7.1-1 Eigenfunction expansion -- 1

Consider
$$\begin{cases} Lu \equiv \frac{d}{dx}\left[p(x)\frac{du}{dx}\right]+q(x)u=f(x), \; a<x<b \\ B_1 u = u_0 \\ B_2 u = v_0 \end{cases}$$

regular Sturm-Liouville eigenvalue problem

Associated eigenvalue problem
$$\begin{cases} L\phi(x)+\lambda\rho(x)\phi(x)=0, \; a<x<b \\ B_1\phi = 0 \\ B_2\phi = 0 \end{cases}$$

C^1 : continuously differentiable
C^0 : continuous

Let (i) $p(x)>0$ & $p(x)\in C^1$, $q(x), \rho(x)\in C^0$ and

(ii) B_1 & B_2 be

- unmixed B. C. or
- periodic B. C., with $p(a)=p(b)$

Charles Sturm (1803-1855) and Joseph Liouville (1809-1882) worked in 1836, 1837 for the above boundary value problem.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right]+q(x)\phi(x)+\lambda\rho(x)\phi(x)=0,\qquad p(x)>0\ \ \&\ \ p(x)\in C^1,\quad q(x),\ \rho(x)\in C^0$$

(self-adjoint)

(1) there exists a **countable** set of eigenvalues, which form an increasing sequence: $\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots\to\infty$

real, simple, countable eigenvalues

$$\begin{cases}L\phi(x)+\lambda\rho(x)\phi(x)=0\\ B_1\phi=0\\ B_2\phi=0\end{cases}$$

take complex conjugate $\Rightarrow$

$$\begin{cases}L\bar{\phi}(x)+\bar{\lambda}\rho(x)\bar{\phi}(x)=0\\ B_1\bar{\phi}=0\\ B_2\bar{\phi}=0\end{cases}$$

$(\bar{\lambda},\bar{\phi})$ **is also an eigen-pair**

$$\int_a^b\left\{\bar{\phi}\left[L\phi+\lambda\rho\phi\right]-\phi\left[L\bar{\phi}+\bar{\lambda}\rho\bar{\phi}\right]\right\}\mathrm{d}x=0$$

$$=\int_a^b\left\{\bar{\phi}\lambda\rho\phi-\phi\bar{\lambda}\rho\bar{\phi}\right\}\mathrm{d}x+\cancel{J\left[\phi,\bar{\phi}\right]\Big|_a^b}$$

(Self-adjoint!)

$$=\left(\lambda-\bar{\lambda}\right)\int_a^b\phi\,\rho\,\bar{\phi}\,\mathrm{d}x\qquad\Rightarrow\quad\lambda=\bar{\lambda}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right]+q(x)\phi(x)+\lambda\rho(x)\phi(x)=0$$

(2) If $\phi_i(x)\ \&\ \phi_j(x)$ correspond to $\lambda_i\ \&\ \lambda_j$, and $\lambda_i \neq \lambda_j$ then

$$\int_a^b \rho(x)\phi_i(x)\,\phi_j(x)\,\mathrm{d}x=0$$

$$\begin{cases} L\phi_i(x)+\lambda_i\rho(x)\phi_i(x)=0 \\ B_1\phi_i=0 \\ B_2\phi_i=0 \end{cases} \qquad \begin{cases} L\phi_j(x)+\lambda_j\rho(x)\phi_j(x)=0 \\ B_1\phi_j=0 \\ B_2\phi_j=0 \end{cases}$$

$$\int_a^b\left\{\phi_j\left[L\phi_i+\lambda_i\rho\phi_i\right]-\phi_i\left[L\phi_j+\lambda_j\rho\phi_j\right]\right\}\mathrm{d}x=0$$

$$=\int_a^b\left\{\phi_j\lambda_i\rho\,\phi_i-\phi_i\lambda_j\rho\,\phi_j\right\}\mathrm{d}x+\cancel{J\left[\phi_i,\phi_j\right]\Big|_a^b}$$

