Slido: #ADA2021

CSIE 2136 Algorithm Design and Analysis, Fall 2021

Amortized Analysis

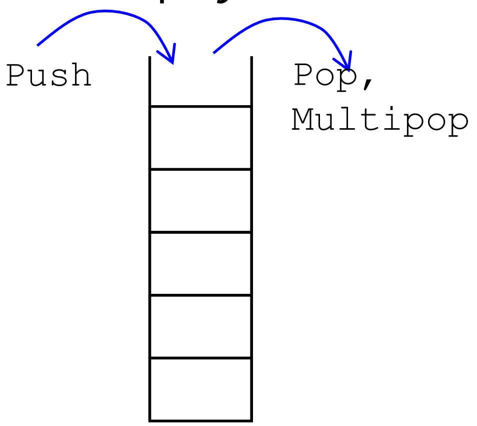
Hsu-Chun Hsiao

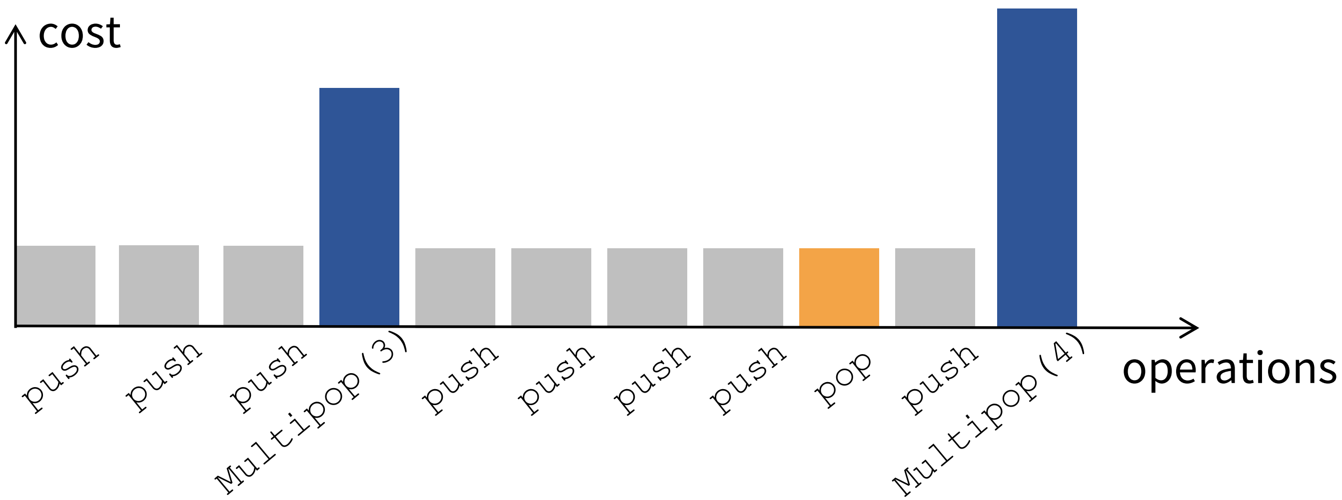
Agenda

- Why amortized analysis (均攤分析)
- Aggregate method (聚集方法)
- Accounting method (記帳方法) or banker's method
- Potential method (位能方法) or physicist's method

Why amortized analysis?

