II. Ordinary differential equations (ODE's) (Part 1)

- Introduction
- Existence & uniqueness of IVP
- System of 1st order ODE's (const. coefficients)
- Green's function – an introduction
- Green's functions
- Alternative theorem & modified Green's functions
- Eigenfunction expansion

Differential Equations and Mathematical models

Many important and significant problems in engineering, physical sciences, and social sciences, when formulated in mathematical terms, require the determination of a function satisfying an equation containing derivatives of the unknown function. Such equations are called differential equations.

Various mathematical equations

Transcendantal equation : $\cos u + \sin u = 1$

Differential equations : $u'' + 4u' + 2u = f(x)$

(with $u,_{xx} = \partial^2 u / \partial x^2$, ...), $u,_{xx} + 2u,_{xy} u,_{yy} = f(x, y)$

Integral equations (Volterra, Fredholm) :

1st kind: $\int u(\xi) K(x;\xi) \mathrm{d}\xi = g(x)$

2nd kind: $u(x) + \int u(\xi) K(x;\xi) \mathrm{d}\xi = g(x)$

Differential-integral equations

ODE for unknown $y = y(x)$

- **General (nonlinear) equation for order n**

$$y^{(n)}(x) = F(x, y(x), y'(x), \ldots\ldots y^{(n-1)}(x))$$
$$= \tilde{F}(y(x), y'(x), \ldots\ldots y^{(n-1)}(x)) + r(x)$$

- **General linear equation for order n**

$$y^{(n)} + P_{n-1}(x) y^{(n-1)} + \ldots\ldots + P_1(x) y' + P_0(x) y = r(x)$$

- **Homogeneous** ⇨ $r(x) = 0$

Examples

Linear equations

$$y'' + 4y' + 2y = r(x), \qquad y'' + 4x^3 y' + (2x-3)y = r(x)$$

(constant coefficients) (variable coefficients)

(possesses n independent solutions, involving n constants of integration, for a n-th order equation)

Nonlinear equations

$$y^2 y'' + 4y^3 y' + 2\cos y = r(x), \qquad y'' + 4y^3 y' + 2\cos y = r(x)$$

(general nonlinear) (quasi-linear – linear in the highest order)

(processes n independent solutions for a n-th order equation plus (sometimes) special additional solutions)

Solutions of ODE's

A **singular solution** of an ordinary differential equation is a solution that is singular, or one for which the initial value problem fails to have a unique solution at some points on the solution.

- **complete solution**
  - ☞ **general solution**
    - **complementary solution (homogeneous solution)**
    - **particular solution**

$$u = u_h + u_p$$ **(only for linear equations)**

  - ☞ **singular solution**

https://en.wikipedia.org/wiki/Singular_solution

- **initial-valued problem (IVP) – specify conditions at one point**

boundary-valued problem (BVP) – specify conditions at multiple points

Solution forms and methods

- explicit solutions

  implicit solutions

- analytic solutions (closed, series, integral forms)

  approximate solutions

  numerical solutions

Solving homogeneous linear equations

- **constant coefficient equations**
- **equidimensional (or Euler) equations**
- **exact equations (integration factor)**
- **reduction of order** (if one solution $y^*(x)$ is known, set $y(x) = u(x)y^*(x)$, the equation governing $u(x)$ is of lower order)
- **transformation to known equations** (Airy equation, parabolic cylinder equation, Bessel equation, …)
- **series solutions**
- **integral transforms (Laplace, Fourier transform, ……)**

Solving inhomogeneous linear equations

- **All first order linear inhomogeneous equations are soluble (integrating factor)**
- **Variation of parameter (second order equation)**
- **Green's function method**
- **Method of integral transform**
- **Reduction of order**
- **Method of undetermined coefficients**
  (determining particular solution of equation with constant coefficients and particular inhomogeneous terms)

Solving first order nonlinear equations (1)
(has few solutions)

■ Bernoulli equation

$$y' = a(x)y + b(x)y^p$$

$p = 0$ ⇨ equation is linear

$p = 1$ ⇨ equation is linear and separable

$p = \text{others}$ ⇨ let $u(x) = [y(x)]^{1-p}$, equation becomes

$$u'(x) = (1-p)a(x)u(x) + (1-p)b(x),$$

which is linear and solvable

Solving first order nonlinear equations (2)

