Green's function

https://en.wikipedia.org/wiki/George_Green_(mathematician)

George Green (1793 – 1841) of England, was practically a self-taught mathematician who made significant contributions to the theory of electricity and magnetism, fluid mechanics, and boundary value problems. He is also the Green of the often used Green's theorem of integral calculus (line integral ↔ surface integral).

George Green

- July 14, 1793 - May 31, 1841
- British mathematician and physicist
- First person to try to explain a mathematical theory of electricity and magnetism
- Almost entirely self-taught! (between ages of 8 – 9)
- Published "*An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*" in 1828.
- Entered Cambridge University as an undergraduate in 1833 at age 40.

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- **https://www.google.com.tw/search?biw=1536&bih=770&tbm=isch&sa=1&q=george+green+the+mathematican&oq=george+green+the+mathematican&gs_l=img.3...25181.28332.0.30729.17.16.0.1.1.0.92.1102.16.16.0....0...1c.1.64.img..0.4.230...0i19k1j0i8i30i19k1j0i24k1.pcDLqmLLMRk#imgrc=K7XiAf_gKQkeXM%3A**

4.1-1 Variation of parameters

$$u=\int_0^\ell G(x;\xi)\,f(\xi)\,d\xi$$

$$G(x;\xi)=\begin{cases}\frac{x\,(\xi-\ell)}{\ell}, & 0<x<\xi<l\\ \frac{\xi\,(x-\ell)}{\ell}, & 0<\xi<x<\ell\end{cases}$$

BVP : 2nd order inhomogeneous ODE

$$\begin{cases}u''=f(x) & ,0<x<\ell\\ u(0)=0\\ u(\ell)=0\end{cases}$$

(linear)

$$u=u_h+u_p$$

u_h : homogeneous solution

$u_h''=0$

u_p : particular solution

$u_p''=f(x)$

$$u_h=A+x\,B$$

$$u_p=C_1(x)+x\,C_2(x)$$

$$u_p'=C_1'(x)+x\,C_2'(x)+C_2(x)$$

$$\text{let}\quad C_1'(x)+x\,C_2'(x)=0$$

$$\therefore\quad u_p'=C_2(x)$$

$$\therefore \text{D.E.}\Rightarrow\qquad C_2'(x)=f(x)$$

$$C_2(x)=\int_0^x f(\xi)\,\mathrm{d}\xi$$

$$C_1(x)=-\int_0^x \xi\,f(\xi)\,\mathrm{d}\xi$$

$$u=A+x\,B+\int_0^x (x-\xi)\,f(\xi)\,\mathrm{d}\xi$$

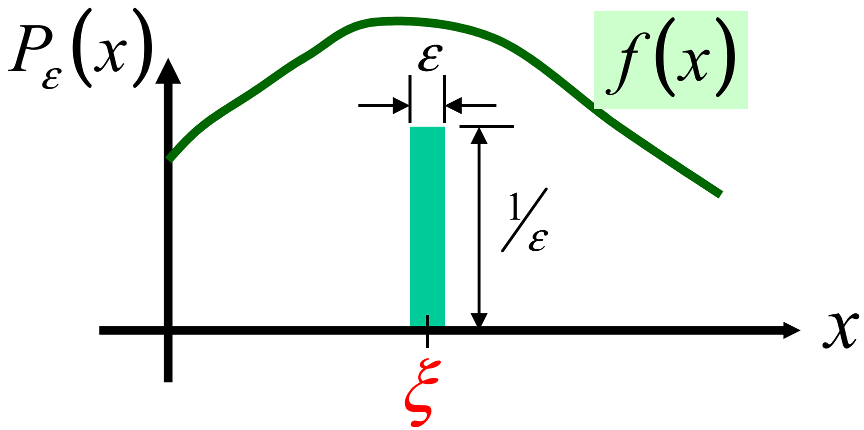
After applying the BC's and rearrangement:

$$u=\int_0^x \frac{\xi(x-\ell)}{\ell}f(\xi)\,\mathrm{d}\xi+\int_x^\ell \frac{x(\xi-\ell)}{\ell}f(\xi)\,\mathrm{d}\xi$$

4.1-2 Accessory problem -- 1

Consider

$$\begin{cases} \dfrac{\mathrm{d}^2 U_\varepsilon(x;\xi)}{\mathrm{d}\,x^2} = P_\varepsilon(x) \\ U_\varepsilon(0;\xi) = 0 \\ U_\varepsilon(\ell;\xi) = 0 \end{cases}$$



where

$$P_\varepsilon(x) = \begin{cases} 0, & 0 < x < \xi - (\varepsilon/2) \\ 1/\varepsilon, & \xi - (\varepsilon/2) < x < \xi + (\varepsilon/2) \\ 0, & \xi + (\varepsilon/2) < x < \ell \end{cases}$$

$$u(x) = \lim_{\substack{\varepsilon\to 0 \\ \Delta\xi_i \to 0}} \sum U_\varepsilon(x;\xi_i) f(\xi_i) \Delta\xi_i$$

$$\overset{?}{=} \int_0^\ell U_\varepsilon(x;\xi) f(\xi)\,\mathrm{d}\xi$$

$$u(x)=\lim_{\substack{\varepsilon\to 0\\ \Delta\xi_i\to 0}}\sum U_\varepsilon(x;\xi_i)f(\xi_i)\Delta\xi_i \overset{?}{=} \int_0^\ell U_\varepsilon(x;\xi)f(\xi)d\xi$$