(Self-adjoint!)

$$=\left(\lambda_i-\lambda_j\right)\int_a^b\phi_i\rho\,\phi_j\,\mathrm{d}x \qquad \Rightarrow \qquad \int_a^b\rho(x)\phi_i(x)\,\phi_j(x)\,\mathrm{d}x=0$$

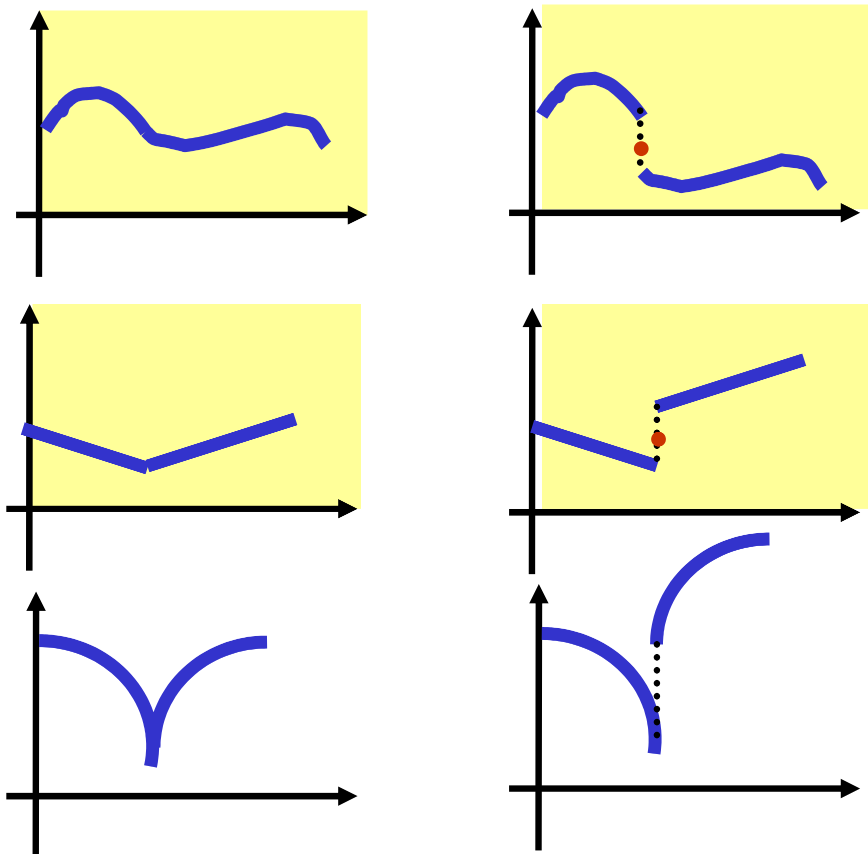
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right]+q(x)\phi(x)+\lambda\rho(x)\phi(x)=0$$

(3) Eigenfunctions $\{\phi_n(x)\}$ forms a complete set in $[a,b]$, then piecewise smooth function $u(x)$ can be expressed as

$$u(x)=\sum_{n=1}^{\infty}c_n\phi_n(x)$$

where $c_n=\int_a^b \rho(x)\,u(x)\,\phi_n(x)\mathrm{d}x$

with $\int_a^b \rho\,\phi_n{}^2\,\mathrm{d}x=1$



7.1-5 Remarks -- 1

$$\begin{cases} Lu \equiv \frac{d}{dx}\left[p(x)\frac{d\phi(x)}{dx}\right] + q(x)\phi(x) + \lambda\rho(x)\phi(x) = 0 & , a < x < b \\ B_1\phi = 0 \\ B_2\phi = 0 \end{cases}$$