Riccati equation

$$y'(x) = a(x)y^2(x) + b(x)y(x) + c(x)$$

$a(x) = 0$ ⇨ equation is linear

$c(x) = 0$ ⇨ equation becomes Bernoulli equation ($p = 2$)

Other cases ⇨ no general technique for obtaining a solution

$$y(x) = -\frac{w'(x)}{a(x)w(x)} \quad \Rightarrow \quad w''(x) - \left[\frac{a'(x)}{a(x)} + b(x)\right]w'(x) + a(x)c(x)w(x) = 0$$

(linear but no analytical solution) (Series solution?)

$$y(x) = y_1(x) + u(x) \quad \Rightarrow \quad u'(x) = \left[b(x) + 2a(x)y_1(x)\right]u(x) + a(x)u^2(x)$$

(given one solution $y_1(x)$) **(which is the Bernoulli equation)**

Solving other nonlinear equations

- **exact equations**
- **substitution**
- **scale invariant equation**
- **……….**

⇨ **the basic idea is to transform the equation into a simpler equation, or reduces the order of the equation**

(see C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers*, McGraw-Hill, 1978).

2. Existence and uniqueness theorem of IVP

- If you can solve the problem and obtain the solution, you do not need the existence and uniqueness theorem.
- However, (i) ∃ many IVP's you cannot solve analytically (need numerical/approximate solution); (ii) furthermore, the problem may have unique solution, many solutions, or no solution.
- It is not *clever* for you to solve "a problem without solution" numerically.
- Thus the existence and uniqueness theorem is important and required (before you solve the problem numerically, say)!

Examples (linear cases)

- $|y'| + |y| = 0, \quad y(0) = 1$

  The IVP has no solution because $y = 0$ is the only solution, which does not satisfy the I.C.

- $y' = x, \quad y(0) = 1$

  The IVP has precisely one solution, $y = x^2/2 + 1$.

- $xy' = y - 1, \quad y(0) = 1$

  The IVP has infinitely many solutions, $y = 1 + cx$, where c is arbitrary.

Solution procedures for simple examples

Ex. Consider $\dfrac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \mathbf{A}\mathbf{y}$

let $\mathbf{y} = \mathrm{e}^{\lambda x}\mathbf{u}$

$$\Rightarrow \quad \lambda \mathrm{e}^{\lambda x}\mathbf{u} = \mathbf{A}\mathbf{u}\mathrm{e}^{\lambda x}$$

$$\therefore \quad \mathbf{A}\mathbf{u} = \lambda \mathbf{u}$$

or $(\mathbf{A} - \lambda \mathbf{I})\mathbf{u} = \mathbf{0}$

solve for λ & $\mathbf{u}$

eigenvalues & eigenvectors

$$\mathbf{y} = \sum_i c_i \mathrm{e}^{\lambda_i x}\mathbf{u}_i$$

$$y'' - 5y' + 6y = 0$$

let $y = a\,\mathrm{e}^{\lambda x}$

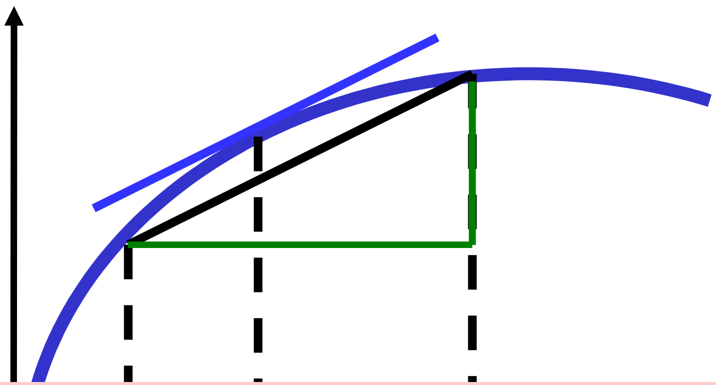
$$\Rightarrow \quad (\lambda^2 - 5\lambda + 6)\,a\,\mathrm{e}^{\lambda x} = 0$$

$$\therefore \quad \lambda^2 - 5\lambda + 6 = 0$$

solve for λ

$$y = a_1 \mathrm{e}^{\lambda_1 x} + a_2 \mathrm{e}^{\lambda_2 x}$$

2.1 Lipschitz condition – 1



If $f'(y)$ exists and is continuous, for $y_1 \le y \le y_2$

$\Rightarrow$ $f(y)$ satisfies Lipschitz condition for $y_1 \le y \le y_2$

***Mean-value theorem*:**

If $f'(y)$ exists and is continuous, for $y_1 \le y \le y_2$

$$\Rightarrow \exists \eta \ni \ |f(y_2) - f(y_1)| = |f'(\eta)| \cdot |y_2 - y_1|, \quad y_1 \le \eta \le y_2$$