4.1-3 Accessory problem -- 2

BVP with 2nd order ODE

$$\begin{cases} u''=f(x) \quad ,0<x<\ell \\ u(0)=0 \\ u(\ell)=0 \end{cases}$$

Consider

$$\begin{cases} \frac{d^2U_\varepsilon(x;\xi)}{dx^2}=P_\varepsilon(x) \\ U_\varepsilon(0;\xi)=0 \\ U_\varepsilon(\ell;\xi)=0 \end{cases}$$

$$\int_0^\ell (U_\varepsilon u''-uU_\varepsilon'')dx \quad =0$$

$$=(U_\varepsilon u'-uU_\varepsilon')\Big|_0^\ell-\int_0^\ell (U_\varepsilon' u'-u'U_\varepsilon')dx$$

$$=\int_0^\ell (U_\varepsilon f-uP_\varepsilon)dx$$

$$u(\xi)=\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} uP_\varepsilon \mathrm{d}x=\int_0^\ell U_\varepsilon(x;\xi)f(x)dx$$

$$\varepsilon\to 0 \qquad x\leftrightarrow\xi$$

$$u(x)=\lim_{\varepsilon\to 0}\int_0^\ell U_\varepsilon(\xi;x)f(\xi)d\xi$$

$$u(x)\overset{?}{=}\int_0^\ell U_\varepsilon(\xi;x)f(\xi)d\xi$$

$$U_\varepsilon(x;\xi)=B_0 x+B_1+\left(\frac{x^2}{2\varepsilon}\right),\ x\in II$$

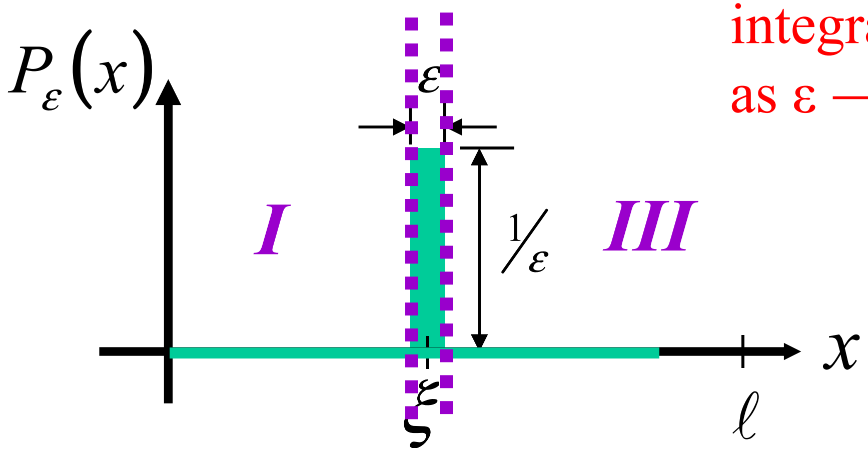
4.1-4 Accessory problem -- 3

Consider

$$\begin{cases}\dfrac{d^2U_\varepsilon(x;\xi)}{dx^2}=P_\varepsilon(x)\\ U_\varepsilon(0;\xi)=0\\ U_\varepsilon(\ell;\xi)=0\end{cases}$$

$$U_\varepsilon(x;\xi)=\begin{cases}x(\xi-\ell)/\ell & ,\ x\in I\\ -\left(\dfrac{1}{2}+\dfrac{\xi}{\varepsilon}-\dfrac{\xi}{l}\right)x+\dfrac{1}{2\varepsilon}\left(\xi-\dfrac{\varepsilon}{2}\right)^2+\left(\dfrac{x^2}{2\varepsilon}\right) & ,\ x\in II\\ \xi(x-\ell)/\ell & ,\ x\in III\end{cases}$$

$$u(\xi)=\lim_{\varepsilon\to 0}\left(\int_0^{\xi-\varepsilon/2}+\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}+\int_{\xi+\varepsilon/2}^{l}\right)U_\varepsilon(x;\xi)f(x)dx$$



The second integral $\to 0$ as $\varepsilon \to 0$

$$u(\xi)=\int_0^\ell U_0(x;\xi)f(x)dx$$

$$U_0(x;\xi)=\begin{cases}x(\xi-\ell)/\ell,\ x<\xi\\ \xi(x-\ell)/\ell,\ \xi<x\end{cases}$$

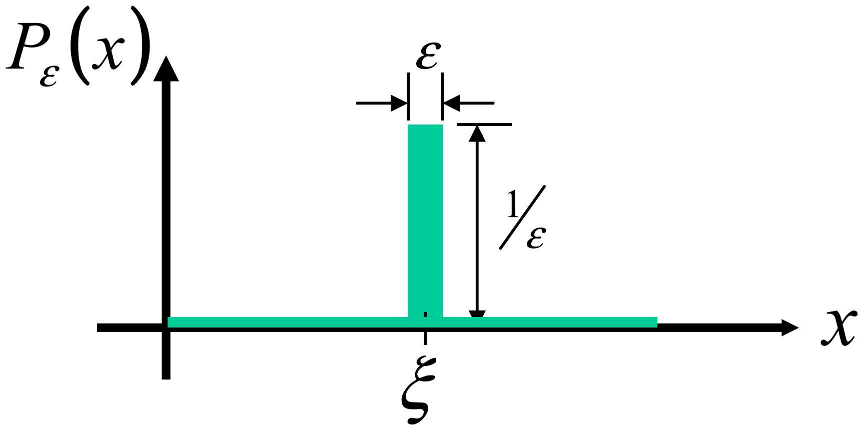
Note for this case: $U_0(\xi;x)=U_0(x;\xi)$