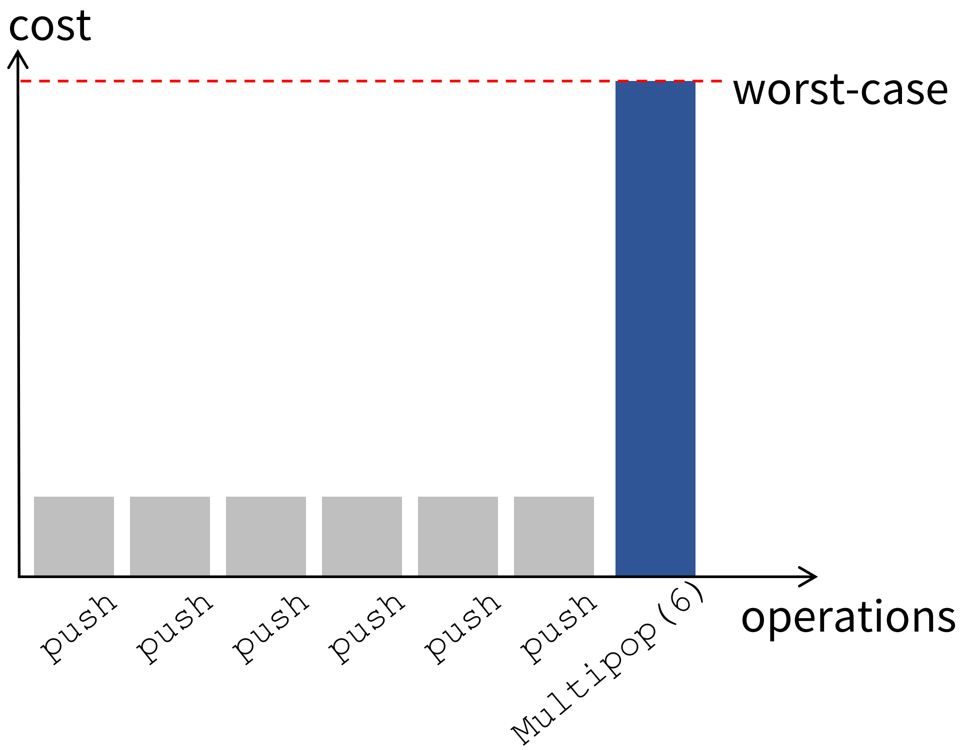
- Example: stack supports `Push`, `Pop`, `Multipop` operations
  - `push(S,x)` $= O(1)$
  - `pop(S)` $= O(1)$
  - `multipop(S,k)` $= O(\min\{|S|, k\})$
- What is the worst-case time of a sequence of n operations on an initially empty stack?





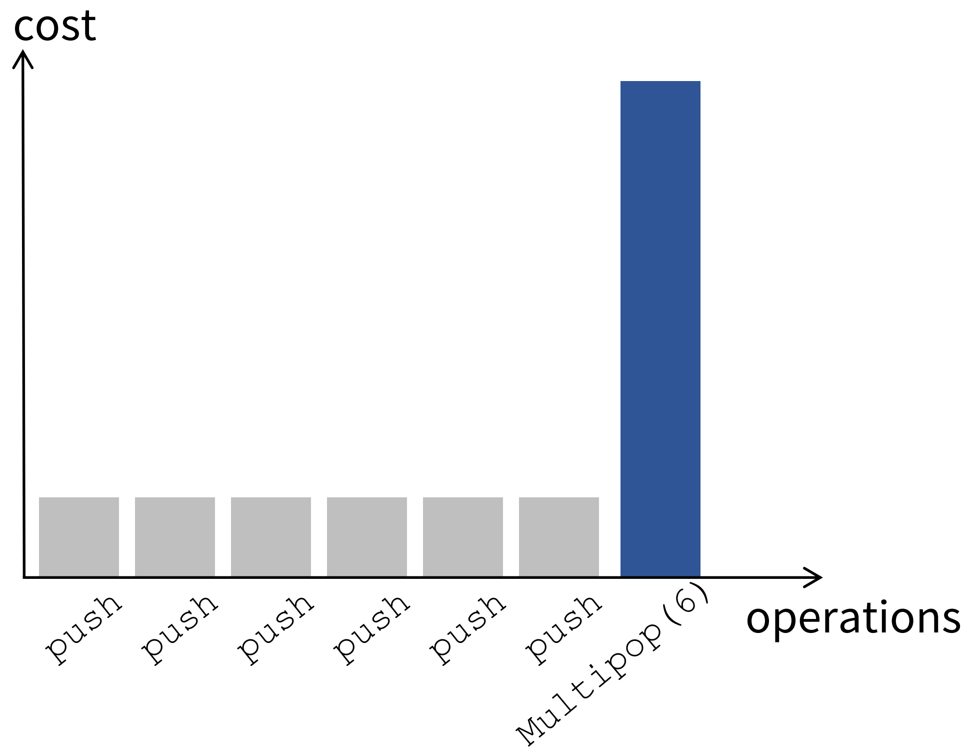
Worst-case analysis on individual operations may be loose

- What is the worst-case time of a sequence of n operations on an initially empty stack?
  - Worst-case time of the n^{th} operation = `multipop(S,n)` $= O(n)$
  - => Worst-case time of a sequence of n operations $= O(n^2)$
- However, this worst-case bound is not tight. Why?
  - The expensive `multipop` operation won't occur frequently!