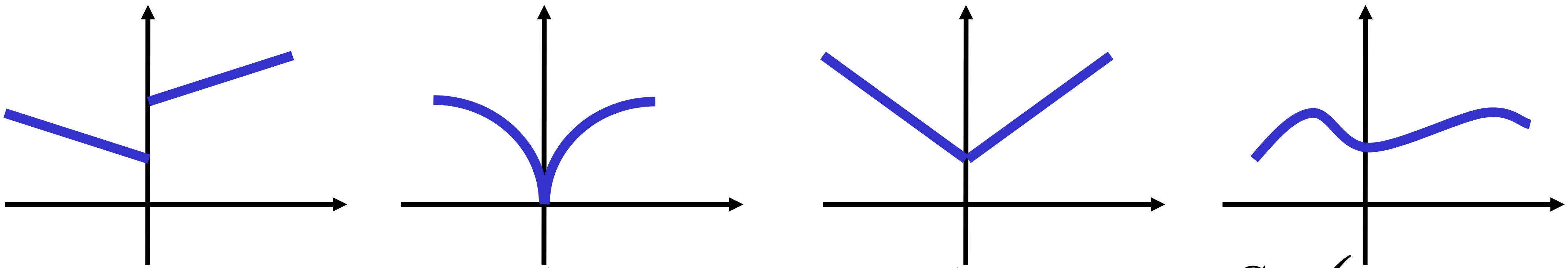
2.1 Lipschitz condition – 2

$f(y)$ satisfies a ***Lipschitz condition*** in a closed interval I

$\Leftrightarrow\ \exists$ a constant C,

$\ni\ \left|f(y_2)-f(y_1)\right| \le C\cdot\left|y_2-y_1\right|,\quad \forall\, y_2, y_1 \in I$

C: continuous
L: Lipschitz condition
D: differentiable



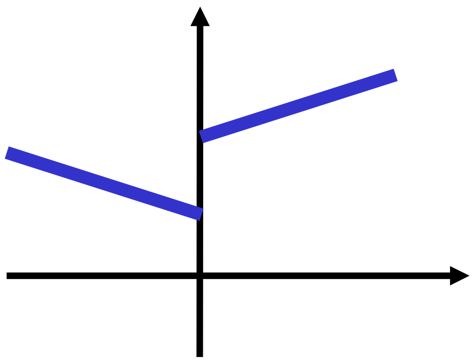
C ×	C ✓	C ✓	C ✓
L ×	**L** ×	**L** ✓	L ✓
D ×	D ×	D ×	D ✓

2.1 Lipschitz condition – 2

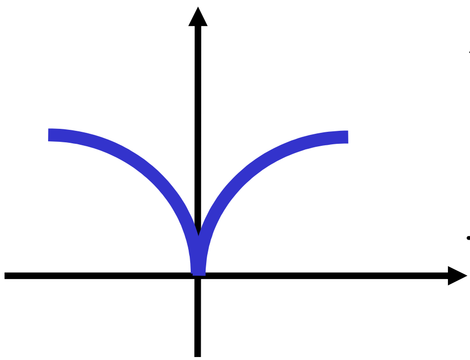
$f(y)$ satisfies a ***Lipschitz condition*** in a closed interval I

$\Leftrightarrow \exists$ a constant C,

$$\ni \quad \left|f(y_2)-f(y_1)\right| \le C \cdot \left|y_2 - y_1\right|, \quad \forall\, y_2, y_1 \in I$$



C ×
L ×
D ×



C ✓
L ×
D ×

Ex. $f(y) = y^{2/3}$,

take $y_1 = 0, \ y_2 = \varepsilon$.