Regular Sturm-Liouville eigenvalue problem

(i) $p(x) > 0$ & $p(x) \in C^1$

$q(x), \rho(x) \in C^0$ and

(ii) B_1 & B_2 be

- unmixed B. C. or
- periodic B. C., with $p(a) = p(b)$

Singular Sturm-Liouville eigenvalue problem

(i) $p(x) = 0$ at some points in $[a,b]$; or

(ii) $q(x)$ is discontinuous in $[a,b]$

(iii) $[a,b]$ is a semi-infinite or an infinite interval

7.1-6 Remarks -- 2

Consider $Lu \equiv a_2(x)u'' + a_1(x)u' + a_0(x)u = f(x) \quad , a < x < b$

If $a_2'(x) \neq a_1(x)$, and $a_2(x) \neq 0$ in $[a,b]$

i.e. L is not formally self-adjoint

To make the equation self-adjoint, find a $r(x)$ such that :

$$r(x)\left[a_2(x)u'' + a_1(x)u' + a_0(x)u\right] = \tfrac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\tfrac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u \quad ??$$

$$\Rightarrow \text{want} \begin{cases} r(x)a_2(x) = p(x) \\ r(x)a_1(x) = p'(x) \end{cases}$$

$$\Rightarrow \quad \frac{p'(x)}{p(x)} = \frac{a_1(x)}{a_2(x)}$$

$$\Rightarrow \quad p'(x) = \tfrac{a_1(x)}{a_2(x)}\, p(x)$$

$$\frac{\mathrm{d}\,p}{p} = \frac{a_1(x)}{a_2(x)}\mathrm{d}\,x$$

$$\therefore \quad p(x) = \exp\left[\int \tfrac{a_1(x)}{a_2(x)}\,\mathrm{d}x\right]$$

$$r(x) = \frac{1}{a_2(x)}\exp\left[\int \tfrac{a_1(x)}{a_2(x)}\,\mathrm{d}x\right]$$

***rLu* is self-adjoint !**

7.1-7 Remarks -- 3

regular Sturm-Liouville eigenvalue problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\left[p\,\frac{\mathrm{d}\phi}{\mathrm{d}x}\right]+q\phi+\lambda\rho\,\phi=0\,,\ a<x<b \\ B_1\phi=0 \\ B_2\phi=0 \end{cases}$$

self-adjoint problem

- real eigenvalues
- eigenfunctions are orthogonal
- eigenfunctions is a complete set

If the eigen-problem is not formally self-adjoint, then the eigen-functions of the problem will not form an orthogonal set. But it could form a bi-orthogonal set with the eigen-function of the adjoint problem.

7.1-8 Remarks -- 4

Consider

(adjoint eigenvalue problem)

$$\begin{cases} L\phi + \lambda\rho\,\phi = 0\,, a < x < b \\ B_1\phi = 0 \\ B_2\phi = 0 \end{cases} \qquad \begin{cases} L^*\psi + \lambda^*\rho\psi = 0\,, a < x < b \\ B_1^*\psi = 0 \\ B_2^*\psi = 0 \end{cases}$$

$$\int_a^b \left\{\psi\left[L\phi + \lambda\rho\phi\right] - \phi\left[L^*\psi + \lambda^*\rho\psi\right]\right\}\mathrm{d}x = 0$$

$$= \int_a^b \left\{\psi\lambda\rho\,\phi - \phi\,\lambda^*\rho\,\psi\right\}\mathrm{d}x + \cancel{J[\phi,\psi]\Big|_a^b}$$

$$= \left(\lambda - \lambda^*\right)\int_a^b \phi\rho\,\psi\,\mathrm{d}x \qquad \Rightarrow \quad \int_a^b \rho(x)\phi(x)\psi(x)\mathrm{d}x = 0$$

($\phi(x)$ and $\psi(x)$ are orthogonal)

7.2-1 BVP (with homogeneous B.C.)

