微積分1–3.3 Derivatives of Trigonometric Functions-Video 1

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Plan of the this Video

In this video, we will cover
materials in 3.3 Derivatives of Trigonometric Functions. We learn how to differentiate
$\sin(x), \cos(x), \tan(x), \csc(x), \sec(x), \cot(x)$.

There are 6 trigonometric functions.

$\sin(x), \cos(x), \tan(x) = \frac{\sin(x)}{\cos(x)}$,

$\csc(x) = \frac{1}{\sin(x)}, \sec(x) = \frac{1}{\cos(x)}, \cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$.

To find the derivatives of trig functions, we just need to find the derivative of $\sin(x), \cos(x)$ and the quotient rule.

There are some important formulae:

(a) $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

(b) $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

(c) $\sin^2(x) + \cos^2(x) = 1$ → 除以 $\cos^2(x) \Rightarrow \frac{\sin^2(x)}{\cos^2(x)} + 1 = \frac{1}{\cos^2(x)}$

(d) $1 + \tan^2(x) = \sec^2(x)$

(e) $1 + \cot^2(x) = \csc^2(x)$

Notation: $\sin^2(x) = (\sin(x))^2$

$1 + \tan^2(x) = \sec^2(x)$

除以 $\sin^2(x) \Rightarrow 1 + \frac{\cos^2(x)}{\sin^2(x)} = \frac{1}{\sin^2(x)} \Rightarrow 1 + \cot^2(x) = \csc^2(x)$

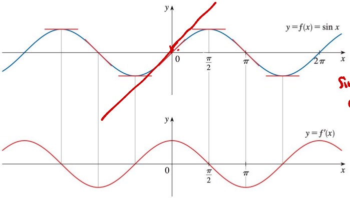
Recall that we have

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \text{ and } \lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0.$$

Watch Sep_23_Video7 from 14:40 or Sep_23_Video7.pdf page 9

If we sketch the graph of the function $f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of $y = f'(x)$. $\cancel{g}f'$ may be the same as the cosine curve.

$f'(0)=1$ $\lim_{h\to 0} \frac{\sin(h)-\sin(0)}{h} = 1$



sin(x) concave down on $[0,\pi]$
$f'(x) \downarrow$

sin(x) concave up on $[\pi, 2\pi]$
$f'(x) \uparrow$

Let' s try to confirm our guess that if $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.

$$\begin{aligned}
&\frac{d}{dx}\sin(x) \\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h}+\frac{\cos(x)\sin(h)}{h} \\
&= \sin(x)\cdot 0+\cos(x)\cdot 1 \\
&= \cos(x).
\end{aligned}$$

Next we show if $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

$$\begin{aligned}
&\frac{d}{dx}\cos(x) \\
&= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)(\cos(h) - 1)}{h} - \frac{\sin(x)\sin(h)}{h} \\
&= \cos(x)\cdot 0 - \sin(x)\cdot 1 \\
&= -\sin(x).
\end{aligned}$$

Now, we use quotient rule $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$ to find the derivative of $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

$$
\begin{aligned}
&\frac{d}{dx}\tan(x) \\
=&\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \\
=&\frac{(\sin(x))'\cos(x) - \sin(x)(\cos(x))'}{\cos^2(x)} \\
=&\frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\
=&\frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
=&\frac{1}{\cos^2(x)} = \sec^2(x).
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\csc(x) &= \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) \\
&= \frac{(1)'\sin(x) - 1(\sin(x))'}{\sin^2(x)} = \frac{-\cos(x)}{\sin^2(x)} \\
&= -\frac{1}{\sin(x)}\frac{\cos(x)}{\sin(x)} = -\csc(x)\cot(x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\sec(x) &= \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) \\
&= \frac{(1)'\cos(x) - 1(\cos(x))'}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} \\
&= \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dx}\cot(x) \\
=&\frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) \\
=&\frac{(\cos(x))'\sin(x)-\cos(x)(\sin(x))'}{\sin^2(x)} \\
=&\frac{-\sin(x)\sin(x)-\cos(x)\cos(x)}{\sin^2(x)} \\
=&\frac{-\sin^2(x)-\cos^2(x)}{\sin^2(x)} \\
=&-\frac{1}{\sin^2(x)}=-\csc^2(x).
\end{aligned}
$$

Derivatives of trig functions

$$\frac{d}{dx}\sin(x) = \cos(x), \qquad \frac{d}{dx}\csc(x) = -csc(x)\cot(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x), \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x), \qquad \frac{d}{dx}\cot(x) = -\csc^2(x).$$

Example. Differentiate the following functions.