Worst-case analysis on individual operations may be loose

- Given a sequence of n operations, their occurrences and costs may depend on others.
- Often happens when operating on the same data structure
- Thus, they should be analyzed together, not individually.



Goal of amortized analysis

- Obtain an asymptotic worst-case bound for a sequence of n operations

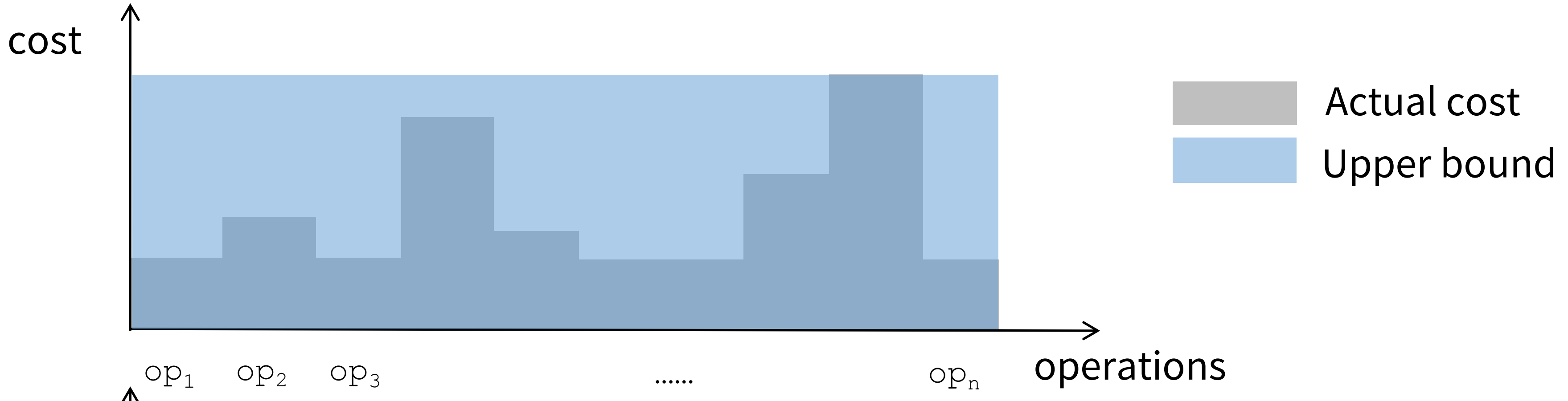
All of the valid operation sequences on stack when $n = 3$:

```
push, push, pop
push, push, multipop(1)
push, push, multipop(2)
push, pop, push
push, multipop(1), push
```