$$u(x)=\int_0^\ell U_0(\xi;x)f(\xi)d\xi$$

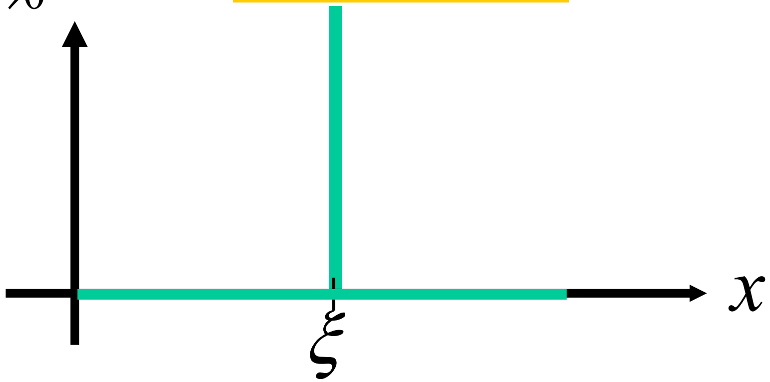
$$\int_{-1}^{3}\left(x^2-1\right)\delta(x-2)\,\mathrm{d}\,x=3$$

$$\int_{-1}^{3}\left(x^2-1\right)\delta(x+2)\,\mathrm{d}\,x=0$$

4.2-1 Dirac delta function -- 1



$$\lim_{\varepsilon\to 0}P_\varepsilon(x)=\delta(x-\xi)$$



$$\delta(x-\xi)=\begin{cases}0, & x\neq\xi\\ \infty, & x=\xi\end{cases}$$

$$\int_a^b\delta(x-\xi)\,\mathrm{d}x=1,\ \xi\in(a,b)$$

generalized function: $\delta, \delta', \delta'', \cdots$

$$\int_a^b\phi(x)\delta(x-\xi)\,\mathrm{d}x=\begin{cases}0, & \xi\notin[a,b]\\ \phi(\xi), & \xi\in(a,b)\end{cases}$$

theory of distribution

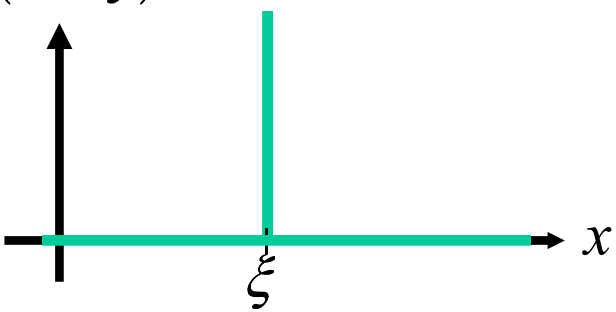
4.2-2 Dirac delta function -- 2

Distributional equality $\longleftrightarrow$

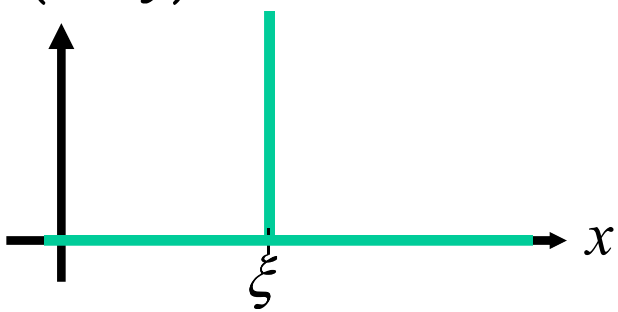
$$\int_{-\infty}^{\infty} \phi(x)\delta(x-\xi)\,\mathrm{d}\,x = \phi(\xi)$$

$$\int_{-\infty}^{\infty} \phi(x)2\delta(x-\xi)\,\mathrm{d}\,x = 2\phi(\xi)$$

$\delta(x-\xi)$



$2\delta(x-\xi)$



$$\delta(x-\xi) \neq 2\delta(x-\xi)$$

$$\int_{-\infty}^{\infty} \phi(x)\Psi_1(x)\,\mathrm{d}\,x = \int_{-\infty}^{\infty} \phi(x)\Psi_2(x)\,\mathrm{d}\,x \qquad \forall\phi(x)$$

$$\Rightarrow \quad \Psi_1(x) = \Psi_2(x)$$

(Equality of 2 generalized functions)

4.2-3 Dirac delta function -- 3

$$\int_a^b \phi(x)\delta(x-\xi)\,\mathrm{d}x = \begin{cases} 0, & \xi \notin [a,b] \\ \phi(\xi), & \xi \in (a,b) \end{cases}$$

$$\delta(-x) = \delta(x)$$

$$\int_{-\infty}^{\infty} \phi(x)\delta(-x)\,\mathrm{d}\,x = \int_{-\infty}^{\infty} \phi(-\eta)\delta(\eta)\,\mathrm{d}\,\eta$$

$$= \phi(0) = \int_{-\infty}^{\infty} \phi(x)\delta(x)\,\mathrm{d}\,x$$

$$\delta(3x) = \frac{1}{3}\delta(x)$$

$$\int_{-\infty}^{\infty} \phi(x)\delta(3x)\,\mathrm{d}\,x = \int_{-\infty}^{\infty} \phi(\eta/3)\,\delta(\eta)\,\mathrm{d}(\eta/3)$$