$$\frac{\left|f(y_2)-f(y_1)\right|}{\left|y_2 - y_1\right|} = \frac{\varepsilon^{2/3}}{\varepsilon} = \varepsilon^{-1/3}$$

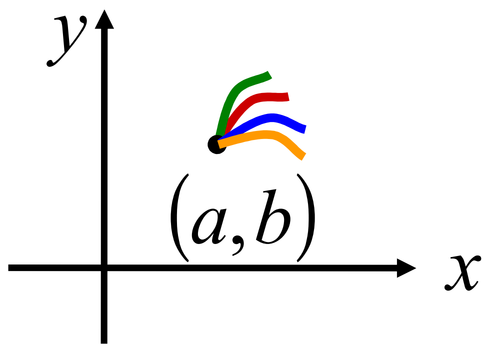
$\to \infty$ as $\varepsilon \to 0$.

Thus Lipschitz condition cannot be satisfied for interval containing 0.

2.2 Existence & Uniqueness theorem of IVP

Consider an **IVP** of function $y(x)$ as

$$\begin{cases} y' = f(x,y), \; x > a & \text{------ D. E.} \\ y(a) = b & \text{------ I. C.} \end{cases}$$



1) If $f(x,y)$ is **continuous** in some closed neighborhood of (a,b) in the (x,y) plane; and

2) $f(x,y)$ satisfies a **Lipschitz condition** in y argument;

$\Rightarrow$ Then there **exists** a **unique** solution $y(x)$ of the IVP in $|x-a| < \delta \quad , \quad \delta > 0$

(If the conditions fail, the IVP may have a unique solution, many solutions, or no solution!)

Remark 2.2–1

Consider an IVP of function $y(x)$ as

$$\begin{cases} y' = f(x, y), \ x > a \\ y(a) = b \end{cases}$$

------ **D. E.**

------ **I. C.**

If $f(x, y) = f(x) + g(x)y$ and is continuous

(IVP with linear 1st order ODE)

$\Rightarrow$ Satisfies Lipschitz condition

$\Rightarrow$ Then the IVP has the unique solution

Remark 2.2–2

Ex. $\begin{cases} y' - k\, y^{1/2} = 0 \quad , x > 0 & \text{------ D. E.} \\ y(0) = 0 & \text{------ I. C.} \end{cases}$

$f(x,y) = k\, y^{1/2}$ is continuous for $x \geq 0$

But $f(x,y)$ does not satisfies Lipschitz condition*

$\Rightarrow$ At $x = 0,\ y = 0$, the theorem does not apply.

Thus the IVP may have more than one solution.

*To see this, consider an interval with $y_1 = 0$ (where I.C. applied) and finite y_2.

$$\frac{\left|f(x,y_2) - f(x,0)\right|}{\left|y_2 - 0\right|} = \left|\frac{k y_2^{1/2} - 0}{y_2 - 0}\right| = \left|k y_2^{-1/2}\right| = C,$$ thus C is unbounded as $y_2 \to 0$.

Remark 2.2–2 （續-1）

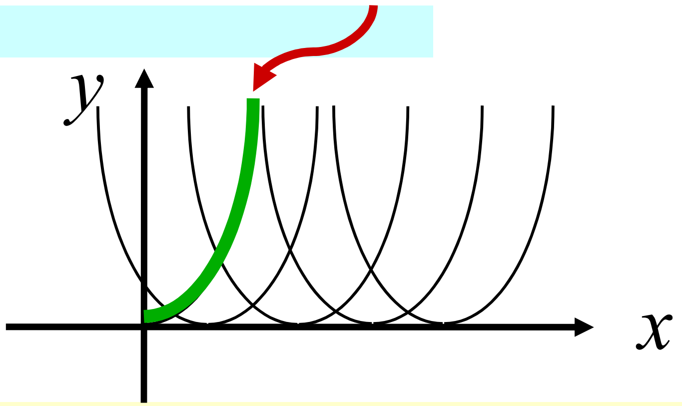
Ex. $\begin{cases} y' - k\,y^{1/2} = 0 & , x > 0 \\ y(0) = 0 \end{cases}$