$$(\mathrm{I})\quad \begin{cases} Lu = f(x) & , a < x < b \\ u(a) = 0 \\ u(b) = 0 \end{cases}$$

$$(\mathrm{II})\quad \begin{cases} Lu_H = 0 & , a < x < b \\ u_H(a) = 0 \\ u_H(b) = 0 \end{cases}$$

$$(\mathrm{III})\quad \begin{cases} L^* u_H{}^* = 0 & , a < x < b \\ u_H{}^*(a) = 0 \\ u_H{}^*(b) = 0 \end{cases}$$

$$(\mathrm{IV})\quad \begin{cases} LG = \delta(x - \xi) \\ G(a;\xi) = 0 \\ G(b;\xi) = 0 \end{cases}$$

7.2-1 BVP (with homogeneous B.C.)

$$(\mathrm{I})\quad \begin{cases} \underline{Lu = f(x)} \ , a < x < b \\ u(a) = 0 \\ u(b) = 0 \end{cases} \qquad \begin{cases} L\underline{\phi_n(x) + \lambda_n \rho(x)\phi_n(x)} = 0 \ , a < x < b \\ \phi_n(a) = 0 \\ \phi_n(b) = 0 \end{cases} \quad \int_a^b \rho(x)\phi_m(x)\phi_n(x) = \delta_{mn}$$

$$u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \qquad \text{where} \quad c_n = \underline{\int_a^b \rho(x)\, u(x)\, \phi_n(x)\,\mathrm{d}x}$$

$$c_n = \frac{\int_a^b \rho(x)\, u(x)\, \phi_n(x)\,\mathrm{d}x}{\int_a^b \rho(\eta)\, \phi_n^2(\eta)\,\mathrm{d}\eta}$$

$$\int_a^b \left\{ \ \phi_n \left[\underline{Lu - f}\right] - u\left[\underline{L\phi_n + \lambda_n \rho \phi_n}\right] \right\} \mathrm{d}x = 0$$

$$= \int_a^b \left\{ -\phi_n f - \underline{u\lambda_n \rho\, \phi_n} \right\} \mathrm{d}x + \cancel{J[u,\phi_n]\Big|_a^b}$$

$$c_n = \frac{\int_a^b \phi_n(x) f(x)\,\mathrm{d}x}{-\lambda_n}$$

$$u(x) = \sum_{n=1}^{\infty} \left[\frac{\int_a^b \phi_n(\eta) f(\eta)\,\mathrm{d}\eta}{-\lambda_n} \right] \phi_n(x)$$

7.2-2 Example 1--1

$$\begin{cases} u''=f(x), \ 0<x<\ell \\ u(0)=0 \\ u(\ell)=0 \end{cases}$$

$$\begin{cases} \phi_n''(x)+\lambda_n\phi_n(x)=0, \ 0<x<\ell \\ \phi_n(0)=0 \\ \phi_n(\ell)=0 \end{cases}$$

eigenvalues & eigenvectors

$$\lambda_n=(n\pi)^2/\ell^2$$

$$\phi_n(x)=\sin\tfrac{n\pi x}{\ell}$$

Method 1:

$$u(x)=\sum_{n=1}^{\infty}c_n\sin\tfrac{n\pi x}{\ell}$$

from D.E.

$$-\sum_{n=1}^{\infty}c_n\left(\tfrac{n\pi}{\ell}\right)^2\sin\tfrac{n\pi x}{\ell}=f(x)$$

$$-c_n\cdot\left(\tfrac{n\pi}{\ell}\right)^2\cdot\tfrac{\ell}{2}=\int_0^{\ell}f(x)\sin\tfrac{n\pi x}{\ell}\,\mathrm{d}x$$

$$u(x)=\frac{-2\ell}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{\int_0^{\ell}f(\eta)\sin\tfrac{n\pi\eta}{\ell}\,\mathrm{d}\eta}{n^2}\right]\sin\tfrac{n\pi x}{\ell}$$

$$\begin{cases} u'' = f(x) , \quad 0 < x < \ell \\ u(0) = 0 \\ u(\ell) = 0 \end{cases} \qquad \begin{cases} \phi_n''(x) + \lambda_n \phi_n(x) = 0 , \quad 0 < x < \ell \\ \phi_n(0) = 0 \\ \phi_n(\ell) = 0 \end{cases}$$