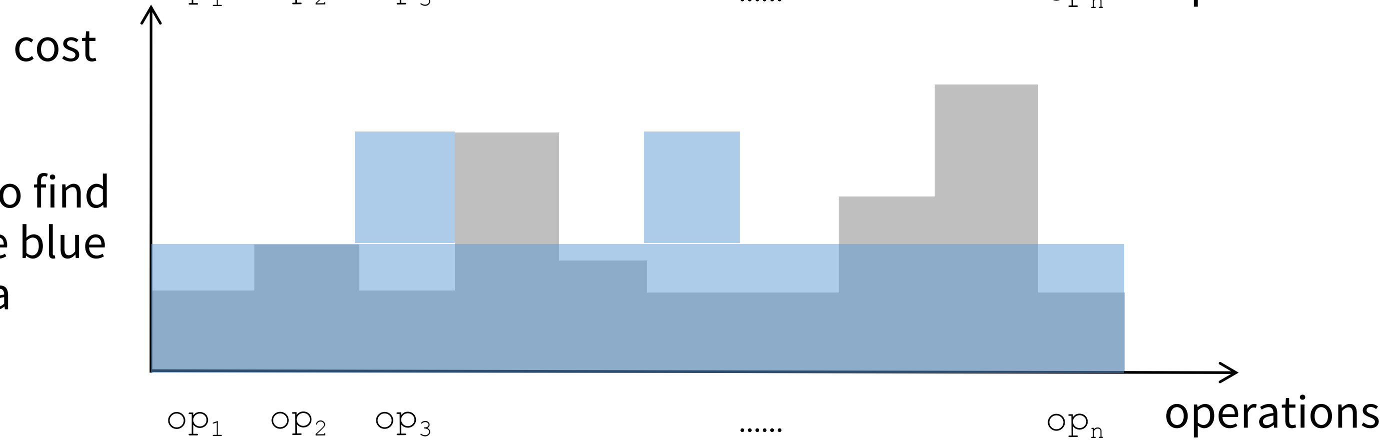
Goal of amortized analysis

- Obtain an asymptotic worst-case bound for a sequence of n operations

Worst-case analysis by considering individual operations



Amortized analysis aims to find a better upper bound (the blue area) of the grey area



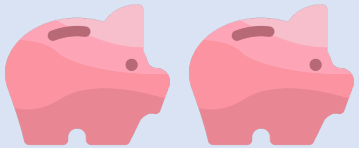
Amortized analysis: 3 common techniques

Aggregate method (聚集方法)

- Determine an upper bound on the cost over any sequence of n operations, $T(n)$
- The average cost per operation is then $T(n)/n$
- All operations have the same amortized cost

Accounting method (記帳方法)

- Each operation is assigned an amortized cost (may differ from the actual cost)
- Each object of the data structure is associated with a credit
- Need to ensure that every object has sufficient credit at any time



Potential method (位能方法)

- Similar to accounting method; each operation is assigned an amortized cost
- The data structure as a whole maintains a credit (i.e., potential)
- Need to ensure that the potential level is nonnegative at any time

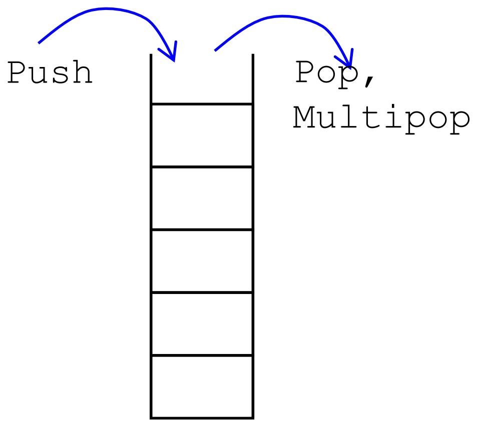
Note: these are for analysis purpose only, not for implementation!

Example #1: stack

A stack is empty initially and supports three types of operations

Implemented using an array or linked list

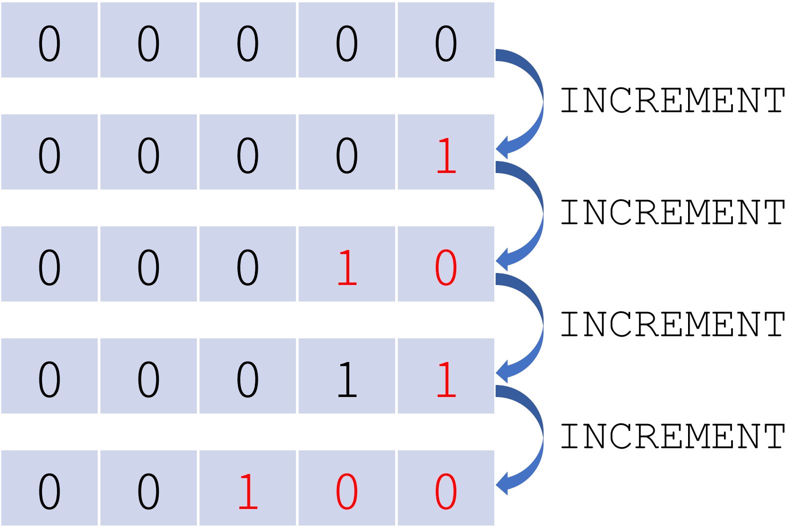
```
MULTIPOP(S,k):
    while not STACK-EMPTY(S) and k > 0
        POP(S)
        k = k - 1
```



Operation type	Cost
`Push(S,x)`	$O(1)$
`Pop(S)`	$O(1)$
`Multipop(S,k)`: pop top k objects at once	$O(\min\{\lvert S\rvert, k\})$

Example #2: k-bit counter

- Counts up from 0 by an operation, INCREMENT
- Implemented using a k-bit array
- If flipping one bit costs $O(1)$, INCREMENT costs $O(k)$




```
INCREMENT(A):
    i = 0
    while i < A.length and A[i] == 1
        A[i] = 0
        i = i + 1
    if i < A.length
        A[i] = 1
```

