微積分1–4.4 Indeterminate Forms and l'Hopital's Rule-Video 1

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Plan of the this Video

In this video, we will cover
materials in 4.4 Indeterminate Forms and l' Hopital' s Rule
We discuss how to determine the limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$
where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or
both $f(x) \to \infty$ or $-\infty$ and $g(x) \to \infty$ or $-\infty$ as $x \to a$.

Suppose we are trying to analyze the behavior of the function $f(x) = \frac{e^x - x - 1}{\sin^2(x)}$. Although f is not defined when $x = 0$, we need to know how f behaves near 0.

In particular, we would like to know the value of the limit $\lim_{x \to 0} \frac{e^x - x - 1}{\sin^2(x)}$. Note that $\lim_{x \to 0} e^x - x - 1 = \lim_{x \to 0} \sin^2(x) = 0$. In computing this limit we can' t apply Law 5 of limits because because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined. We can not determine this limit right away.

We can't use other algebraic method or known limits to determine this limit. We need to use new method to determine this method.

limit.

In general, if we have a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an indeterminate form of type $\frac{0}{0}$.

In general, if we have a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ where both $f(x) \to \infty$ or $-\infty$ and $g(x) \to \infty$ or $-\infty$ as $x \to a$, then this limit may or may not exist and is called an indeterminate form of type $\frac{\infty}{\infty}$.

需要更多的分析来求 limit.

L' Hopital' s Rule Suppose f and g are continuous near a and are differentiable near a except possibly at a. Also $g'(x) \neq 0$ near a except possibly at a. Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty.$$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if the limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists (or is ∞ or $-\infty$).

Proof: We prove a simpler case. For the special case in which $f(a) = g(a) = 0$, f' and g' are continuous and $g'(a) \neq 0$. t is easy to see why l' Hospital' s Rule is true. In fact, using the alternative form of the definition of a derivative, we have

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-f(a)}{x-a}} = \lim_{x \to a} \frac{f(x)}{g(x)}.$$

Remark: 1. If it is not indefinite form, we can not apply L'Hopital's rule. $\lim_{x\to 5} \frac{x-3}{x-4} = 2$.

If we blindly apply L'Hopital's rule, $\lim_{x\to 5} \frac{(x-3)'}{(x-4)'} = \lim_{x\to 5} \frac{1}{1} = 1$ which is the wrong answer. The problem is that the original limit problem is not indefinite form.

2. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ doesn't exist, $\lim_{x\to a} \frac{f(x)}{g(x)}$ could still exist.

For example, consider the limit problem $\lim_{x\to\infty} \frac{2x+\sin(x)}{2x+\cos(x)}$.

Note that $\lim_{x\to\infty} 2x + \sin(x) = \infty$ and $\lim_{x\to\infty} 2x + \cos(x) = \infty$. So it it indefinite form. However, $\lim_{x\to\infty} \frac{(2x+\sin(x))'}{(2x+\cos(x))'} = \lim_{x\to\infty} \frac{2+\cos(x)}{2-\sin(x)}$. Note that this limit doesn't exist.

However,

$$\lim_{x\to\infty} \frac{2x+\sin(x)}{2x+\cos(x)} = \lim_{x\to\infty} \frac{2x(1+\frac{\sin(x)}{2x})}{2x(1+\frac{\cos(x)}{2x})} = \lim_{x\to\infty} \frac{(1+\frac{\sin(x)}{2x})}{(1+\frac{\cos(x)}{2x})} = 1.$$

Note that $0 \le |\frac{\sin(x)}{2x}| \le |\frac{1}{2x}|$. So $\lim_{x\to\infty} \frac{\sin(x)}{2x} = 0$. Similarly, $\lim_{x\to\infty} \frac{\cos(x)}{2x} = 0$.

3. If $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \infty$ or $\lim_{x\to a} \frac{f'(x)}{g'(x)} = -\infty$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \infty$ or $\lim_{x\to a} \frac{f(x)}{g(x)} = -\infty$.

To use l'Hopital's rule,
(a) we first need to make sure that it is of the indefinite form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
(b) First we check if we can find the limit by simple method (like rationalization or factoring out the largest terms). If it can't be found by simple method, then we check if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exist or not. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.
$\lim_{x\to a} \frac{f'(x)}{g'(x)}$ doesn't exist, then we can't say anything about $\lim_{x\to a} \frac{f(x)}{g(x)}$.
If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ is still indefinite form, then we can simplify the expression $\frac{f'(x)}{g'(x)} = F(x)\frac{h(x)}{k(x)}$ where $\lim_{x\to a} F(x)$ exists.
If $\lim_{x\to a} \frac{h(x)}{k(x)}$ is still indefinite form then we can try l'Hopital's rule again to see if $\lim_{x\to a} \frac{h'(x)}{k'(x)}$ exists or not. If $\lim_{x\to a} \frac{h'(x)}{k'(x)}$ exists then
$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{h'(x)}{k'(x)} \lim_{x\to a} F(x)$.
Sometime, we need to use l'Hopital's rule several times.

