微積分1–3.6 Derivatives of Logarithmic and Inverse Trigonometric Functions-Video 2

崔茂培 Mao-Pei Tsui

臺大數學系

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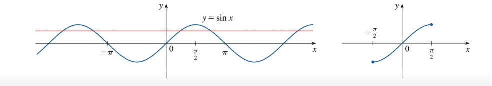
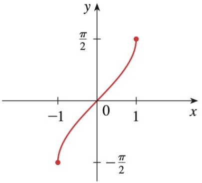
Plan of the this Video

In this video, we will cover
materials in 3.6 Derivatives of Logarithmic and Inverse Trigonometric Functions. We learn how to differentiate Inverse Trigonometric Functions

We first discuss the derivatives of inverse sine functions.

Recall that $f(x) = \sin(x)$ is one to one on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and its range is $[-1, 1]$. Its inverse is denoted by $f^{-1}(x) = \sin^{-1} x$ or $\arcsin(x)$ whose domain is $[-1, 1]$ and whose range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ We know that $\arcsin(\sin(x)) = x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\sin(\arcsin(x)) = x$ for $x \in [-1, 1]$.

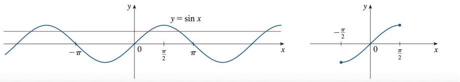


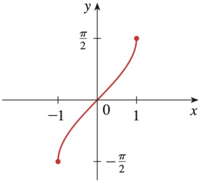


To find the derivative of $\arcsin(x)$, we use the equation $\sin(\arcsin(x)) = x$. We first differentiate both sides w.r.t. x $\frac{d}{dx}\sin(\arcsin(x)) = \frac{d}{dx}x$. Using the chain rule on the left, we get. $\cos(\arcsin(x))\frac{d}{dx}\arcsin(x) = 1$. Thus $\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}$

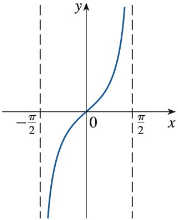
Recall that $\sin^2(\arcsin(x)) + \cos^2(\arcsin(x)) = 1$. We have $[\cos(\arcsin(x))]^2 = 1 - x^2$ and $\cos(\arcsin(x)) = \sqrt{1 - x^2}$ (since $\arcsin(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\cos(\arcsin(x)) \geq 0$).

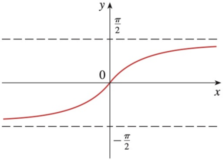
Hence $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$.





Let $f(x) = \tan(x)$ is one to one on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and its range is $(-\infty, \infty)$. Its inverse is denoted by $f^{-1}(x) = \tan^{-1} x$ or $\arctan(x)$ whose domain is $(-\infty, \infty)$ and whose range is $(-\frac{\pi}{2}, \frac{\pi}{2})$. We know that $\arcsin(\tan(x)) = x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\tan(\arctan(x)) = x$ for $x \in (-\infty, \infty)$

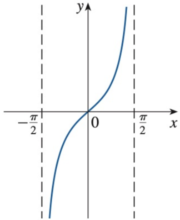


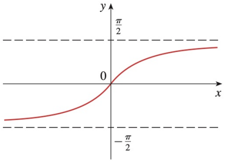


To find the derivative of $\arctan(x)$, we use the equation $\tan(\arctan(x)) = x$. We first differentiate both sides w.r.t. x $\frac{d}{dx}\tan(\arctan(x)) = \frac{d}{dx}x$. Using the chain rule on the left, we get. $\sec^2(\arctan(x))\frac{d}{dx}\arctan(x) = 1$. Thus
$\frac{d}{dx}\arctan(x) = \frac{1}{\sec^2(\arctan(x))}$

Recall that $1 + \tan^2(\arctan(x)) = \sec^2(\arctan(x))$. We have $[\sec(\arctan(x))]^2 = 1 + x^2$.

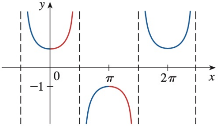
Hence $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.





To find the derivative of $\operatorname{arcsec}(x)$, we use the equation $\sec(\operatorname{arcsec}(x)) = x$. We first differentiate both sides w.r.t. x $\frac{d}{dx}\sec(\operatorname{arcsec}(x)) = \frac{d}{dx}x$. Using the chain rule on the left, we get. $\sec(\operatorname{arcsec}(x))\tan(\operatorname{arcsec}(x))\frac{d}{dx}\operatorname{arcsec}(x) = 1$. Thus $\frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{\sec(\operatorname{arcsec}(x))\tan(\operatorname{arcsec}(x))}$.