More examples

- Dynamic table (insertion only, insertion and deletion)
- Dynamic binary search [Problem 17.2]
- Queue with two stacks
- Disjoint-set implementation (linked list with weighted union, forest with union-by-rank and path compression) [Ch. 21.4]
- Splay tree
- Cuckoo hashing [1]

[1]: https://web.stanford.edu/class/archive/cs/cs166/cs166.1146/lectures/13/Small13.pdf

Aggregate Method (聚集方法)

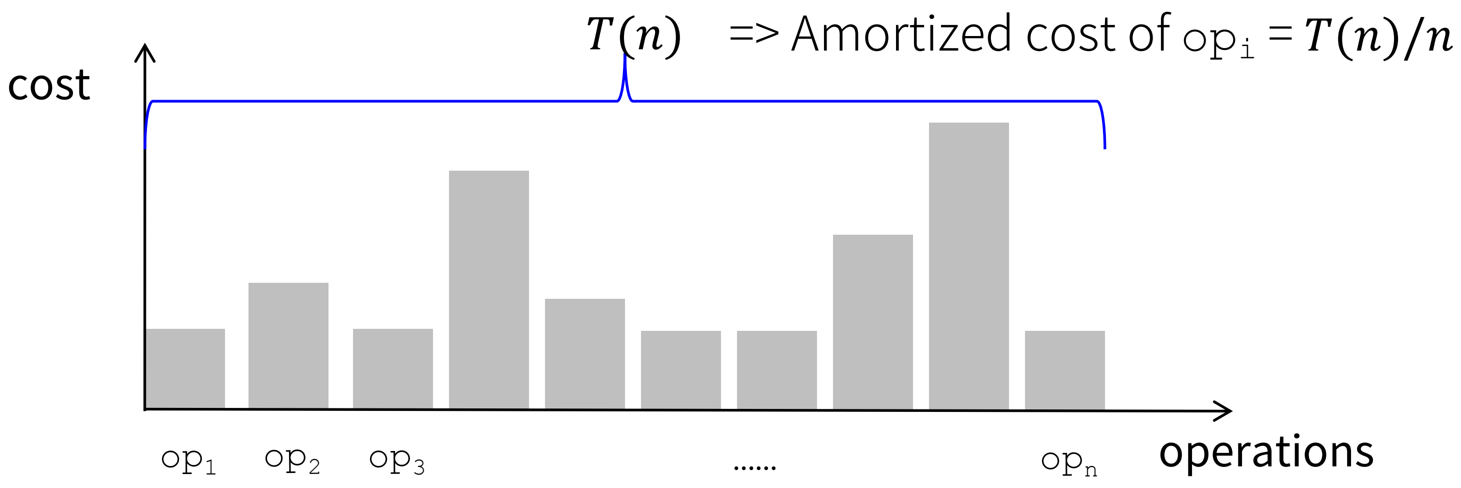
Chapter 17.1

Aggregate method

<u>Idea:</u> 直接觀察 n 次操作的總花費的上限

<u>Approach:</u>

1. Determine an upper bound $T(n)$ on the cost of any sequence of n operations
2. Calculate the amortized cost per operation as $T(n)/n$
   - All operations have the **same amortized cost**



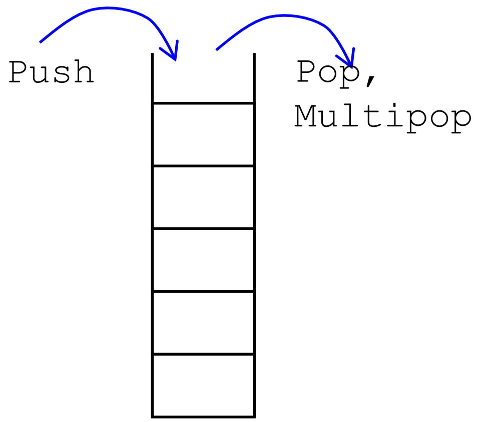
Aggregate method for stack

Observations:

1. Total cost = # of popped objects + # of pushed objects
2. For a sequence of n operations, maximum # of pushed objects is n
3. # of popped objects $\leq$ # of pushed objects
   - 出來的不可能比進去的多