First, we show that some of the limits that we found before can be determined by l'Hopital's rule easily Example: Find the following limits.

(a) $\lim_{x\to 0} \frac{\sin(ax)}{bx}$ where $b \neq 0$

(b) $\lim_{x\to c} \frac{x^n - c^n}{x^m - c^m}$ where m, n are nonzero integers and $c \neq 0$.

Solution: (a) Since $\lim_{x\to 0} \sin(ax) = 0$ and $\lim_{x\to 0} bx = 0$, we know that $\lim_{x\to 0} \frac{\sin(ax)}{bx}$ is the indefinite form $\frac{0}{0}$.

$\cos(0) = 1$

We compute $\lim_{x\to 0} \frac{[\sin(ax)]'}{(bx)'} = \lim_{x\to 0} \frac{a\cos(ax)}{b} = \frac{a}{b}$.

By l'Hopital's rule, we have $\lim_{x\to 0} \frac{\sin(ax)}{bx} = \lim_{x\to 0} \frac{[\sin(ax)]'}{(bx)'} = \frac{a}{b}$.

(b) Since $\lim_{x\to c} x^n - c^n = 0$ and $\lim_{x\to c} x^m - c^m = 0$, we know that $\lim_{x\to c} \frac{x^n-c^n}{x^m-c^m}$ is the indefinite form $\frac{0}{0}$.

We compute $\lim_{x\to c} \frac{(x^n-c^n)'}{(x^m-c^m)'} = \lim_{x\to c} \frac{nx^{n-1}}{mx^{m-1}} = \frac{nc^{n-1}}{mc^{m-1}}$.

By l'Hopital's rule, we have
$\lim_{x\to c} \frac{x^n-c^n}{x^m-c^m} = \lim_{x\to c} \frac{(x^n-c^n)'}{(x^m-c^m)'} = \frac{nc^{n-1}}{mc^{m-1}}$.

Example: Find the following limits.
(a) $\lim_{x\to 0}\frac{\ln(1+x)}{x}$ (b) $\lim_{x\to 0}\frac{e^x-1-x}{\sin(x)^2}$ (c) $\lim_{x\to 0}\frac{\cos(x)-1}{\sin(x)^2}$
(d) $\lim_{x\to\infty}\frac{x^2}{e^x}$ (e) $\lim_{x\to\infty}\frac{e^x}{\ln x}$ (f) $\lim_{x\to\infty}\frac{\ln(x)}{\sqrt{x}}$
(g) $\lim_{x\to\infty}\frac{x^2}{e^x}$ (h) $\lim_{x\to\infty}\frac{x^{1.5}}{e^x}$
Solution: (a) Since $\lim_{x\to 0}\ln(1+x)=0$ and $\lim_{x\to 0}x=0$, we know that $\lim_{x\to 0}\frac{\ln(1+x)}{x}$ is the indefinite form $\frac{0}{0}$.

We compute $\lim_{x\to 0}\frac{[\ln(1+x)]'}{x'}=\lim_{x\to 0}\frac{\frac{1}{1+x}}{1}=1$. By l'Hopital's rule, we have $\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{[\ln(1+x)]'}{x'}=1$.

(b) Since $\lim_{x\to 0} e^x - 1 - x = 0$ and $\lim_{x\to 0} \sin^2(x) = 0$, $\lim_{x\to 0} \frac{e^x-1-x}{\sin(x)^2}$ is the indefinite form $\frac{0}{0}$.

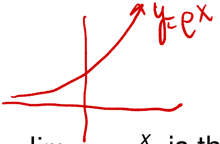
We compute
$\lim_{x\to 0} \frac{(e^x-1-x)'}{(\sin^2(x))'} = \lim_{x\to 0} \frac{e^x-1}{2\sin(x)\cos(x)} = \lim_{x\to 0} \frac{e^x-1}{\sin(x)} \frac{1}{2\cos(x)}$.

$\lim_{x\to 0} \frac{(e^x-1)'}{(\sin(x))'} = \lim_{x\to 0} \frac{e^x}{\cos(x)} = 1$.