$$= \phi(0)/3 = \tfrac{1}{3}\int_{-\infty}^{\infty} \phi(x)\delta(x)\,\mathrm{d}\,x$$

$$f(x)\delta(x) = f(0)\delta(x)$$

$$x^2\delta'(x) = \ldots.$$

$$H'(x) = \delta(x)$$

$$\int_{-\infty}^{\infty} \phi(x)H'(x)\,\mathrm{d}\,x = \phi(x)H(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi'(x)H(x)\,\mathrm{d}\,x$$

$$= \phi(\infty) - \int_0^{\infty} \phi'(x)\,\mathrm{d}x = \phi(\infty) - \left[\phi(x)\right]_0^{\infty}$$

$$= \phi(0) = \int_{-\infty}^{\infty} \phi(x)\delta(x)\,dx$$

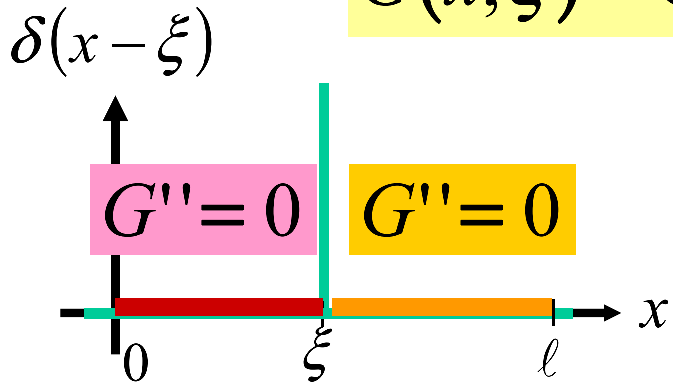
4.3-1 Green's function -- 1

$$G(x;\xi)=\begin{cases} x(\xi-\ell)/\ell, & x<\xi \\ \xi(x-\ell)/\ell, & \xi<x \end{cases}$$

$$G(x;\xi)=G(\xi;x)$$

Consider

$$\begin{cases} \dfrac{\mathrm{d}^2 G(x;\xi)}{\mathrm{d}x^2}=\delta(x-\xi) \\ G(0;\xi)=0 \\ G(\ell;\xi)=0 \end{cases}$$



$$G(x;\xi)=\begin{cases} A_0 x + A_1 & ,0<x<\xi \\ C_0 x + C_1 & ,\xi<x<\ell \end{cases}$$

by B.C.

$$G(x;\xi)=\begin{cases} A_0 x & ,0<x<\xi \\ C_0(x-\ell) & ,\xi<x<\ell \end{cases}$$

Continuity condition:

$$G\big|_{x=\xi-}=G\big|_{x=\xi+} \Rightarrow A_0\,\xi = C_0(\xi-\ell)$$

Jump condition:

(by integrating the equation of G)

$$G'\big|_{x=\xi+}-G'\big|_{x=\xi-}=1$$

$$\Rightarrow C_0 - A_0 = 1$$

4.3-2 Green's function -- 2

BVP with 2nd order ODE

$$\begin{cases} u'' = f(x) \quad , 0 < x < \ell \\ u(0) = 0 \\ u(\ell) = 0 \end{cases}$$

Consider

$$\begin{cases} \dfrac{d^2 G(x;\xi)}{d\,x^2} = \delta(x-\xi) \\ G(0;\xi) = 0 \\ G(\ell;\xi) = 0 \end{cases}$$

$$\int_0^\ell \left(G\frac{d^2 u}{d\,x^2} - u\frac{d^2 G}{d\,x^2} \right) d\,x \quad = 0$$

$$= \int_0^\ell \left[G(x;\xi) f(x) - u(x)\delta(x-\xi) \right] d\,x$$

$$u(\xi) = \int_0^\ell G(x;\xi) f(x) d\,x$$

$x \leftrightarrow \xi \qquad u(x) = \int_0^\ell G(\xi;x) f(\xi) d\xi$

If $\quad G(x;\xi) = G(\xi;x)$

$$u(x) = \int_0^\ell G(x;\xi) f(\xi) d\,\xi$$

$G(x;\xi) \neq G(\xi;x)$?? $u(x) \sim G(x;\xi)$?? $G\big|_{x=\xi-} = G\big|_{x=\xi+}$ $G'\big|_{x=\xi+} - G'\big|_{x=\xi-} = -1$

4.3-3 Green's function -- 3

$G_h = A + B\,\mathrm{e}^{-x}$

BVP with 2nd order ODE

$$\begin{cases} -u'' - u' = f(x)\ ,\ 0 < x < 1 \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

Associated Green's function prob.

$$\begin{cases} -G'' - G' = \delta(x - \xi) \\ G(0;\xi) = 0 \\ G'(1;\xi) = 0 \end{cases}$$

$$G(x;\xi) = \begin{cases} A_1 + B_1\,\mathrm{e}^{-x} & ,0 < x < \xi \\ A_2 + B_2\,\mathrm{e}^{-x} & ,\xi < x < 1 \end{cases}$$

by B.C.

$$G(x;\xi) = \begin{cases} A_1\left(1 - \mathrm{e}^{-x}\right) & ,x < \xi \\ A_2 & ,\xi < x \end{cases}$$

