微積分1–4.7 Optimization Problems-Video 1

崔茂培 Mao-Pei Tsui

臺大數學系

October 28, 2021

Plan of the this Video

In this video, we will cover
materials in 4.7 Optimization Problems In this video, we will discuss optimization problem.

How to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized?

Let’ s recall the problem-solving principles.
Steps In Solving Optimization Problems

1. Understand the ProblemThe first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given constraint conditions?

2. Draw a pictureto represent the problem

3. Identify the variables $(x, y, \cdots)$
Identify the constraint equations
Identify the function (quantity) that one wants to optimize

4. Use the constraint equation to express the "objective function" (the function that we want to optimize) in terms of one variable

5. Find the domain of the objective function and use calculus to find the absolute maximum/minimum of the problem

a. If the domain is a closed and bounded interval then find the critical number in the interval. Evaluate the objective function at end points and critical number(s). The largest number is the absolute maximum. The smallest number is the absolute minimum.

b. If the domain is open or a half interval, then use the first derivative test (Find the intervals where it is increasing or decreasing).

Example. A box with a square and open top have a volume of $32000cm^3$. Find the dimension of the box that minimize the amount of the materials used.
Solution. Let the width of the base is x and the height of the box is h. Constraint volume =32000
$x^2 h = 32000$
Objective function to be minimize = the area of the open box $= x^2 + 4xh$ (area of the base + 4 area of the side).

From the constraint equation, we have $h = \frac{32000}{x^2}$
Area function $= x^2 + 4xh = x^2 + 4x \cdot \frac{32000}{x^2} = x^2 + \frac{128000}{x}$

So want to minimize the area function $f(x) = x^2 + \frac{128000}{x}$.
Since x is the width, we have $x > 0$.
So want to minimize $f(x) = x^2 + \frac{128000}{x}$ on $(0, \infty)$.

Recall $f(x) = x^2 + \frac{128000}{x}$. f is continuous and differentiable on $(0, \infty)$. Compute $f'(x) = 2x - \frac{128000}{x^2}$.

$f'(x) = 0$ when $x^3 = 64000$ and $x = 40$. Since $f'(x) = \frac{2(x^3 - 64000)}{x^2}$, we have $f'(x) < 0$ on $(0, 40)$ and $f'(x) > 0$ on $(40, \infty)$. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 + \frac{128000}{x} = \infty$. So f is decreasing on $(0, 40)$ and increasing on $(0, \infty)$. Thus f achieves its absolute minimum at $x = 40$.

Recall $x^2 h = 32000$. We have $h = \frac{32000}{40^2} = 20$.

So $x = 40$ and $h = 20$. The minimal area $= x^2 + 4x^2 h = 1600 + 4 \cdot 40^2 \cdot 20 = 1600 + 128000 = 129600 cm^3$.