from I.C. $y = {(k\,x)^2}/{4}$

$$\frac{dy}{dx} = k\,y^{1/2} \quad \Rightarrow \quad y^{-1/2}\,dy = k\,dx$$

$$2y^{1/2} = k\,x + c$$

$$\Rightarrow \quad y = \frac{(k\,x + c)^2}{4}$$



general solution: 'every solution' is given by some appropriate choice of c

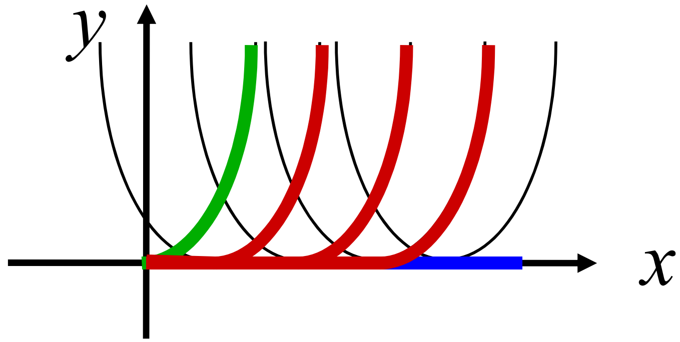
Singular solution: envelope of the general solution.

General solution may not be a complete solution.

Remark 2.2–2 （續-2）

Finding the envelope of one-parameter family of curves

let $\phi(x, y\ ; c) \equiv \dfrac{(kx+c)^2}{4} - y = 0$



envelope is given by

$$\begin{cases} \phi(x, y; c) = 0 \\ \dfrac{\partial \phi}{\partial c} = 0 \end{cases}$$

$\Rightarrow (kx+c)^2 - 4y = 0$

$\Rightarrow 2(kx+c)/4 = 0$

$\Rightarrow \quad y = 0$

singular solution

Remark 2.2–2 （續-3）

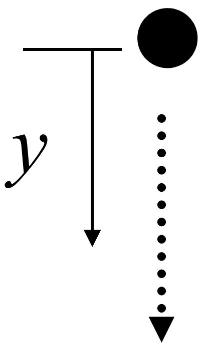
Physical interpretation:

$$\begin{cases} y' - k\, y^{1/2} = 0 & , x > 0 \\ y(0) = 0 \end{cases}$$

$$y'' = \frac{d\,y'}{d\,x} = \frac{d\,y'}{d\,y}\frac{d\,y}{d\,x} = y'\frac{d\,y'}{d\,y}$$

$\Rightarrow$ Solution is not unique

Drop a ball from rest at time "$x = 0$"



$$\begin{cases} my'' = mg & , x > 0 \\ y(0) = y'(0) = 0 \end{cases}$$

$$\Rightarrow \quad y = g\,x^2 / 2$$

(The green curve on the last page – unique solution)

$$\begin{cases} \frac{1}{2} m \left(y'\right)^2 = mgy, \ x > 0 > 0 \\ y(0) = 0 \end{cases}$$

energy conservation: KE = PE

(Convert to a nonlinear equation: multiple solutions)

$$|\mathbf{f}_2-\mathbf{f}_1| \equiv \sum_i^n |f_{2i}-f_{1i}|$$

Remark 2.2 – 3

$$|\mathbf{y}_2-\mathbf{y}_1| \equiv \sum_i^n |y_{2i}-y_{1i}|$$

IVP of function $y(x)$ as

$$\begin{cases} y' = f(x,y) & , x > a \\ y(a) = b \end{cases}$$

IVP of system of equations

$$\begin{cases} \mathbf{y}' = \mathbf{f}(x,\mathbf{y}) & , x > a \\ \mathbf{y}(a) = \mathbf{b} \end{cases}$$

1) If $\mathbf{f}(x,\mathbf{y})$ is continuous in some closed neighborhood of $(a,\mathbf{b})$ in the $(x,\mathbf{y})$ plane; and

2) $\mathbf{f}(x,\mathbf{y})$ satisfies a Lipschitz condition in $\mathbf{y}$ argument;

$\Rightarrow$ Then there exists a unique solution $\mathbf{y}(x)$ of the IVP in $|x-a| < \delta \quad , \quad \delta > 0$