Method 2:

$$u(x) = \sum_{n=1}^{\infty} c_n \sin\tfrac{n\pi x}{\ell} \qquad c_n = \frac{2}{\ell}\int_0^\ell u(\eta)\sin\tfrac{n\pi\eta}{\ell}\,\mathrm{d}\eta$$

$$\int_0^\ell \{\phi_n [u'' - f] - u[\phi_n'' + \lambda_n \rho \phi_n]\}\mathrm{d}x = 0$$

$$= \int_0^\ell \{-\phi_n f - u\lambda_n \phi_n\}\mathrm{d}x + \cancel{J[u, \phi_n]\Big|_0^\ell}$$

$$\therefore \quad -c_n \cdot \left(\tfrac{n\pi}{\ell}\right)^2 \cdot \tfrac{\ell}{2} = \int_0^\ell f(x)\sin\tfrac{n\pi x}{\ell}\,\mathrm{d}x$$

$$u(x) = \cdots$$

$$\begin{cases} u'' = f(x), \quad 0 < x < \ell \\ u(0) = 0 \\ u(\ell) = 0 \end{cases}$$

$$\begin{cases} u(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{\ell} \\ c_n = \frac{2}{\ell}\int_0^{\ell} u(x)\sin\frac{n\pi x}{\ell}\,dx \end{cases}$$

Method 3:

$$\begin{cases} c_n = \frac{2}{\ell}\int_0^{\ell} u(x)\sin\frac{n\pi x}{\ell}\,dx \\ u(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{\ell} \end{cases}$$

finite transform !!
c.f. Laplace or Fourier transform

Apply transform $\frac{2}{\ell}\int_0^{\ell} (*)\sin\frac{n\pi x}{\ell}\,dx$ to the D.E.

$$\int_0^{\ell} u''(x)\sin\frac{n\pi x}{\ell}\,dx = \int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Integration by parts + B.C.

$$-c_n \cdot \left(\frac{n\pi}{\ell}\right)^2 \cdot \frac{\ell}{2} = \int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

$$u(x) = \cdots$$

7.3-1 BVP (with inhomogeneous B.C.)

$$\begin{cases} u'' = f(x) \ , \ 0 < x < \ell \\ u(0) = u_0 \\ u(\ell) = v_0 \end{cases}$$

$$\begin{cases} \phi_n''(x) + \lambda_n \phi_n(x) = 0 \ , \ 0 < x < \ell \\ \phi_n(0) = 0 \\ \phi_n(\ell) = 0 \end{cases}$$

eigenvalues & eigenvectors

$$\lambda_n = (n\pi)^2 / \ell^2$$

$$\phi_n(x) = \sin \tfrac{n\pi x}{\ell}$$

Method 1:

$$u(x) = \sum_{n=1}^{\infty} d_n \sin \tfrac{n\pi x}{\ell}$$

from D.E.

$$-\sum_{n=1}^{\infty} d_n \left(\tfrac{n\pi}{\ell}\right)^2 \sin \tfrac{n\pi x}{\ell} = f(x)$$

$$-d_n \cdot \left(\tfrac{n\pi}{\ell}\right)^2 \cdot \tfrac{\ell}{2} = \int_0^{\ell} f(x) \sin \tfrac{n\pi x}{\ell} \, \mathrm{d}x$$

$$u(x) = \frac{-2\ell}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\int_0^{\ell} f(\eta) \sin \frac{n\pi\eta}{\ell} \, \mathrm{d}\eta}{n^2} \right] \sin \tfrac{n\pi x}{\ell}$$

????

$$\begin{cases} u''=f(x) \ , \ 0<x<\ell \\ u(0)=u_0 \\ u(\ell)=v_0 \end{cases} \qquad \begin{cases} \phi_n''(x)+\lambda_n\phi_n(x)=0 \ , \ 0<x<\ell \\ \phi_n(0)=0 \\ \phi_n(\ell)=0 \end{cases}$$