Continuity cond.: $\Rightarrow A_1\left(1 - \mathrm{e}^{-\xi}\right) = A_2$

Jump cond.: $\Rightarrow 0 - (+A_1)\mathrm{e}^{-\xi} = -1$

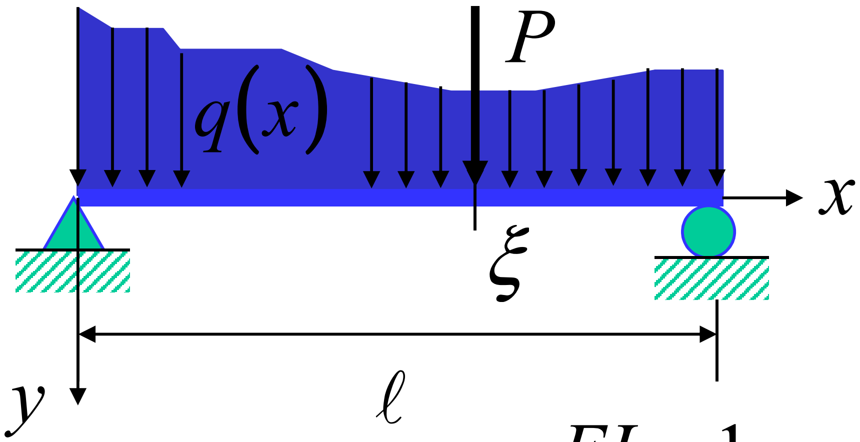
$$G(x;\xi) = \begin{cases} \mathrm{e}^{\xi} - \mathrm{e}^{-(x-\xi)} & ,x < \xi \\ \mathrm{e}^{\xi} - 1 & ,\xi < x \end{cases}$$

$$u(x) \sim G(x;\xi)??$$

4.3-4 Green's function -- 4

Simply supported beam

$$\begin{cases} G^{(iv)} = P\,\delta(x-\xi) \ , 0 < x < \ell \\ G(0) = 0 \quad , \quad G''(0) = 0 \\ G(\ell) = 0 \quad , \quad G''(\ell) = 0 \end{cases}$$



$EI = 1$

$$G(x;\xi) = \begin{cases} A_0 + A_1 x + A_2 x^2 + A_3 x^3 \quad , \quad x < \xi \\ B_0 + B_1 x + B_2 x^2 + B_3 x^3 \quad , \quad \xi < x \end{cases}$$

Continuity cond.: $G\big|_{x=\xi-} = G\big|_{x=\xi+}$ $\quad G'\big|_{x=\xi+} = G'\big|_{x=\xi-}$ $\quad G''\big|_{x=\xi+} = G''\big|_{x=\xi-}$

Jump cond.: $G'''\big|_{x=\xi+} - G'''\big|_{x=\xi-} = P$

5.1-1 Adjoint operator -- 1

differential operator

$$\begin{cases} Lu = f(x) \quad , \quad a < x < b & \text{--- D.E.} \\ \mathbf{B}u = 0 \quad \text{(n conditions)} & \text{--- I.C., B.C.} \end{cases}$$

$$L(\bullet) = \sum_{k=0}^{n} a_k(x) \frac{\mathrm{d}^k(\bullet)}{\mathrm{d}x^k}$$

e.g.

$$Lu \equiv a_n u^{(n)} + a_{n-1} u^{(n-1)} + \cdots + a_1 u' + a_0 u$$

$$B_1 u \equiv u(0) \qquad \text{(Here } a = 0,\ b = \ell\text{)}$$

$$B_2 u \equiv u'(0) + u(\ell)$$

$$\cdots\cdots$$

(linear operator)

5.1-1 Adjoint operator -- 2

differential operator

$$\begin{cases} Lu = f(x) \quad , \quad a < x < b & \text{--- D.E.} \\ \mathbf{B}u = 0 \quad \text{(n conditions)} & \text{--- I.C., B.C.} \end{cases}$$

$$L(\bullet) = \sum_{k=0}^{n} a_k(x) \frac{\mathrm{d}^k(\bullet)}{\mathrm{d}\,x^k}$$

$$\int_a^b v(x) Lu(x)\mathrm{dx} = \left[\text{boundary terms}\right]_a^b + \int_a^b u(x) L^* v(x)\mathrm{dx}$$

$$\int_a^b v(x) Lu(x)\mathrm{dx} = \mathrm{J}(u,v)\Big|_a^b + \int_a^b u(x) L^* v(x)\mathrm{dx}$$

L^* **: adjoint operator of** L

$$L^*(\bullet) = \sum_{k=0}^{n} (-1)^k \frac{\mathrm{d}^k[a_k(x)(\bullet)]}{\mathrm{d}\,x^k}$$

$\mathbf{B}^* v = 0$ **: adjoint B.C. of** $\mathbf{B}u = 0$

$\ni \quad \mathrm{J}(u,v)\Big|_a^b = 0$

If $L = L^* \iff L$: formally self-adjoint operator

If $L = L^*$ & $\mathbf{B} = \mathbf{B}^* \iff$ BVP : self-adjoint system

5.1-2 Adjoint operator -- 3

e.g. $\quad Lu = a_2 u'' + a_1 u' + a_0 u$

$$\int vLu\,\mathrm{d}x = \int v\left(a_2 u'' + a_1 u' + a_0 u\right)\mathrm{d}x$$ (integration by parts, twice)

$$= \left[a_2\left(vu' - uv'\right) + \left(a_1 - a_2'\right)uv\right] + \int u\left[\left(a_2 v\right)'' - \left(a_1 v\right)' + a_0 v\right]\mathrm{d}x$$

$$\Rightarrow L^* v = \left(a_2 v\right)'' - \left(a_1 v\right)' + a_0 v = a_2 v'' + \left(2a_2' - a_1\right)v' + \left(a_2'' - a_1' + a_0\right)v$$

