微積分1–4.2 The Mean Value Theorem-Video 1

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Plan of the this Video

In this video, we will cover materials in 4.2 The Mean Value Theorem. ~~We discuss how to determine the absolute maximum/ minimum of a continuous functions on a closed and bounded interval.~~

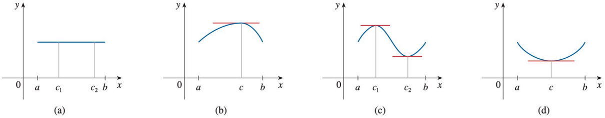
Discuss Rolle Th

and Mean-Value Th.

From extreme value theorem, we know that if f is continuous on $[a, b]$ and differentiable on (a, b) and f achieves an absolute maximum or minimum at $c \in (a, b)$ then $f'(c) = 0$.

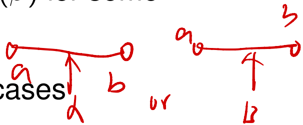
The following Rolle Theorem is a result that implies such a case could happen.

Rolle Theorem Let f be continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$. Then there exists c with $a < c < b$ such that $f'(c) = 0$.
Rolle Theorem is obvious in terms of picture.



Now we prove this Theorem.
Proof. Since f is continuous on $[a, b]$, f achieves its absolute maximum $f(\alpha)$ and its absolute minimum $f(\beta)$ for some $\alpha, \beta \in [a, b]$. We have $f(\beta) \leq f(\alpha)$.

We can divdie the discussion into several cases



Case 1. $\alpha \in (a, b)$ or $\beta \in (a, b)$. If $\alpha \in (a, b)$ then $f(\alpha)$ is also a local maximum. Since f is differentiable on (a, b) then $f'(\alpha) = 0$. Similarly, if $\beta \in (a, b)$ then $f'(\beta) = 0$.

Case 2. $\alpha = a$ or $\alpha = b$ and $\beta = a$ or $\beta = b$
Using $f(a) = f(b) = C$, then $f(\alpha) = f(\beta) = C$. Thus $C = f(\beta) \leq f(x) \leq f(\alpha) = C$ and $f(x) = C$ for all $x \in [a, b]$. Then $f'(x) = 0$ for all $x \in (a, b)$

From the discussion above, we can find at least one point $c \in (a, b)$ such that $f'(c) = 0$.

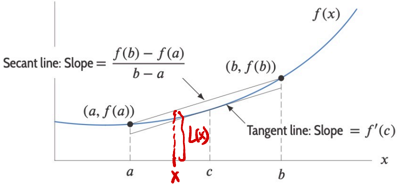
Our main use of Rolle's Theorem is in proving the following important theorem.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b). Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark: Mean-Value Theorem says that there is at least one point $c \in (a, b)$ such that
the slope of the tangent line to the graph at $(c, f(c))$ is the same as the slope of the secant line $\frac{f(b)-f(a)}{b-a}$.



The point c with $f'(c) = \frac{f(b)-f(a)}{b-a}$

Proof. Note that the linear equation of the secant line thru $(a, f(a))$ and $(b, f(b))$ is $L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. Obviously, $f(x)$ intersects the secant line at $x = a$ and $x = b$, i.e. $f(a) = L(a)$ and $f(b) = L(b)$. L cts and differentiable

Let $g(x) = f(x) - L(x)$. Then $g(a) = f(a) - L(a) = 0$ and $g(b) = f(b) - L(b) = 0$. We know that g is continuous on (a, b) and g is diferentiable on (a, b).

By Rolle Theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$

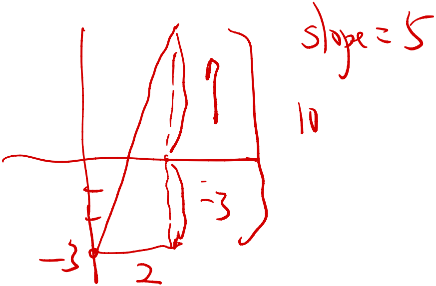
Recall that $g(x) = f(x) - L(x) = f(x) - [f(a) + \frac{f(b)-f(a)}{b-a}(x-a)]$ and $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Thus $g'(c) = 0$ implies that $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. Thus $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Assume f is cts, differentiable for all x.

