微積分1–3.4 The Chain Rule-Video 1

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Plan of the this Video

In this video, we will cover
materials in 3.4 The Chain Rule. We learn how to differentiate the composition of functions.

Important for deep learning

Suppose you are asked to differentiate the function $F(x) = \sin(x^2 + 1)$

Note that $F(x) = f(g(x))$ where $g(x) = x^2 + 1$ and $f(x) = \sin(x)$.
$F(x)$ is the composition of f and g.
It turns out that the derivative of the composite function $f \circ g$ can be expressed in a simple way. This fact is one of the most important of the differentiation rules and is called the Chain Rule.

Let's try to differentiate $f(x) = \sin(x^2 + 1)$ by definition.

$f(\boxed{x+h}) = \sin((x+h)^2+1)$

$$
\begin{aligned}
&\frac{d}{dx}\sin(x^2+1) \\
&= \lim_{h\to 0} \frac{\sin((x+h)^2+1) - \sin(x^2+1)}{h} \\
&= \lim_{h\to 0} \frac{\sin((x+h)^2+1) - \sin(x^2+1)}{(x+h)^2+1-(x^2+1)} \frac{(x+h)^2+1-(x^2+1)}{h} \\
&= \lim_{y\to x^2+1} \frac{\sin(y) - \sin(x^2+1)}{y-(x^2+1)} \lim_{h\to 0} \frac{(x+h)^2+1-(x^2+1)}{h} \\
&= \cos(x^2+1)\frac{d}{dx}(x^2+1) \\
&= \cos(x^2+1)(2x)
\end{aligned}
$$

$y = (x+h)^2+1$

$h \to 0 \Rightarrow y \to x^2+1$

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product $F'(x) = f'(g(x))g'(x)$.

Proof.

$h \to 0 \Rightarrow \underbrace{g(x+h)}_{y} \to g(x)$

$y \to g(x)$

$$\frac{d}{dx} f(g(x))$$
$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x)))}{h}$$
$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x)))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h}$$

Note that

$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x)))}{g(x+h) - g(x)} = \lim_{y\to g(x)} \frac{f(y) - f(g(x)))}{y - g(x)} = f'(g(x))$ and $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$.

Thus

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

Remark: In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions,then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

We have the following formula.

1. $(f(x)^r)' = rf(x)^{r-1}f'(x)$.
2. $(e^{f(x)})' = e^{f(x)}f'(x)$
3. $\frac{d}{dx}\sin(f(x)) = \cos(f(x))f'(x)$.
4. $\frac{d}{dx}\cos(f(x)) = -\sin(f(x))f'(x)$
5. $\frac{d}{dx}\tan(f(x)) = \sec^2(f(x))f'(x)$
6. $\frac{d}{dx}\csc(f(x)) = -\csc(f(x))\cot(f(x))f'(x)$
7. $\frac{d}{dx}\sec(f(x)) = \sec(f(x))\tan(f(x))f'(x)$
8. $\frac{d}{dx}\cot(f(x)) = -\csc^2(f(x))f'(x)$.

$\frac{1}{2} - 1 = -\frac{1}{2}$

Example:

1. $[(x^2+1)^{\frac{1}{2}}]' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(x^2+1)' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}2x = x(x^2+1)^{-\frac{1}{2}}.$
2. $(e^{\sin(x)})' = e^{\sin(x)}(\sin(x))' = e^{\sin(x)}\cos(x)$
3. $\frac{d}{dx}\sin(3x) = \cos(3x)(3x)' = 3\cos(3x)$.
4. $\frac{d}{dx}\cos(x^2) = -\sin(x^2)(x^2)' = -2x\sin(x^2)$
5. $\frac{d}{dx}\tan(x^2) = \sec^2(x^2)(x^2)' = \sec^2(x^2)2x$
6. $\frac{d}{dx}\csc(x^2) = -\csc(x^2)\cot(x^2)(x^2)' = -\csc(x^2)\cot(x^2)2x$
7. $\frac{d}{dx}\sec(x^2) = \sec(x^2)\tan(x^2)(x^2)' = \sec(x^2)\tan(x^2)2x$
8. $\frac{d}{dx}\cot(x^2) = -\csc^2(x^2)(x^2)' = -\csc^2(x^2)2x$.