Remark 2.2 – 4

IVP with high order ODE

$$\begin{cases} y^{(n)} = f\left(x, y, y', \cdots, y^{(n-1)}\right) \quad , x > a \\ y(a) = y_0 \\ y'(a) = y'_0 \\ \vdots \\ y^{(n-1)}(a) = y_0^{(n-1)} \end{cases}$$

let
$$\begin{cases} y = y_1 \\ y' = y_2 \\ y'' = y_3 \\ \vdots \\ y^{(n-1)} = y_n \end{cases} \Rightarrow \begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_{(n-1)}' = y_n \end{cases}$$

$$y_n' = f\left(x, y_1, y_2, \cdots, y_{(n-1)}\right)$$

IVP of system of equations

$$\begin{cases} \mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \; x > a \\ \mathbf{y}(a) = \mathbf{b} \end{cases}$$

2.3-1 Proof

A. Existence:

a. construct a sequence of approximate solutions

b. show it converges to an exact solution

B. Uniqueness:

2.3-2 Proof (existence – 1)

Consider an IVP of function $y(x)$ as

$$\begin{cases} y' = f(x, y), \ x > a \\ y(a) = b \end{cases} \Rightarrow \quad y(x) = b + \int_a^x f[\xi, y(\xi)]\,\mathrm{d}\xi$$

Construct a successive approximation (Picard's iteration method)

let $y_0(x) = b$

$$y_1(x) = b + \int_a^x f(\xi, y_0)\,\mathrm{d}\xi$$

$$y_2(x) = b + \int_a^x f(\xi, y_1)\,\mathrm{d}\xi$$

$$y_n(x) = b + \int_a^x f(\xi, y_{n-1})\,\mathrm{d}\xi$$

$y_n(x) \to y(x)$ as $n \to \infty$?

(If yes, a solution can be found at least by Picard's iteration method, or say, there exists a solution.)

2.3-3 uniform convergence (1)

convergence: (conditional) convergence, absolute convergence

$$\Rightarrow \sum a_k + \sum b_k = \sum (a_k + b_k)$$ (convergent)

$$\Rightarrow \sum a_k \cdot \sum b_k = a_1 b_1 + a_1 b_2 + \cdots + a_i b_j + \cdots$$ (absolutely convergent)

infinite series $\sum a_k$

1. comparison test
2. ratio test $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$
3. root test $\lim_{k\to\infty} \sqrt[k]{a_k}$

< 1 convergent,

> 1 divergent,

$= 1$? ? ?

2.3-3 uniform convergence (2)

uniform convergence, pointwise convergence

$$\frac{d}{dx}\sum a_k(x) = \sum a'_k(x)$$

$$\lim_{n\to\infty}\int \quad \to \quad \int \lim_{n\to\infty}$$

Consider an infinite series of function

$$S(x) = f_1(x) + f_2(x) + \cdots + f_k(x) + \cdots$$

$S_k(x)$: partial sum of its first k-terms. The series converges

to $S(x) \Leftrightarrow$ for any given $\varepsilon > 0,\ \exists N > 0$ **(N independent of x)**

$$\ni \left|S(x) - S_k(x)\right| < \varepsilon, \quad \forall k > N$$

2.3-4 M-test (comparison test)

A series of function

$$S(x) = f_1(x) + f_2(x) + \cdots + f_k(x) + \cdots$$

converges uniformly in the interval of interest,

IF: $\exists$ (absolutely) convergent series Q

$$Q = Q_1 + Q_2 + \cdots Q_k + \cdots$$

and $0 \le |f_i(x)| \le Q_i \quad , \forall\, x \in$ (the interval of interest)

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

2.3-5 Proof (existence – 2)

Consider $$S(x) = y_0(x) + [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)] + \cdots$$

$$|y_1(x) - y_0(x)| = \left| \int_a^x f[\xi, y_0(\xi)]\,\mathrm{d}\xi \right| \le \int_a^x |f[\xi, y_0(\xi)]|\,\mathrm{d}\xi \le M\,|x-a| = \frac{M}{C}\frac{C|x-a|}{1!}$$

$$|y_2(x) - y_1(x)| = \left| \int_a^x \left\{ f[\xi, y_1(\xi)] - f[\xi, y_0(\xi)] \right\} \mathrm{d}\xi \right| \le \int_a^x |f[\xi, y_1(\xi)] - f[\xi, y_0(\xi)]|\,\mathrm{d}\xi$$

$$\le C\int_a^x |y_1(\xi) - y_0(\xi)|\,\mathrm{d}\xi \le CM\frac{|x-a|^2}{2} = \frac{M}{C}\frac{C^2|x-a|^2}{2!}$$

$$|y_3(x) - y_2(x)| = \cdots \le C\int_a^x |y_2(\xi) - y_1(\xi)|\,\mathrm{d}\xi \le C^2 M\frac{|x-a|^3}{3!} = \frac{M}{C}\frac{C^3|x-a|^3}{3!}$$