Method 2:

$$u(x)=\sum_{n=1}^{\infty} d_n \sin\frac{n\pi x}{\ell} \qquad d_n=\frac{2}{\ell}\int_0^{\ell} u(\eta)\sin\frac{n\pi\eta}{\ell}\,\mathrm{d}\eta$$

$$\int_0^{\ell}\{\phi_n[u''-f]-u[\phi_n''+\lambda_n\phi_n]\}\mathrm{d}x=0$$

$$=\int_0^{\ell}\{-\phi_n f-u\lambda_n\phi_n\}\mathrm{d}x-(u\phi_n'-\phi_n u')\Big|_0^{\ell}$$

$$\therefore -d_n\cdot\left(\frac{n\pi}{\ell}\right)^2\cdot\frac{\ell}{2}=\int_0^{\ell} f(x)\ \sin\frac{n\pi x}{\ell}\mathrm{d}x-\frac{n\pi}{\ell}\left[u_0-(-1)^n v_0\right]$$

$$d_n=-\frac{2\ell}{\pi^2}\frac{1}{n^2}\int_0^{\ell} f(x)\ \sin\frac{n\pi x}{\ell}\mathrm{d}x+\frac{2}{n\pi}\left[u_0-(-1)^n v_0\right]$$

$$\begin{cases} u'' = f(x)\ ,\ 0 < x < \ell \\ u(0) = u_0 \\ u(\ell) = v_0 \end{cases}$$

7.3-3 Example 2--3

Method 3: Apply transform $\frac{2}{\ell}\int_0^\ell (*) \sin\frac{n\pi x}{\ell}\mathrm{d}x$ to the D.E.

$$\int_0^\ell u''(x)\sin\frac{n\pi x}{\ell}\mathrm{d}x = \int_0^\ell f(x)\sin\frac{n\pi x}{\ell}\mathrm{d}x$$

Integration by parts

$$-c_n \cdot \left(\frac{n\pi}{\ell}\right)^2 \cdot \frac{\ell}{2} + \frac{n\pi}{\ell}\left[u_0 - (-1)^n v_0\right] = \int_0^\ell f(x)\sin\frac{n\pi x}{\ell}\mathrm{d}x$$

$$u(x) = \cdots$$

$$\begin{cases} u'' = f(x)\ , \quad 0 < x < \ell \\ u(0) = u_0 \\ u(\ell) = v_0 \end{cases}$$

7.3-4 Example 2--4

Method 4: Let $u(x) = U(x) + u_0\left(1 - \frac{x}{\ell}\right) + v_0 \frac{x}{\ell}$

$$\begin{cases} U'' = f(x)\ , \quad 0 < x < \ell \\ U(0) = 0 \\ U(\ell) = 0 \end{cases}$$

Converge faster (see lecture notes)

↓

$$\therefore u(x) = u_0 \left(1 - \frac{x}{\ell}\right) + v_0 \frac{x}{\ell} + \frac{-2\ell}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\int_0^{\ell} f(\eta) \sin \frac{n\pi\eta}{\ell} \, d\eta}{n^2} \right] \sin \frac{n\pi x}{\ell}$$

From method 2:

$$u(x) = \frac{-2\ell}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\int_0^{\ell} f(\eta) \sin(n\pi\eta / l) \, d\eta}{n^2} \right] \sin \frac{n\pi x}{l} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[u_0 - (-1)^n v_0 \right] \sin \frac{n\pi x}{l}$$

7.4-1 Example 3--1 (not-self-adjoint)

$$\begin{cases} u''+4u'+3u=3\ ,\ \ 0<x<1 \\ u(0)=0 \\ u(1)=0 \end{cases}$$

Method 1:

$$u = u_h + u_p = A e^{-x} + B e^{-3x} + 1$$

By using of B.C., one has…….

