微積分1–4.2 The Mean Value Theorem-Video 1

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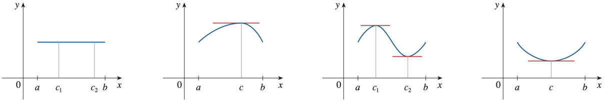
Plan of the this Video

In this video, we will cover
materials in 4.2 The Mean Value Theorem. We discuss Rolle Theorem and Mean value Theorem.

From extreme value theorem, we know that if f is continuous on $[a, b]$ and differentiable on (a, b) and f achieves an absolute maximum or minimum at $c \in (a, b)$ then $f'(c) = 0$.

The following Rolle Theorem is a result that implies such a case could happen.

Rolle Theorem Let f be continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$. Then there exists c with $a < c < b$ such that $f'(c) = 0$.
Rolle Theorem is obvious in terms of picture.



(a) (b) (c) (d)

Now we prove this Theorem.
Proof. Since f is continuous on $[a, b]$, f achieves its absolute maximum $f(\alpha)$ and its absolute minimum $f(\beta)$ for some $\alpha, \beta \in [a, b]$. We have $f(\beta) \leq f(\alpha)$.

We can divdie the discussion into several cases

Case 1. $\alpha \in (a, b)$ or $\beta \in (a, b)$. If $\alpha \in (a, b)$ then $f(\alpha)$ is also a local maximum. Since f is differentiable on (a, b) then $f'(\alpha) = 0$. Similarly, if $\beta \in (a, b)$ then $f'(\beta) = 0$.

Case 2. $\alpha = a$ or $\alpha = b$ and $\beta = a$ or $\beta = b$
Using $f(a) = f(b) = C$, then $f(\alpha) = f(\beta) = C$. Thus $C = f(\beta) \leq f(x) \leq f(\alpha) = C$ and $f(x) = C$ for all $x \in [a, b]$. Then $f'(x) = 0$ for all $x \in (a, b)$

From the discussion above, we can find at least one point $c \in (a, b)$ such that $f'(c) = 0$.

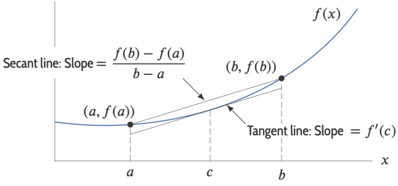
Our main use of Rolle's Theorem is in proving the following important theorem.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b). Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark: Mean-Value Theorem says that there is at least one point $c \in (a, b)$ such that
the slope of the tangent line to the graph at $(c, f(c))$ is the same as the slope of the secant line $\frac{f(b)-f(a)}{b-a}$.



The point c with $f'(c) = \frac{f(b)-f(a)}{b-a}$

Proof. Note that the linear equation of the secant line thru $(a, f(a))$ and $(b, f(b))$ is $L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. Obviously, $f(x)$ intersects the secant line at $x = a$ and $x = b$, i.e. $f(a) = L(a)$ and $f(b) = L(b)$.

Let $g(x) = f(x) - L(x)$. Then $g(a) = f(a) - L(a) = 0$ and $g(b) = f(b) - L(b) = 0$. Note that L is continuous and differentiable everywhere. We know that g is continuous on (a, b) and g is differentiable on (a, b).

By Rolle Theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$

Recall that $g(x) = f(x) - L(x) = f(x) - [f(a) + \frac{f(b)-f(a)}{b-a}(x-a)]$ and $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Thus $g'(c) = 0$ implies that $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. Thus $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Example. Suppose f is continuous and differentiable everywhere, $f(0) = -3$ and $f'(x) \leq 5$ for all x. How large can $f(2)$ possibly be?
Solution: We have that $\frac{f(2)-f(0)}{2-0} = f'(c)$ for some $c \in (0, 2)$.
Now $f'(c) \leq 5$. So $f(2) - f(0) \leq 2f'(c) \leq 10$. So
$f(2) \leq f(0) + 10 = -3 + 10 = 7$.
So $f(2)$ is at most 7.

Example.(a) Using the Mean Value Theorem, prove that $|\cos(x) - \cos(y)| \leq |x - y|$ for any real numbers x, y.
(b) Hence, compute the limit $\lim_{x\to\infty} \cos(\sqrt{2021 + x}) - \cos(\sqrt{x})$.
Solution. (a) By MVT, $\frac{\cos(x)-\cos(y)}{x-y} = -\sin(c)$ for any $x < y$. So $|\frac{\cos(x)-\cos(y)}{x-y}| = |-\sin(c)| \leq 1$ and $|\cos(x) - \cos(y)| \leq |x - y|$.
If $x = y$ then $|\cos(x) - \cos(y)| \leq |x - y|$ is obvious true. If $x > y$ then it can be proved similarly.
(b) We know that

$$
\begin{aligned}
0 \leq |\cos(\sqrt{2021 + x}) - \cos(\sqrt{x})| &\leq |\sqrt{2021 + x} - \sqrt{x}| \\
=& |\frac{(\sqrt{2021 + x} - \sqrt{x})(\sqrt{2021 + x} + \sqrt{x})}{(\sqrt{2021 + x} + \sqrt{x})}| \\
=& |\frac{202}{(\sqrt{2021 + x} + \sqrt{x})}|
\end{aligned}
\tag{1}
$$

Since $\lim_{x\to\infty} |\frac{202}{(\sqrt{2021+x}+\sqrt{x})}| = 0$. By squeeze Theorem, we have $\lim_{x\to\infty} |\cos(\sqrt{2021 + x}) - \cos(\sqrt{x})| = 0$ and $\lim_{x\to\infty} \cos(\sqrt{2021 + x}) - \cos(\sqrt{x}) = 0$.

The Mean Value Theorem can be used to establish some of the basic facts of differential calculus.
One of these basic facts is the following theorem.

Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b). Proof. Let $x_1 < x_2$ with $a < x_1 < x_2 < b$. By mean value theorem, we have $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$ for some $c \in (x_1, x_2)$. Now $f'(x) = 0$ for all $x \in (a, b)$. Then $\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$. So $f(x_2) = f(x_1)$. This means that any two values in (a, b) ate the same. Then f must be a constant in (a, b).

Example. Let k be a constant. Suppose $f'(x) = kf(x)$ for all x then there exists a constant c such that $f(x) = ce^{kx}$.
Proof. Let $g(x) = f(x)e^{-kx}$. Then
$g'(x) = f'(x)e^{-kx} + f(x)(e^{-kx})' = kf(x)e^{-kx} - kf(x)e^{-kx} = 0$.
So there exists a constant c such that $g(x) = f(x)e^{-kx} = c$, i.e. $f(x) = ce^{kx}$.