(An absolutely and uniformly convergent series through the M-test) ↓

By mathematical induction $$|y_n(x) - y_{n-1}(x)| \le \frac{M}{C}\frac{C^n|x-a|^n}{n!}$$

$$S(x) \le b - \frac{M}{C} + \frac{M}{C}e^{C|x-a|}$$

2.3-5 Proof (existence – 3)

- **S(x) is an absolutely and uniformly convergent series through the Weierstrass M-test**
- $\lim_{k\to\infty} y_k(x) = y(x)$ $\lim_{k\to\infty} f(x, y_k(x)) = f(x, y(x))$
- $$y(x) = \lim_{n\to\infty} y_n(x) = b + \lim_{n\to\infty} \int_a^x f(\xi, y_{n-1})\,\mathrm{d}\xi$$
  $$= b + \int_a^x \lim_{n\to\infty} f(\xi, y_{n-1})\,\mathrm{d}\xi = b + \int_a^x f(\xi, y)\,\mathrm{d}\xi$$

(A solution can be found at least by the Picard iteration procedure)

2.3-6 Proof (uniqueness)

Let $Y(x)$ & $y(x)$ be two solutions to the IVP

$$|Y(x)-y(x)| = \left|\int_a^x \{f[\xi, Y(\xi)] - f[\xi, y(\xi)]\}\,\mathrm{d}\xi\right| \le C\int_a^x |Y(\xi)-y(\xi)|\,\mathrm{d}\xi$$

$$\Rightarrow r(x) \le C\int_a^x r(\xi)\,\mathrm{d}\xi \quad \left[\text{ where } \; r(x) = |Y(x)-y(x)| \;\right]$$

let $\quad u(x) = \int_a^x r(\xi)\mathrm{d}\xi \ge 0$ **($u(x)$ is non-negative)**

$\therefore \dfrac{\mathrm{d}u}{\mathrm{d}x} \le Cu(x)$ $\quad \left[\dfrac{\mathrm{d}u}{\mathrm{d}x} - Cu(x)\right]\mathrm{e}^{-C|x-a|} \le 0$ Integrate from a to x $\quad u(x')\mathrm{e}^{-C|x'-a|}\Big|_{x'=a}^{x'=x} \le 0$

$u(x) \le u(a)e^{C|x-a|} = 0$ (because $u(a)=0$), $\quad \therefore \; Y(x) = y(x)$

Well-posedness of IVP

- IVP = DE + IC's
- The IVP is said to be well-posed in a domain D when (i) there is one and only one solution $y = y(x,b)$ in D of the given DE for each given $(a,b) \in D$; and (ii) when this solution varies continuously with b, i.e., the solution depends continuously on the initial value.

Recall the IVP:

$$\begin{cases} y' = f(x,y), & x > a \\ y(a) = b \end{cases}$$

2.4 Well-posedness theorem

Consider an ODE of function $y(x)$ as

$$y' = f(x, y), \; x > a$$

Let $y_1(x)$ & $y_2(x)$ be two solutions of the ODE in a domain D

where $f(x, y)$ satisfies a Lipschitz condition in y argument,

then

$$\left| y_2(x) - y_1(x) \right| \leq \mathrm{e}^{C|x-a|} \left| y_2(a) - y_1(a) \right|$$

for a finite C.

(Aim of the theorem: to see the effect of the variation of the IC on the solution; the variation of the solutions is bounded if the difference in IC's is finite)

Example

Consider $y'' - \frac{(1+x)}{x} y' + \frac{y}{x} = 0,$

The general solution is $y(x) = C_1 e^x + C_2(1+x)$

(1) $y(1) = 1, y'(1) = 2 \Rightarrow C_1 = 3/e, C_2 = -1$

$y = \frac{3}{e} e^x - (1+x)$ **(well-posed)**

(2) $y(0) = 1,\ y'(0) = 2 \ \Rightarrow\ C_1 + C_2 = 1,\ C_1 + C_2 = 2$

(cannot determine C_1 and C_2, solution does not exist, ill-posed !)