Recall that $1 + \tan^2(\operatorname{arcsec}(x)) = \sec^2(\operatorname{arcsec}(x))$. We have $[\tan(\operatorname{arcsec}(x))]^2 = x^2 - 1$ and $\tan(\operatorname{arcsec}(x)) = \sqrt{x^2 - 1}$. Thus $\frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{x\sqrt{x^2-1}}$.



$y = \csc^{-1}x \; (|x| \geqslant 1) \iff \csc y = x$ and $y \in (0, \pi/2] \cup (\pi, 3\pi/2]$

$y = \sec^{-1}x \; (|x| \geqslant 1) \iff \sec y = x$ and $y \in [0, \pi/2) \cup [\pi, 3\pi/2)$

$y = \cot^{-1}x \; (x \in \mathbb{R}) \iff \cot y = x$ and $y \in (0, \pi)$

We can derive the derivative of other inverse functions in a similar way. We summarize the result here.

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arcsin(f(x)) = \frac{f'(x)}{\sqrt{1-f^2(x)}}$$

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arccos(x) = -\frac{f'(x)}{\sqrt{1-f^2(x)}}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}, \quad \frac{d}{dx}\arctan(f(x)) = \frac{f'(x)}{1+f^2(x)}$$

$$\frac{d}{dx}\text{arccsc}(x) = -\frac{1}{x\sqrt{x^2-1}}, \quad \frac{d}{dx}\text{arccsc}(f(x)) = -\frac{f'(x)}{f(x)\sqrt{f^2(x)-1}}$$

$$\frac{d}{dx}\text{arcsec}(x) = \frac{1}{x\sqrt{x^2-1}}, \quad \frac{d}{dx}\text{arcsec}(f(x)) = \frac{f'(x)}{f(x)\sqrt{f^2(x)-1}}$$

$$\frac{d}{dx}\text{arccot}(x) = -\frac{1}{1+x^2}, \quad \frac{d}{dx}\text{arccot}(f(x)) = -\frac{f'(x)}{1+f^2(x)}$$

Example. Differentiate the following functions.

(a) $\frac{1}{\arcsin(\sqrt{1-x^2})}$ (b) $\ln\arctan(x^3)$ (c) $\arctan(e^{x^2})$ (d) $e^{\arccos(x^2)}$

Solution: (a)

$$\frac{d}{dx}\frac{1}{\arcsin(\sqrt{1-x^2})} = -\frac{[\arcsin(\sqrt{1-x^2})]'}{[\arcsin(\sqrt{1-x^2})]^2}$$

$$= -\frac{\frac{[(1-x^2)^{\frac{1}{2}}]'}{\sqrt{1-(1-x^2)}}}{[\arcsin(\sqrt{1-x^2})]^2} = -\frac{\frac{\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)}{|x|}}{[\arcsin(\sqrt{1-x^2})]^2}$$

$$= \frac{x}{|x|\sqrt{1-x^2}[\arcsin(\sqrt{1-x^2})]^2}.$$

(b)

$$\frac{d}{dx}\ln\arctan(x^3) = \frac{1}{\arctan(x^3)}[\arctan(x^3)]'$$

$$= \frac{1}{\arctan(x^3)}\frac{3x^2}{1+x^6} = \frac{3x^2}{\arctan(x^3)(1+x^6)}.$$

(c)

$$\frac{d}{dx}\arctan(e^{x^2}) = \frac{1}{1+(e^{x^2})^2}[e^{x^2}]'$$
$$= \frac{1}{1+e^{2x^2}}e^{x^2}2x = \frac{2xe^{x^2}}{1+e^{2x^2}}.$$

(d)

$$\frac{d}{dx}e^{\arccos(x^2)} = e^{\arccos(x^2)}[\arccos(x^2)]'$$
$$= e^{\arccos(x^2)}\cdot(-1)\frac{2x}{\sqrt{1-x^4}} = -\frac{2xe^{\arccos(x^2)}}{\sqrt{1-x^4}}.$$