Recall that if $b = e^{\ln(b)}$ then $b^x = (e^{\ln b})^x = e^{x\ln(b)}$. Hence $b^x = e^{x\ln(b)}$.

$(a^a)^c = a^{a \cdot c}$

Thus

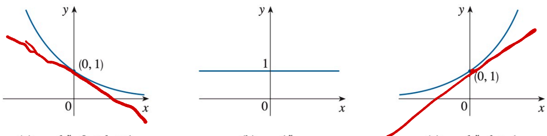
b is a constant

$\frac{d}{dx}b^x = (e^{\ln(b)x})' = e^{\ln(b)x}(x\ln(b))' = e^{\ln(b)x} \cdot \ln b = \ln(b)b^x$.

Recall that we have shown $\frac{d}{dx}b^x = \lim_{h\to 0}\frac{b^h - 1}{h}b^x$. Thus we have $\lim_{h\to 0}\frac{b^h-1}{h} = \ln(b)$.

$\lim_{h\to 0}\frac{2^h-1}{h} = \ln(2)$

slope $= \ln b$.



slope $= \ln(b)$

(a) $y = b^x,\ 0 < b < 1$ (b) $y = 1^x$ (c) $y = b^x,\ b > 1$

$3^{\sin x} = (e^{\ln 3})^{\sin(x)} = e^{(\ln 3)\sin(x)}$

Hence $\frac{d}{dx}(b^{f(x)}) = \ln(b)b^{f(x)}f'(x)$.

Example. $\frac{d}{dx}(3^{\sin x}) = [e^{(\sin(x)\ln 3)}]' = e^{(\sin(x)\ln 3)}(\sin(x)\ln 3)'$

$= e^{(\sin(x)\ln 3)}\cos(x)\ln 3 = 3^{\sin x}\cos(x)\ln 3.$

Example. Find the derivatives of the following functions.

(a) $3^{x^3} + x^{3^x}$ (b) $\sqrt{x^2 + \sqrt[3]{x}}$. (c) $5^{\cot^3(4x)}$

(d) $(x^3 - 4x + 1)^{2021}$ (e) $e^{\sqrt{x^2-1}} \sec(\sin(5x))$ (f) $\frac{\tan(\cos((x^2+1)))}{(x^2+1)^5}$

Solution; (a) Recall that $3^{x^3} = e^{x^3 \ln 3}$ and $3^x = e^{x \ln 3}$.

$$
\begin{aligned}
&\frac{d}{dx}\left(3^{x^3} + x^{3^x}\right) \\
=&(e^{x^3 \ln 3})' + (x^{e^{x \ln 3}})' \\
=&e^{x^3 \ln 3}(x^3 \ln 3)' + x^{e^{x \ln 3}-1}(e^{x \ln 3})' \\
=&e^{x^3 \ln 3} 3x^2 \ln 3 + x^{e^{x \ln 3}-1} e^{x \ln 3} \ln 3 \\
=&3^{x^3} 3x^2 \ln 3 + \cancel{x^{3^x}} 3^x \ln 3
\end{aligned}
$$

x^{3^x-1}

(b) We want to differentiate $\sqrt{x^2+\sqrt[3]{x}}$. Note that $\sqrt[3]{x}=x^{\frac{1}{3}}$ and $\sqrt{x^2+\sqrt[3]{x}}=(x^2+x^{\frac{1}{3}})^{\frac{1}{2}}$. Recall that $3^{x^3}=e^{x^3\ln 3}$ and $3^x=e^{x\ln 3}$.

$$
\begin{aligned}
&\frac{d}{dx}\sqrt{x^2+\sqrt[3]{x}}\\
=&\frac{d}{dx}(x^2+x^{\frac{1}{3}})^{\frac{1}{2}}\\
=&\frac{1}{2}(x^2+x^{\frac{1}{3}})^{-\frac{1}{2}}(x^2+x^{\frac{1}{3}})'\\
=&\frac{1}{2}(x^2+x^{\frac{1}{3}})^{-\frac{1}{2}}(2x+\frac{1}{3}x^{-\frac{2}{3}})
\end{aligned}
$$

