} f(x)$ not exist?
(b) At what numbers a is f not continuous?
(c) At what numbers a does $\lim_{x\to a} f(x)$ exist but f is not continuous at a?



Sol.
(a) From the graph we see that the limit does not exist at $a = 1$ since the function increases without bound from the left and from the right. Also, the limit does not exist at $a = 5$ since the left and right limits are not the same.
(b) f is not continuous at $a = 1$ and $a = 5$ since the limits do not exist by part (a). Also, f is not continuous at $a = 3$ since $\lim\limits_{x\to 3} f(x) \neq f(3)$
(c) From previous two parts we know that the limit of f exists but f is not continuous at $a = 3$.

26. $f(x) = \dfrac{x^2 - 7x + 12}{x - 3}$
(a) Show that f has a removable discontinuity at $x = 3$.
(b) Redefine $f(3)$ so that f is continuous at $x = 3$ (and thus the discontinuity is "removed").

Sol.
(a)
$f(x) = \dfrac{x^2 - 7x + 12}{x - 3} = \dfrac{(x-3)(x-4)}{x-3} = x - 4$ for $x \neq 3$. $f(3)$ is undefined, so f is discontinuous at $x = 3$.
Further, $\lim_{x\to 3} f(x) = 3 - 4 = -1$. Since f is discontinuous at $x = 3$ but $\lim_{x\to 3} f(x)$ exists, f has

a removable discontinuity at $x = 3$.

(b)

If f is redefined to be -1 at $x = 3$, then f will be equivalent to the function $g(x) = x - 4$, which is continuous everywhere (and is thus continuous at $x = 3$).

42. Show that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geqslant \pi/4 \end{cases}$$

Sol.

By Theorem 7, the trigonometric functions are continuous (on their domains). So f is continuous on $(-\infty, \pi/4) \cup (\pi/4, \infty)$.

$\lim_{x\to(\pi/4)^+} f(x) = \lim_{x\to(\pi/4)^+} \sin x = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ since the sine function is continuous at $\pi/4$.

Similarly, we may obtain that $\lim_{x\to(\pi/4)^-} f(x) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

Thus, $\lim_{x\to(\pi/4)} f(x)$ exists and equals $f(\pi/4)$. Therefore, f is also continuous at $\pi/4$.

So f is continuous on $(-\infty, \infty)$.

44. Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

$$f(x) = \begin{cases} 2^x & \text{if } x \leqslant 1 \\ 3 - x & \text{if } 1 < x \leqslant 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

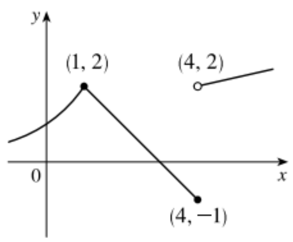
Sol.

f is continuous on $(-\infty, 1), (1, 4), \text{and}(4, \infty)$, where it is an exponential, a polynomial, and a root function, respectively.

For $x = 1$ and $x = 4$, consider the following table:

	x=1	x=4
function value	2	-1
left limit	2	-1
right limit	2	2

So we have that f is also continuous at $x = 1$. f is not continuous at $x = 4$, but it is continuous from the left at 4.



48. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leqslant x < 3 \\ 2x - a + b & \text{if } x \geqslant 3 \end{cases}$$

Sol.
We only need to tackle the cases when $x = 2$ and $x = 3$.
At $x = 2$:
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{x^2-4}{x-2} = \lim_{x\to 2^-} \dfrac{(x+2)(x-2)}{x-2} = \lim_{x\to 2^-} (x+2) = 4$
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} (ax^2 - bx + 3) = 4a - 2b + 3$
We must have that $4a - 2b + 3 = 4$, or $4a - 2b = 1$.

At $x = 3$:
$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$
$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} (2x - a + b) = 6 - a + b$
We must have that $9a - 3b + 3 = 6 - a + b$, or $10a - 4b = 3$.

Solving these two equations and we obtain that $a = b = \frac{1}{2}$. So f is continuous everywhere when $a = b = \frac{1}{2}$.

54. Suppose f is continuous on [1,5] and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.
Sol.
Suppose that $f(3) < 6$. By the Intermediate Value Theorem applied to the continuous function f on the closed interval $[2, 3]$ **(specify every condition when applying theorems!)**, the fact that $f(2) = 8 > 6$ and $f(3) < 6$ implies that there is a number c in $(2, 3)$ such that $f(c) = 6$. This contradicts the fact that the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. Hence, our supposition that $f(3) < 6$ was incorrect. It follows that $f(3) \geq 6$. But $f(3) \neq 6$ because again the only solutions of $f(x) = 6$ are $x = 1$ and $x = 4$. Therefore, $f(3) < 6$.

58. Use the Intermediate Value Theorem to show that there is a solution of the given equation in the specified interval.
$\sin x = x^2 - x, \quad (1, 2)$
Sol.
Let $f(x) = \sin x - x^2 + x$ which is continuous on the interval $[1, 2]$.
Consider $f(1) = \sin 1 \approx 0.84$ and $f(2) = \sin 2 - 2 \approx -1.09$. Since $f(1) > 0 > f(2)$, by the Intermediate Value Theorem there is a number c in $(1, 2)$ such that $f(c) = 0$, which is the desired solution for $\sin x = x^2 - x$.