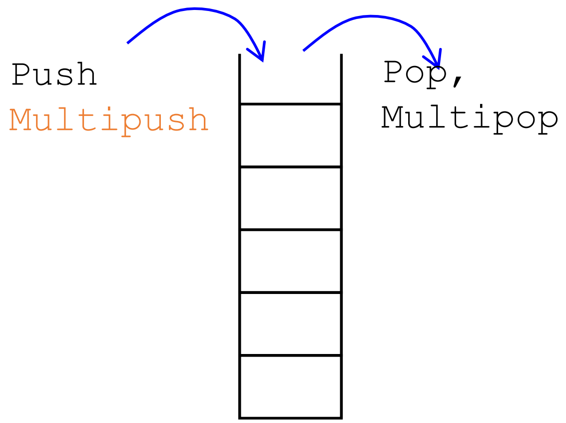
=> Total cost for an entire sequence is $O(n)$

=> Amortized cost per operation is $O(n)/n = O(1)$



Consider a variant of the stack data structure supporting `mutlipush(S, X)`, where X is a set of objects. Does this stack variant also have $O(1)$ amortized cost per operation?

No.



Operation type	Cost
`Push(S,x)`	$O(1)$
`Pop(S)`	$O(1)$
`Multipop(S,k)`: pop top k objects at once	$O(\min\{\|S\|, k\})$
`Multipush(S,X)`: X is a set of objects	$O(\|X\|)$

Aggregate method for k-bit counter

Counter value	A[3]	A[2]	A[1]	A[0]	Total cost of first n operations
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11
8	1	0	0	0	15

Total cost = # of bit flips = # of RED in the table

Aggregate method for k-bit counter

Counter value	A[3]	A[2]	A[1]	A[0]	Total cost of first n operations
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11
8	1	0	0	0	15

Flip every increment
Flip every 2 increments
Flip every 4 increments
Flip every 8 increments

Aggregate method for k-bit counter

Observation: Total # of bit flips in n increment operations

$$= n + \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{4}\rfloor + \cdots + \lfloor\frac{n}{2^{k-1}}\rfloor$$

$$< 2n$$

- => Total cost of the sequence is $O(n)$
- => Amortized cost per operation is $O(n)/n = O(1)$

Consider a variant of the k-bit counter data structure where flipping the ith bit costs 2^i instead of $O(1)$.
Does this variant also have $O(1)$ amortized cost per operation?
If not, what is its amortized cost?

No.

Total # of bit flips in n increment operations

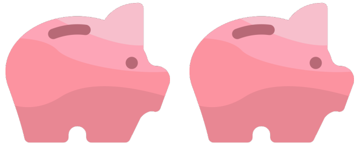
$$= n * 2^0 + \left\lfloor \frac{n}{2} \right\rfloor * 2^1 + \left\lfloor \frac{n}{4} \right\rfloor * 2^2 + \cdots + \left\lfloor \frac{n}{2^{k-1}} \right\rfloor * 2^{k-1}$$
$$\leq kn$$

Amortized cost per operation is $O(k)$

Accounting Method (記帳方法)

Chapter 17.2

Accounting method: idea



- 猜測每種操作的均攤費用
- 想像資料結構中每個物件有一個帳戶，初始金額為零
- 對某個物件進行操作時，
    - 若實際費用比均攤費用低，就存錢到其帳戶
    - 若實際費用比均攤費用高，從其帳戶裡拿錢補貼
- 若進行任意 n 個操作的過程不會造成任何帳戶透支，則為合理的均攤費用

- <u>跟 aggregate method 的主要差別</u>
    - 不同操作可以有不同的均攤費用
    - 先猜測每種操作的均攤費用，再推算 $T(n)$

Accounting method: approach

1. Guess each operation type's amortized cost
2. Validity check: Check if the per-op amortized costs are valid
   - Assume every object initially has credit $= 0$
   - Let c_i and $\widehat{c_i}$ be the actual and amortized costs of the i^{th} op, respectively
   - Check if it has sufficient credit (≥ 0) for any sequence of n ops
     - If actual cost < amortized cost ($c_i < \widehat{c_i}$), the difference becomes credit
     - If actual cost > amortized cost ($c_i > \widehat{c_i}$), then withdraw stored credit
   - If the check fails, go back to Step 1
3. Calculate the total cost $T(n) = \Sigma_{i=1}^{n} \widehat{c_i}$

Following the accounting method, show that $T(n) = \Sigma_{i=1}^{n} \widehat{c_i}$ is an upper bound for the actual cost $\Sigma_{i=1}^{n} c_i$.