If $a_2' = a_1 \quad \Leftrightarrow \quad L = L^*$, i.e. L formally self-adjoint

i.e. $\quad Lu = \left(a_2 u'\right)' + a_0 u$

More on adjoint operator (1)

(related to the finding of the integrating factor)

Consider $L(u) = a_2(x)u'' + a_1(x)u' + a_0(x)u = 0,$

The equation is exact if it can be expressed as

$$L(u) = a_2(x)u'' + a_1(x)u' + a_0(x)u = 0 = \frac{d}{dx}\left[A(x)u' + B(x)u\right]$$

Then finding the solution of $L(u) = 0$ **(second order)**

is equivalent to finding the solution of

$$A(x)u' + B(x)u = C = \text{constant} \qquad \textbf{(first order)}$$

We have $a_2 = A,\ a_1 = A' + B,\ a_0 = B'$**, and need**

$$a_2'' - a_1' + a_0 = 0 \qquad \text{(for the equation } L(u) = 0 \text{ to be exact)}$$

More on adjoint operator (2)

If $L(u)=0$ **is not exact, we may find an integrating factor** $v(x)$ **to make the equation exact differential, i.e.,**

$$v(x)\left[a_2(x)u''+a_1(x)u'+a_0u\right]=\frac{d}{dx}\left[A(x)u'+B(x)u\right]$$

$$\Rightarrow va_2=A,\ va_1=A'+B,\ va_0=B'$$

$$\Rightarrow (va_1)'=A''+B'=(va_2)''+(va_0)$$

$$\Rightarrow L^*(v)\equiv(a_2v)''-(a_1v)'+(a_0v)=0$$

The integrating factor $v(x)$ **of** $L(u)=0$ **satisfy** $L^*(v)=0$.

$\int_0^1 vLu\,\mathrm{d}x =$

5.1-3 Adjoint operator -- 3

$$\begin{cases} Lu \equiv u'' = f(x) \;, 0 < x < 1 \\ B_1 u \equiv u(0) = 0 \\ B_2 u \equiv u(1) = 0 \end{cases}$$

self-adjoint problem !!

$$\int_0^1 v u''\,\mathrm{d}x = [vu' - uv']\big|_0^1 + \int_0^1 u v''\,\mathrm{d}x$$

$$\Rightarrow \; L^* = L$$

$$= \underline{v(1)}u'(1) - \cancel{u(1)}v'(1) - [\underline{v(0)}u'(0) - \cancel{u(0)}v'(0)] + \int_0^1 u v''\,\mathrm{d}x$$

$$\begin{cases} B_1^{\,*} v \equiv v(0) = 0 \\ B_2^{\,*} v \equiv v(1) = 0 \end{cases}$$

$$\Rightarrow \; \mathbf{B}^* = \mathbf{B}$$

5.1-4 Adjoint operator -- 4

$$\begin{cases} -u''-u'=f(x) \ , 0<x<1 \\ u(0)+u'(0)=0 \\ u(1)-u'(1)=0 \end{cases}$$

$$\Rightarrow \quad L^{*}(\bullet)\equiv -(\bullet)''+(\bullet)'\neq L$$

$$\int_0^1 v\cdot(-u''-u')\mathrm{d}x = \left[-(vu'-uv')-uv\right]\Big|_0^1+\int_0^1 u(-v''+v')\mathrm{d}x$$

$$J=-v(1)u'(1)+u(1)v'(1)-u(1)v(1)-\left[-v(0)\cancel{u'}(0)+u(0)v'(0)-\cancel{u}(0)v(0)\right]$$

$$=-u(1)\left[2v(1)-v'(1)\right]-u(0)v'(0)$$

$$\begin{cases} 2v(1)-v'(1)=0 \\ v'(0)=0 \end{cases}$$

$$\Rightarrow \quad \mathbf{B}^{*}\neq \mathbf{B}$$

5.2-2 Remarks -- 1

Consider ODE

$$L u = f(x) \;, a < x < b$$

Ex. Consider $x^2 \dfrac{\mathrm{d}\,u(x)}{\mathrm{d}\,x} = 0$

It can be verified that

$$u(x) = C_1 + C_2\, H(x) + C_3\, \delta(x)$$

- classical solution — **(strong solution)** $u(x)$ is sufficiently differentiable
- weak solution — $u(x)$ is locally integrable, not sufficiently differentiable
- **distributional solution** — $u(x)$ is a generalized function

5.2-2 Remarks -- 2

$L = L^*$

homogeneous B.C.