Method 2: Green's function approach

$$u = 3\int_0^1 G(x;\xi)\mathrm{d}\xi$$

$$\begin{cases} u''+4u'+3u=3 \ , \ 0<x<1 \\ u(0)=0 \\ u(1)=0 \end{cases}$$

Example 3-2 (not-self-adjoint)

Method 3:

Associated eigenvalue problem

$$\begin{cases} \phi''+4\phi'+\underline{\lambda\phi}=0 \ , \ 0<x<1 \\ \phi(0)=0 \\ \phi(1)=0 \end{cases}$$

eigenvalues & eigenvectors

$$\lambda_n=(n\pi)^2+4$$

$$\phi_n(x)=e^{-2x}\sin n\pi x$$

$$u(x)=e^{-2x}\sum_{n=1}^{\infty}A_n\sin n\pi x$$

from D.E.

$$e^{-2x}\sum_{n=1}^{\infty}A_n\left(-n^2\pi^2-1\right)\sin n\pi x=3$$

Orthogonality of eigenfuntions ??

$$A_n=???$$

$$\begin{cases} u''+4u'+3u=3 \ , \ 0<x<1 \\ u(0)=0 \\ u(1)=0 \end{cases}$$

Example 3-3 (not-self-adjoint)

Associated eigenvalue problem

$$\begin{cases} \phi''+4\phi'+\lambda\phi=0 \ , \ 0<x<1 \\ \phi(0)=0 \\ \phi(1)=0 \end{cases}$$

$$\int_0^1 \phi_n(x)\phi_m(x)\mathrm{d}x \neq 0$$

Adjoint eigenvalue problem

$$\begin{cases} \psi''-4\psi'+\lambda\psi=0 \ , \ 0<x<1 \\ \psi(0)=0 \\ \psi(1)=0 \end{cases}$$

$$\phi_n(x)=e^{-2x}\sin n\pi x$$

$$\lambda_n=(n\pi)^2+4$$

$$\psi_n(x)=e^{2x}\sin n\pi x$$

$$e^{-2x}\sum_{n=1}^{\infty}A_n\left(-n^2\pi^2-1\right)\sin n\pi x=3$$

From orthogonality of eigenfuntions

$$A_n\cdot\left(-n^2\pi^2-1\right)\cdot\frac{1}{2}=3\int_0^1 e^{2x}\sin n\pi x\,\mathrm{d}x \qquad \therefore A_n=\cdots$$

$$\begin{cases} u''+4u'+3u=3 \ , \ 0<x<1 \\ u(0)=0 \\ u(1)=0 \end{cases}$$

Associated eigenvalue problem

$$\begin{cases} \phi''+4\phi'+\lambda\phi=0 \ , \ 0<x<1 \\ \phi(0)=0 \\ \phi(1)=0 \end{cases}$$

Method 4:

Associated eigenvalue problem

$$\begin{cases} e^{4x}\phi''+4e^{4x}\phi'+\lambda e^{4x}\phi=0 \ , \ 0<x<1 \\ \phi(0)=0 \\ \phi(1)=0 \end{cases}$$

$(e^{4x} = \text{integrating factor})$

$$\lambda_n=(n\pi)^2+4 \qquad \phi_n(x)=e^{-2x}\sin n\pi x$$

$$u(x)=e^{-2x}\sum_{n=1}^{\infty}A_n\sin n\pi x$$

from D.E.

$$e^{2x}\sum_{n=1}^{\infty}A_n\left(-n^2\pi^2-1\right)\sin n\pi x=3$$

Orthogonality of eigenfuntions OK!!

7.4-5 Example 3--5 (not-self-adjoint)

Method 5:

Assume $u(x) = e^{-2x} \sum_{n=1}^{\infty} A_n \sin n\pi x = e^{-2x} w(x)$

$$\begin{cases} w'' - w = 3e^{2x} \quad , \quad 0 < x < 1 \\ w(0) = 0 \\ w(1) = 0 \end{cases}$$

(Self-adjoint now!)

$$w(x) = \sum_{n=1}^{\infty} B_n \sin n\pi x$$

from D.E. $\cdots$