SOLUTION TO CALCULUS HOMEWORK 1

(1) $\boxed{\text{10 pts}}$ Let

$$f(x) = \begin{cases} -3x, & x < 1; \\ ax^2 + b, & 1 \le x < 2; \\ x^3, & 2 \le x. \end{cases}$$

Determine a and b if both $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} f(x)$ exist.

Solution: Since both $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} f(x)$ exist, we have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \quad \text{and} \quad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x).$$

Moreover, evaluate four limits,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -3x = -3$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} ax^2 + b = a + b$$
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} ax^2 + b = 4a + b$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^3 = 8.$$

Therefore, we need to solve

$$\begin{cases} a + b = -3 \\ 4a + b = 8 \end{cases}.$$

Thus, $a = \frac{11}{3}$ and $b = \frac{-20}{3}$.

(2) Evaluating the following limits

(a) $\boxed{15 \text{ pts}}$ $\lim_{x\to 2^-} \frac{-x^2+4}{x^2-5x+6}$, $\lim_{x\to 2^+} \frac{-x^2+4}{x^2-5x+6}$, $\lim_{x\to 2} \frac{-x^2+4}{x^2-5x+6}$

(b) $\boxed{15 \text{ pts}}$ $\lim_{x\to 0} \frac{|3x-1|-|3x+1|}{\sin(2x)}$

(c) $\boxed{15 \text{ pts}}$ $\lim_{x\to 1^-} \frac{[x^2-1]}{x^2-3x+2}$, $\lim_{x\to 1^+} \frac{[x^2-1]}{x^2-3x+2}$, $\lim_{x\to 1} \frac{[x^2-1]}{x^2-3x+2}$ where [ ] is the greatest integer function.

Solution:

(a) For convenience, consider the following

$$\frac{-x^2+4}{x^2-5x+6} = \frac{-(x+2)(x-2)}{(x-3)(x-2)} = \frac{-(x+2)}{x-3} \quad \text{for } x \neq 2\,.$$

Hence,

$$\lim_{x\to 2^-} \frac{-x^2+4}{x^2-5x+6} = 4, \quad \lim_{x\to 2^+} \frac{-x^2+4}{x^2-5x+6} = 4 \quad \text{and} \quad \lim_{x\to 2} \frac{-x^2+4}{x^2-5x+6} = 4\,.$$

Remark: If you say "since $\lim_{x\to 2} \frac{-x^2+4}{x^2-5x+6} = 4$, $\lim_{x\to 2^-} \frac{-x^2+4}{x^2-5x+6} = \lim_{x\to 2^+} \frac{-x^2+4}{x^2-5x+6} = 4$", you should **prove** $\lim_{x\to 2} \frac{-x^2+4}{x^2-5x+6}$ exists, instead of only evaluating.

(b) If x is sufficient close to 0, *i.e.* $x \to 0$, then

$$3x-1 < 0 \quad \text{and} \quad 3x+1 > 0\,.$$

Hence,

$$\begin{aligned}\lim_{x\to 0} \frac{|3x-1|-|3x+1|}{\sin(2x)} &= \lim_{x\to 0} \frac{-(3x-1)-(3x+1)}{\sin(2x)} \\ &= \lim_{x\to 0} \frac{-6x}{\sin(2x)} = -3\lim_{x\to 0} \frac{2x}{\sin(2x)} = -3\end{aligned}$$

(c) - As $x \to 1^-$, $x^2-1 \to 0^-$, which implies

$$[x^2-1] = -1\,.$$

Thus,

$$\lim_{x\to 1^-} \frac{[x^2-1]}{x^2-3x+2} = \lim_{x\to 1^-} \frac{-1}{x^2-3x+2} = -\infty$$

- As $x \to 1^+$, $x^2-1 \to 0^+$, which implies

$$[x^2-1] = 0\,.$$

Thus,

$$\lim_{x\to 1^+} \frac{[x^2-1]}{x^2-3x+2} = \lim_{x\to 1^+} \frac{0}{x^2-3x+2} = 0$$

- Since $\lim_{x\to 1^+} \frac{[x^2-1]}{x^2-3x+2} \neq \lim_{x\to 1^-} \frac{[x^2-1]}{x^2-3x+2}$, the limit $\lim_{x\to 1} \frac{[x^2-1]}{x^2-3x+2}$ doesn't exist.

(3) $\boxed{\text{15 pts}}$ Find the equation of the line tangent to $y = \sqrt{3x^2 + 2x - 1}$ at $(1, 2)$.

Solution: Let m be slope of tangent line,

$$\begin{aligned} m &= \lim_{x \to 1} \frac{y-2}{x-1} = \lim_{x \to 1} \frac{\sqrt{3x^2+2x-1}-2}{x-1} \\ &= \lim_{x \to 1} \frac{3x^2+2x-5}{(x-1)(\sqrt{3x^2+2x-1}+2)} \\ &= \lim_{x \to 1} \frac{(x-1)(3x+5)}{(x-1)(\sqrt{3x^2+2x-1}+2)} \\ &= \lim_{x \to 1} \frac{(3x+5)}{(\sqrt{3x^2+2x-1}+2)} = 2 \end{aligned}$$

Now, since the tangent line pass (1, 2), the tangent line is $y - 2 = 2(x - 1)$. Thus, the tangent line is $y = 2x$.

(4) Evaluating the following limits

(a) $\boxed{15 \text{ pts}}$ $\lim_{x\to 0} \frac{\tan(2x)}{\sqrt{3x+4}-2}$

(b) $\boxed{15 \text{ pts}}$ $\lim_{x\to 0} \frac{\cos(x)-1}{\sin(x\sin(x))}$.

Solution:

(a)

$$
\begin{aligned}
&\lim_{x\to 0} \frac{\tan(2x)}{\sqrt{3x+4}-2} \\
=&\lim_{x\to 0} \frac{\tan(2x)(\sqrt{3x+4}+2)}{(\sqrt{3x+4}-2)(\sqrt{3x+4}+2)} \\
=&\lim_{x\to 0} \frac{\tan(2x)(\sqrt{3x+4}+2)}{3x} \\
=&\lim_{x\to 0} \frac{\sin(2x)(\sqrt{3x+4}+2)}{\cos(2x)3x} \\
=&\lim_{x\to 0} \frac{\sin(2x)}{2x}\frac{2x(\sqrt{3x+4}+2)}{\cos(2x)3x} \\
=&1\cdot\frac{2\cdot 4}{1\cdot 3} = \frac{8}{3}
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\lim_{x\to 0} \frac{\cos(x)-1}{\sin(x\sin(x))} = \lim_{x\to 0} \frac{x\sin(x)}{\sin(x\sin(x))}\frac{(\cos(x)-1)(\cos(x)+1)}{x\sin(x)(\cos(x)+1)} \\
=&\lim_{x\to 0} \frac{x\sin(x)}{\sin(x\sin(x))}\frac{\cos^2(x)-1}{x\sin(x)(\cos(x)+1)} \\
=&\lim_{x\to 0} \frac{x\sin(x)}{\sin(x\sin(x))}\frac{-\sin^2(x)}{x\sin(x)(\cos(x)+1)} \\
=&\lim_{x\to 0} \frac{x\sin(x)}{\sin(x\sin(x))}\frac{-\sin(x)}{x}\frac{1}{(\cos(x)+1)} \\
=&1\cdot(-1)\cdot\frac{1}{2} = -\frac{1}{2}
\end{aligned}
$$