(a) $\frac{1+\tan(x)}{1-\tan(x)}$ (b) $\sin(x)\cos(x)$ (c) $\frac{\csc(x)}{1+\cot(x)}$.

Solution:

$$
\begin{aligned}
& \frac{d}{dx}\frac{1+\tan(x)}{1-\tan(x)} \\
= \; & \frac{(1+\tan(x))'(1-\tan(x)) - (1+\tan(x))(1-\tan(x))'}{(1-\tan(x))^2} \\
= \; & \frac{\sec^2(x)(1-\tan(x)) - (1+\tan(x))(-\sec^2(x))}{(1-\tan(x))^2} \\
= \; & \frac{\sec^2(x) - \sec^2(x)\tan(x) + \sec^2(x) + \sec^2(x)\tan(x))}{(1-\tan(x))^2} \\
= \; & \frac{2\sec^2(x)}{(1-\tan(x))^2}.
\end{aligned}
$$

$$
\begin{aligned}
& \frac{d}{dx}\sin(x)\cos(x) \\
= \quad & (\sin(x))'\cos(x) + \sin(x)(\cos(x))' \\
= \quad & \cos(x)\cos(x) + \sin(x)(-\sin(x)) \\
= \quad & \cos^2(x) - \sin^2(x).
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\frac{\csc(x)}{1+\cot(x)} &= \frac{(\csc(x))'(1+\cot(x)) - \csc(x)(1+\cot(x))'}{(1+\cot(x))^2} \\
= \quad & \frac{-\csc(x)\cot(x)(1+\cot(x)) - \csc(x)(-\csc^2(x))}{(1+\cot(x))^2} \\
= \quad & \frac{-\csc(x)\cot(x) - \csc(x)\cot^2(x) + \csc^3(x)}{(1+\cot(x))^2} \\
= \quad & \frac{-\csc(x)\cot(x) - \csc(x)(\csc^2(x) - 1) + \csc^3(x)}{(1+\cot(x))^2} \\
= \quad & \frac{\csc(x)(1-\cot(x))}{(1+\cot(x))^2}.
\end{aligned}
$$

$1+\cot^2(x) = \csc^2(x)$

$\cot^2(x) = \csc^2(x) - 1$

$= -\csc(x)\cot(x) + \csc(x)$

Example: Let $f(x) = e^x \cos(x)$. Show that $f''(x) - 2f'(x) + 2f(x) = 0$. ← (differential equation)

Solution: We first compute

$$\begin{aligned} f'(x) &= (e^x \cos(x))' = (e^x)' \cos(x) + e^x (\cos(x))' \\ &= e^x \cos(x) - e^x \sin(x). \end{aligned}$$

Next, we compute

$$\begin{aligned} f''(x) &= (e^x \cos(x) - e^x \sin(x))' = (e^x \cos(x))' - (e^x \sin(x))'; \\ &= e^x \cos(x) - e^x \sin(x) - e^x \sin(x) - e^x \cos(x) \\ &= -2e^x \sin(x). \end{aligned}$$

$$(e^x \sin(x))' = (e^x)' \sin(x) + e^x (\sin(x))'$$
$$= e^x \sin x + e^x \cos(x)$$

Thus

$$\begin{aligned} & f''(x) - 2f'(x) + 2f(x) \\ = \; & -2e^x \sin(x) - 2(e^x \cos(x) - e^x \sin(x)) + 2e^x \cos(x) \\ = \; & 0. \end{aligned}$$