So $\lim_{x\to 0} \frac{(e^x-1-x)'}{(\sin^2(x))'} = \lim_{x\to 0} \frac{e^x}{\cos(x)} . \lim_{x\to 0} \frac{1}{2\cos(x)} = \frac{1}{2}$.

By l'Hopital's rule, we have
$\lim_{x\to 0} \frac{e^x-1-x}{\sin^2(x)} = \lim_{x\to 0} \frac{(e^x-1-x)'}{(\sin(x)^2)'} = \frac{1}{2}$.

(c) Since $\lim_{x\to 0}\cos(x) - 1 = 0$ and $\lim_{x\to 0}\sin(x)^2 = 0$,
$\lim_{x\to 0}\frac{\cos(x)-1}{\sin(x)^2}$ is the indefinite form $\frac{0}{0}$.
We compute
$\lim_{x\to 0}\frac{(\cos(x)-1)'}{(\sin(x)^2)'} = \lim_{x\to 0}\frac{-\sin(x)}{2\sin(x)\cos(x)} = \lim_{x\to 0}\frac{-1}{2\cos(x)} = -\frac{1}{2}$.
By l'Hopital's rule, we have
$\lim_{x\to 0}\frac{\cos(x)-1}{\sin(x)^2} = \lim_{x\to 0}\frac{(\cos(x)-1)'}{(\sin(x)^2)'} = -\frac{1}{2}$.



(d) Since $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to\infty}\frac{x}{e^x}$ is the indefinite form $\frac{\infty}{\infty}$.
We compute $\lim_{x\to\infty}\frac{(x)'}{(e^x)'} = \lim_{x\to\infty}\frac{1}{e^x} = 0$.

By l'Hopital's rule, we have $\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{x'}{(e^x)'} = 0$.

(e) Since $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} \ln x = \infty$, $\lim_{x\to\infty} \frac{e^x}{\ln x}$ is the indefinite form $\frac{\infty}{\infty}$.

We compute $\lim_{x\to\infty} \frac{(e^x)'}{(\ln x)'} = \lim_{x\to\infty} \frac{e^x}{\frac{1}{x}} = \lim_{x\to\infty} xe^x = \infty$.

By l'Hopital's rule, we have $\lim_{x\to\infty} \frac{e^x}{\ln x} = \lim_{x\to\infty} \frac{(e^x)'}{(\ln x)'} = \infty$. Note that ∞ or $-\infty$ is considered as a limit.

(f) Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} \sqrt{x} = \infty$, $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}}$ is the indefinite form $\frac{\infty}{\infty}$.

We compute $\lim_{x\to\infty} \frac{(\ln(x))'}{(\sqrt{x})'} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$.

By l'Hopital's rule, we have $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{(\ln x)'}{(\sqrt{x})'} = 0$.

(g) Since $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to\infty} \frac{x^2}{e^x}$ is the indefinite form $\frac{\infty}{\infty}$.

We compute $\lim_{x\to\infty} \frac{(x^2)'}{(e^x)'} = \lim_{x\to\infty} \frac{2x}{e^x} = \frac{\infty}{\infty}$ indefinite form.

$\lim_{x\to\infty} \frac{(2x)'}{(e^x)'} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$

So $\lim_{x\to\infty} \frac{(x^2)'}{(e^x)'} = 0.$

By l'Hopital's rule, $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$

(h) Since $\lim_{x\to\infty} x^{1.5} = \infty$ and $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to\infty} \frac{x^{1.5}}{e^x}$ is the indefinite form $\frac{\infty}{\infty}$.

We compute $\lim_{x\to\infty} \frac{(x^{1.5})'}{(e^x)'} = \lim_{x\to\infty} \frac{1.5x^{0.5}}{e^x} = \frac{\infty}{\infty}$ indefinite form.

$\lim_{x\to\infty} \frac{(x^{0.5})'}{(e^x)'} = \lim_{x\to\infty} \frac{0.5x^{-0.5}}{e^x} = \lim_{x\to\infty} \frac{0.5}{x^{0.5}e^x} = 0.$

By l'Hopital's rule, we have $\lim_{x\to\infty} \frac{x^{1.5}}{e^x} = 0.$

Remark: $\lim_{x\to\infty} \frac{x^q}{e^x} = 0$ and $\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$ for any $q > 0, p > 0$.

So $e^x >> x^p >> \ln x$ as $x \to \infty$.

$p>0$

$\frac{(\ln x)'}{(x^p)'} = \frac{\frac{1}{x}}{p x^{p-1}} = \frac{1}{p x^p}$