Consider BVP

$$\begin{cases} (a_2 u')' + a_0 u = f(x) \quad , a < x < b \\ \mathbf{B} u = 0 \end{cases}$$

Different Boundary conditions:

- **unmixed** ✓ $\begin{cases} \alpha_1 u(a) + \alpha_2 u'(a) = 0 \\ \beta_1 u(b) + \beta_2 u'(b) = 0 \end{cases}$
- **periodic** ✓ $\begin{cases} u(a) = u(b) \\ u'(a) = u'(b) \end{cases}$ with $a_2(a) = a_2(b)$
- **initial conditions** ✗ $\begin{cases} u(a) = 0 \\ u'(a) = 0 \end{cases}$

General homogeneous B.C.'s for 2nd order linear equation

$$Lu = a_2(x)u''(x) + a_1(x)u'(x) + a_0(x)u(x) = f(x)$$
$$B_1 u = \alpha_{11}u(a) + \alpha_{12}u'(a) + \beta_{11}u(b) + \beta_{12}u'(b) = 0$$
$$B_2 u = \alpha_{21}u(a) + \alpha_{22}u'(a) + \beta_{21}u(b) + \beta_{22}u'(b) = 0$$

$$\text{unmixed} \Rightarrow \beta_{11} = \beta_{12} = \alpha_{21} = \alpha_{22} = 0$$
$$\text{I.C.'s} \Rightarrow \alpha_{12} = \beta_{11} = \beta_{12} = \alpha_{21} = \beta_{21} = \beta_{22} = 0$$

$$u(x) \sim G(x;\xi)??$$

$$\int_a^b v(x) L u(x) \mathrm{d}x = \mathrm{J}(u,v)\Big|_a^b + \int_a^b u(x) L^* v(x) \mathrm{d}x$$

5.2-1 Adjoint Green's function -- 1

$$\begin{cases} L u = f(x) \quad , a < x < b \\ B_1 u = 0 \\ B_2 u = 0 \end{cases} \qquad \begin{cases} L G = \delta(x-\xi) \\ B_1 G = 0 \\ B_2 G = 0 \end{cases} \qquad \begin{cases} L^* G^* = \delta(x-\eta) \\ B_1^* G^* = 0 \\ B_2^* G^* = 0 \end{cases}$$

$$\int_a^b \left[G^* L u - u L^* G^* \right] \mathrm{d}x = \mathrm{J}(u, G^*)\Big|_a^b = 0$$

$$\Rightarrow u(\eta) = \int_a^b G^*(x;\eta) f(x) \mathrm{d}x \qquad \Rightarrow u(x) = \int_a^b G^*(\eta;x) f(\eta) \mathrm{d}\eta$$

$$\int_a^b \left[G^*(x;\eta) L G(x;\xi) - G(x;\xi) L^* G^*(x;\eta) \right] \mathrm{d}x = \mathrm{J}(G, G^*)\Big|_a^b = 0$$

(for homogeneous B.C.'s)

$$\Rightarrow G(\eta;\xi) = G^*(\xi;\eta) \qquad \Rightarrow u(x) = \int_a^b G(x;\xi) f(\xi) \mathrm{d}\xi$$

$u(x) \sim G(x;\xi)$??

$\Rightarrow u(x) = \int_a^b G(x;\eta) f(\eta) \mathrm{d}\eta + \cdots$???

5.3-1 Inhomogeneous B.C. -- 1

$$\begin{cases} \underline{L u = f(x)} \quad , a < x < b \\ B_1 u = u_0 \\ B_2 u = v_0 \end{cases}$$
(inhomogeneous)

$$\begin{cases} L G = \delta(x-\xi) \\ B_1 G = 0 \\ B_2 G = 0 \end{cases}$$
(homogeneous)

$$\begin{cases} \underline{L^* G^* = \delta(x-\eta)} \\ B_1^* G^* = 0 \\ B_2^* G^* = 0 \end{cases}$$
(homogeneous)

$$\int_a^b \left[G^* Lu - u L^* G^*\right] \mathrm{d}x = \mathrm{J}\left(u, G^*\right)\Big|_a^b \neq 0$$

$$\Rightarrow G(\eta;\xi) = G^*(\xi;\eta)$$

Equation (*)

$$\int_a^b \left[G^*(x;\eta) \underline{Lu(x)} - u(x) \underline{L^* G^*(x;\eta)}\right] \mathrm{d}x = \mathrm{J}\left[u(x), G^*(x;\eta)\right]\Big|_{x=a}^b$$

$$u(\eta) = \int_a^b G^*(x;\eta) f(x) \mathrm{d}x - \mathrm{J}\left[u(x), G^*(x;\eta)\right]\Big|_{x=a}^b$$

$\eta \leftrightarrow x \; ; \; \eta \to \xi$

Using Equation (*)

$$u(x) = \int_a^b G(x;\xi) f(\xi) \mathrm{d}\xi - \mathrm{J}\left[u(\xi), G(x;\xi)\right]\Big|_{\xi=a}^b$$

For second order problem: $J(u,v)=\left[a_2(vu'-uv')+(a_1-a_2')uv\right]$

5.3-1 Inhomogeneous B.C. -- 2

$$\begin{cases} Lu=f(x) \ , a<x<b \\ B_1 u=u_0 \\ B_2 u=v_0 \end{cases} \qquad \begin{cases} LG=\delta(x-\xi) \\ B_1 G=0 \\ B_2 G=0 \end{cases} \qquad \begin{cases} L^* G^*=\delta(x-\eta) \\ B_1^* G^*=0 \\ B_2^* G^*=0 \end{cases}$$

$$u(x)=\int_a^b G(x;\xi)f(\xi)\mathrm{d}\xi-\mathrm{J}\left[u(\xi),G(x;\xi)\right]\Big|_{\xi=a}^{b}$$

$$\mathrm{J}\left[u(\xi),G(x;\xi)\right]=a_2(\xi)\left[G(x;\xi)\frac{\mathrm{d}u}{\mathrm{d}\xi}-u(\xi)\frac{\mathrm{d}G(x;\xi)}{\mathrm{d}\xi}\right]+\left[a_1(\xi)-a_2'(\xi)\right]u(\xi)G(x;\xi)$$