Example. Suppose $f(0) = -3$ and $f'(x) \leq 5$ for all x. How large can $f(2)$ possibly be?

≤ 5

Solution: We have that $\frac{f(2)-f(0)}{2-0} = f'(c)$ for some $c \in (0, 2)$.
Now $f'(c) \leq 5$. So $f(2) - f(0) \leq 2f'(c) \leq 10$. So
$f(2) \leq f(0) + 10 = -3 + 10 = 7$.
So $f(2)$ is at most 7.



Example.(a) Using the Mean Value Theorem, prove that $|\cos(x) - \cos(y)| \leq |x - y|$ for any real numbers x, y.
(b) Hence, compute the limit $\lim_{x\to\infty} \cos(\sqrt{2021 + x}) - \cos(\sqrt{x})$.

Solution. (a) By MVT, $\frac{\cos(x)-\cos(y)}{x-y} = -\sin(c)$ for any $x < y$. So $|\frac{\cos(x)-\cos(y)}{x-y}| = |-\sin(c)| \leq 1$ and $|\cos(x) - \cos(y)| \leq |x - y|$.
If $x = y$ then $|\cos(x) - \cos(y)| \leq |x - y|$ is obvious true. If $x > y$ then it can be proved similarly.

$f(x) = \cos(x)$
$f'(x) = -\sin(x)$

$\frac{f(x)-f(y)}{x-y} = f'(c) = -\sin(c)$
$c \in (x, y)$

if $y < x$
$\left|\frac{f(y)-f(x)}{y-x}\right| \leq 1$
$|\cos(y) - \cos(x)| \leq |y-x|$

(b) We know that

$$
\begin{aligned}
0 &\leq |\cos(\sqrt{2021 + x}) - \cos(\sqrt{x})| \leq |\sqrt{2021 + x} - \sqrt{x}| \\
&= \left|\frac{(\sqrt{2021 + x} - \sqrt{x})(\sqrt{2021 + x} + \sqrt{x})}{(\sqrt{2021 + x} + \sqrt{x})}\right| \qquad (1) \\
&= \left|\frac{2021}{(\sqrt{2021 + x} + \sqrt{x})}\right|
\end{aligned}
$$

$\lim_{x\to\infty} |f(x)| = 0 \Rightarrow \lim_{x\to\infty} f(x) = 0$

Since $\lim_{x\to\infty} \left|\frac{2021}{(\sqrt{2021+x}+\sqrt{x})}\right| = 0$. By squeeze Theorem, we have $\lim_{x\to\infty} |\cos(\sqrt{2021 + x}) - \cos(\sqrt{x})| = 0$ and $\lim_{x\to\infty} \cos(\sqrt{2021 + x}) - \cos(\sqrt{x}) = 0$.

$-|f(x)| \leq f(x) \leq |f(x)|$

The Mean Value Theorem can be used to establish some of the basic facts of differential calculus.
One of these basic facts is the following theorem.

Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b). Proof. Let $x_1 < x_2$ with $a < x_1 < x_2 < b$. By mean value theorem, we have $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$ for some $c \in (x_1, x_2)$. Now $f'(x) = 0$ for all $x \in (a, b)$. Then $\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$. So $f(x_2) = f(x_1)$. This means that any two values in (a, b) ate the same. Then f must be a constant in (a, b).

k is a constant and

Example. Suppose $f'(x) = kf(x)$ for all x then there exists a constant c such that $f(x) = ce^{kx}$.
Proof. Let $g(x) = f(x)e^{-kx}$. Then
$g'(x) = f'(x)e^{-kx} + f(x)(e^{-kx})' = kf(x)e^{-kx} - kf(x)e^{-kx} = 0$.
So there exists a constant c such that $g(x) = f(x)e^{-kx} = c$, i.e. $f(x) = ce^{kx}$.