(c) We want to differentiate $5^{\cot^3(4x)}$ Note that $5^{\cot^3(4x)} = e^{\cot^3(4x)\ln 5}$ $5^{\cot^3(4x)} = (e^{\ln 5})^{\cot^3(4x)}$

$$
\begin{aligned}
&\frac{d}{dx} 5^{\cot^3(4x)} \\
=&\frac{d}{dx} e^{\cot^3(4x)\ln 5} \\
=&e^{\cot^3(4x)\ln 5}(\cot^3(4x)\ln 5)' \\
=&e^{\cot^3(4x)\ln 5}((\cot(4x))^3\ln 5)' \\
=&(\ln 5)e^{\cot^3(4x)\ln 5} 3\cot^2(4x)(\cot(4x))' \\
=&(\ln 5)e^{\cot^3(4x)\ln 5} 3\cot^2(4x)(-\csc^2(4x))\cdot(4x)' \\
=&(\ln 5)e^{\cot^3(4x)\ln 5} 3\cot^2(4x)(-\csc^2(4x))\cdot 4 \\
=&-12(\ln 5)5^{\cot^3(4x)}\cot^2(4x)\csc^2(4x)
\end{aligned}
$$

(d) We want to differentiate $(x^3 - 4x + 1)^{2021}$

$$\begin{aligned}
&\frac{d}{dx}(x^3 - 4x + 1)^{2021} \\
=&2021(x^3 - 4x + 1)^{2020}(x^3 - 4x + 1)' \\
=&2021(x^3 - 4x + 1)^{2020}(3x^2 - 4)
\end{aligned}$$

(e) We want to differentiate $e^{\sqrt{x^2-1}}\sec(\sin(5x))$.

$$\frac{d}{dx}e^{\sqrt{x^2-1}}\sec(\sin(5x))$$

$$=(e^{(x^2-1)^{\frac{1}{2}}})'\sec(\sin(5x))+e^{\sqrt{x^2-1}}(\sec(\sin(5x)))'$$

$$=e^{(x^2-1)^{\frac{1}{2}}}((x^2-1)^{\frac{1}{2}})'\,\sec(\sin(5x))+e^{\sqrt{x^2-1}}\sec(\sin(5x))\tan(\sin(5x))(\sin(5x))'$$

$$=e^{(x^2-1)^{\frac{1}{2}}}\frac{1}{2}(x^2-1)^{-\frac{1}{2}}(x^2-1)'\cdot\sec(\sin(5x))$$

$$+e^{\sqrt{x^2-1}}\sec(\sin(5x))\tan(\sin(5x))(\cos(5x))(5x)'$$

$$=e^{(x^2-1)^{\frac{1}{2}}}\frac{1}{2}(x^2-1)^{-\frac{1}{2}}2x\cdot\sec(\sin(5x))$$

$$+e^{\sqrt{x^2-1}}\sec(\sin(5x))\tan(\sin(5x))(\cos(5x))5$$

(f) We want to differentiate $\frac{\tan(\cos((x^2+1)))}{(x^2+1)^5}$

$$\frac{d}{dx}\frac{\tan(\cos((x^2+1)))}{(x^2+1)^5}$$

$$=\frac{[\tan(\cos((x^2+1)))]'(x^2+1)^5-\tan(\cos((x^2+1)))[(x^2+1)^5]'}{(x^2+1)^{10}}$$

$$=\frac{\sec^2(\cos((x^2+1)))[\cos((x^2+1))]'(x^2+1)^5}{(x^2+1)^{10}}$$

$$-\frac{\tan(\cos((x^2+1)))5(x^2+1)^4(x^2+1)'}{(x^2+1)^{10}}$$

$(x^2+1)'$ =

$$=\frac{\sec^2(\cos((x^2+1)))[-\sin((x^2+1))](2x)(x^2+1)^5}{(x^2+1)^{10}}$$

$$-\frac{\tan(\cos((x^2+1)))5(x^2+1)^4 2x}{(x^2+1)^{10}}$$