5.3-2 Inhomogeneous B.C. -- 3

$$J[u(\xi),G(x;\xi)]=a_2(\xi)\left[G(x;\xi)\frac{\mathrm{d}u}{\mathrm{d}\xi}-u(\xi)\frac{\mathrm{d}G(x;\xi)}{\mathrm{d}\xi}\right]+[a_1(\xi)-a_2'(\xi)]u(\xi)G(x;\xi)$$

$$\begin{cases} u''=f(x) \quad ,0<x<\ell \\ u(0)=u_1 \\ u(\ell)=u_2 \end{cases}$$

Associated Green's function problem

$$\begin{cases} G''=\delta(x-\xi) \quad ,0<x,\xi<\ell \\ G(0;\xi)=0 \quad ;G(\ell;\xi)=0 \end{cases}$$

$$\Rightarrow G(x;\xi)=\begin{cases} \frac{x(\xi-\ell)}{\ell} & ,0<x<\xi \\ \frac{\xi(x-\ell)}{\ell} & ,\xi<x<\ell \end{cases} \Rightarrow \frac{\mathrm{d}G(x;\xi)}{\mathrm{d}\xi}=\begin{cases} x/\ell \ , & 0<x<\xi<l \\ (x-\ell)/\ell \ , & 0<\xi<x<\ell \end{cases}$$

$$J[u(\xi),G(x;\xi)]\Big|_{\xi=0}^{\ell}=\left[\cancel{G(x;\ell)\frac{\mathrm{d}u(\ell)}{\mathrm{d}\xi}}-u(\ell)\frac{\mathrm{d}G(x;\ell)}{\mathrm{d}\xi}\right]-\left[\cancel{G(x;0)\frac{\mathrm{d}u(0)}{\mathrm{d}\xi}}-u(0)\frac{\mathrm{d}G(x;0)}{\mathrm{d}\xi}\right]$$

$$=-\left[u_2\frac{x}{\ell}+u_1\frac{(\ell-x)}{\ell}\right]$$

$$u(x)=\int_a^b G(x;\xi)f(\xi)\mathrm{d}\xi+\left[u_2\frac{x}{\ell}+u_1\frac{(\ell-x)}{\ell}\right]$$

5.3-3 Inhomogeneous B.C. -- 4

Ex. $\begin{cases} u'' = f(t), \ 0 < t < \infty \\ u(0) = u_0 \\ u'(0) = v_0 \end{cases}$

Associated Green's function problem

$$\begin{cases} G'' = \delta(t-\xi), \quad 0 < t, \xi < \infty \\ G(0;\xi) = 0; \quad G'(0;\xi) = 0 \end{cases}$$

$$G(t;\xi) = \begin{cases} A_1 + A_2 t, \ 0 < t < \xi \\ B_1 + B_2 t, \ \xi < t < \infty \end{cases}$$

by I.C.

$$G(t;\xi) = \begin{cases} 0, \ 0 < t < \xi \\ B_1 + B_2 t, \ \xi < t < \infty \end{cases}$$

by continuity condition

$$G(t;\xi) = \begin{cases} 0, \ 0 < t < \xi \\ B_2 (t-\xi), \ \xi < t < \infty \end{cases}$$

by jump condition

$$G(t;\xi) = \begin{cases} 0, \ 0 < t < \xi \\ t-\xi, \ \xi < t < \infty \end{cases}$$

5.3-3 Inhomogeneous B.C. -- 6

$$u(x)=\int_a^b G(x;\xi)\,f(\xi)\,\mathrm{d}\xi-\mathrm{J}\left[u(\xi),G(x;\xi)\right]\Big|_{\xi=a}^{b}$$

$$\mathrm{J}\left[u(\xi),G(x;\xi)\right]=a_2(\xi)\left[G(x;\xi)\frac{\mathrm{d}u}{\mathrm{d}\xi}-u(\xi)\frac{\mathrm{d}G(x;\xi)}{\mathrm{d}\xi}\right]+\left[a_1(\xi)-a_2'(\xi)\right]u(\xi)G(x;\xi)$$

$$\int_0^\infty\left[G^*(t;\xi)\,f(t)-u(t)\,\delta(t-\xi)\right]\mathrm{d}t=\left[G^*(t;\xi)\frac{\mathrm{d}u}{\mathrm{d}t}-u(t)\frac{\mathrm{d}G^*(t;\xi)}{\mathrm{d}t}\right]\Bigg|_0^\infty$$

$$u(t)=\int_0^\infty G(t;\xi)\,f(\xi)\,\mathrm{d}\xi-\left[G(t;\xi)\frac{\mathrm{d}u}{\mathrm{d}\xi}-u(\xi)\frac{\mathrm{d}G(t;\xi)}{\mathrm{d}\xi}\right]\Bigg|_{\xi=0}^{\infty}$$

$$G(t;\xi)=\begin{cases}0, & 0<t<\xi\\ t-\xi, & \xi<t<\infty\end{cases}$$

$$\frac{\mathrm{d}G(t;\xi)}{\mathrm{d}\xi}=\begin{cases}0, & 0<t<\xi\\ -1, & \xi<t<\infty\end{cases}$$

$$u(t)=\int_0^t(t-\xi)\,f(\xi)\,\mathrm{d}\xi+v_0t+u_0$$