(3) $y(0) = 1,\ y'(0) = 1 \ \Rightarrow\ C_1 + C_2 = 1,\ C_1 + C_2 = 1$

(can determine infinite pairs of the C_1 and C_2, there exist infinite solutions, ill-posed !)

Proof of the well-posedness

$$\begin{cases} y' = f(x,y), & x > a \\ y(a) = b \end{cases} \Rightarrow \quad y(x) = b + \int_a^x f[\xi, y(\xi)]\,\mathrm{d}\xi$$

With $x = a,\ y = y_1(a) = b_1 \quad \to \quad y_1(x) = b_1 + \int_a^x f\left[\xi, y_1(\xi)\right]\,\mathrm{d}\xi$

$x = a,\ y = y_2(a) = b_2 \quad \to \quad y_2(x) = b_2 + \int_a^x f\left[\xi, y_2(\xi)\right]\,\mathrm{d}\xi$

Then $y_2 - y_1 = b_2 - b_1 + \int_a^x f\left[\xi, y_2(\xi)\right] - f\left[\xi, y_1(\xi)\right]\mathrm{d}\xi$

Picard Iteration:

$$\left|y_2^0 - y_1^0\right| = \left|b_2 - b_1\right|$$

$$\left|y_2^1 - y_1^1\right| \le \left|b_2 - b_1\right| + \int_a^x \left|f\left[\xi, y_2^0(\xi)\right] - f\left[\xi, y_1^0(\xi)\right]\right|\mathrm{d}\xi \le \left|b_2 - b_1\right| + \int_a^x C\left|y_2^0 - y_1^0\right|\mathrm{d}\xi$$

$$= \left|b_2 - b_1\right| + C\left|b_2 - b_1\right|\left|x - a\right|$$

$$\left|y_2^2-y_1^2\right|\le\left|b_2-b_1\right|+\int_a^x\left|f\left[\xi,y_2^1(\xi)\right]-f\left[\xi,y_1^1(\xi)\right]\right|\mathrm{d}\xi\le\left|b_2-b_1\right|+\int_a^x C\left|y_2^1-y_1^1\right|\mathrm{d}\xi$$

$$=\left|b_2-b_1\right|+C\left|b_2-b_1\right|\left|x-a\right|+C^2\left|b_2-b_1\right|\frac{\left|x-a\right|^2}{2!}$$

$$\left|y_2^3-y_1^3\right|\le\left|b_2-b_1\right|+\int_a^x\left|f\left[\xi,y_2^2(\xi)\right]-f\left[\xi,y_1^2(\xi)\right]\right|\mathrm{d}\xi\le\left|b_2-b_1\right|+\int_a^x C\left|y_2^2-y_1^2\right|\mathrm{d}\xi$$

$$=\left|b_2-b_1\right|+C\left|b_2-b_1\right|\left|x-a\right|+C^2\left|b_2-b_1\right|\frac{\left|x-a\right|^2}{2!}C^3\left|b_2-b_1\right|\frac{\left|x-a\right|^3}{3!}$$

....

$$\left|y_2^n-y_1^n\right|\le\left|b_2-b_1\right|+\int_a^x\left|f\left[\xi,y_2^{n-1}(\xi)\right]-f\left[\xi,y_1^{n-1}(\xi)\right]\right|\mathrm{d}\xi\le\left|b_2-b_1\right|+\int_a^x C\left|y_2^{n-1}-y_1^{n-1}\right|\mathrm{d}\xi$$

$$=\left|b_2-b_1\right|+C\left|b_2-b_1\right|\left|x-a\right|+C^2\left|b_2-b_1\right|\frac{\left|x-a\right|^2}{2!}+\cdots+C^n\left|b_2-b_1\right|\frac{\left|x-a\right|^n}{n!}$$

As $n\to\infty$, $\left|y_2(x)-y_1(x)\right|\le e^{C|x-a|}\left|b_2-b_1\right|=e^{C|x-a|}\left|y_2(a)-y_1(a)\right|$