Hint: Show that for any sequence of n operations, if every object has sufficient credit (≥ 0) throughout the execution, then $\Sigma_{i=1}^{n} (\widehat{c_i} - c_i) \geq 0$.

Accounting method for stack

1. Guess per-op amortized costs:

Operation	Actual cost	Amortized cost	Credit change
`push(S,x)`	1	2	存 1 元在 x 的帳戶裡
`pop(S)`	1	0	從 popped object 的帳戶領 1 元
`multipop(S,k)`	$min\{\|S\|, k\}$	0	從每個 popped object 的帳戶領 1 元

Accounting method for stack

1. Guess per-op amortized costs:

Operation	Actual cost	Amortized cost	Credit change
`push(S,x)`	1	2	存 1 元在 x 的帳戶裡
`pop(S)`	1	0	從 popped object 的帳戶領 1 元
`multipop(S,k)`	$min\{\|S\|, k\}$	0	從每個 popped object 的帳戶領 1 元

2. Validity check: Show that every object has credit $\geq \mathbf{0}$
   - push: the pushed object is deposited \$1 credit
   - pop and multipop: use the credit stored with the popped object
   - There is always enough credit to pay for each operation

Accounting method for stack

1. Guess per-op amortized costs:

Operation	Actual cost	Amortized cost	Credit change
`push(S,x)`	1	2	存 1 元在 x 的帳戶裡
`pop(S)`	1	0	從 popped object 的帳戶領 1 元
`multipop(S,k)`	$min\{\|S\|, k\}$	0	從每個 popped object 的帳戶領 1 元

2. Show that every object has credit ≥ 0
   - **push**: the pushed object is deposited $1 credit
   - **pop and multipop**: use the credit stored with the popped object
   - There is always enough credit to pay for each operation
3. Per-op amortized costs are all $O(1)$, so total cost is $T(n) = O(n)$

Accounting method for k-bit counter

1. Guess per-op amortized costs:

Operation	Actual cost	Amortized cost	Credit change
`INCREMENT`	# of bits flipped	?	
i^{th} bit 0 -> 1	1	\$2	存 1 元在 i^{th} bit 裡
i^{th} bit 1 -> 0	1	\$0	用掉存在 i^{th} bit 的 1 元

Accounting method for k-bit counter

1. Guess per-op amortized costs:

Operation	Actual cost	Amortized cost	Credit change
`INCREMENT`	# of bits flipped	\$2	
i^{th} bit 0 -> 1	1	\$2	存 1 元在 i^{th} bit 裡
i^{th} bit 1 -> 0	1	\$0	用掉存在 i^{th} bit 的 1 元

2. Validity check:
   - Counter 起始值為 00⋯0
   - 每次 `INCREMENT` 都會把一個 0 設成 1，可能把很多 1 設成 0
   - 可在被設為 1 的 bit 存一元
   - 把 1 設成 0 時，花掉存在這個 bit 的一元即可

Accounting method for k-bit counter

1. Guess per-op amortized costs:

Operation	Actual cost	Amortized cost	Credit change
`INCREMENT`	# of bits flipped	$2	
i^{th} bit 0 -> 1	1	$2	存 1 元在 i^{th} bit 裡
i^{th} bit 1 -> 0	1	$0	用掉存在 i^{th} bit 的 1 元

2. Validity check:
   - Counter 起始值為 00⋯0
   - 每次 `INCREMENT` 都會把一個 0 設成 1，可能把很多 1 設成 0
   - 可在被設為 1 的 bit 存一元
   - 把 1 設成 0 時，花掉存在這個 bit 的一元即可
3. Per-op amortized cost is $O(1)$ => total cost $T(n) = O(n)$

Potential Method (位能方法)

Chapter 17.3

Potential method: idea

- 想像將資料結構的狀態對應到位能，初始值為 0
- 對資料結構的狀態做操作時，
  - 若實際所需能量比均攤值低，將多餘能量的儲存成位能
  - 若實際所需能量比均攤值高，用儲存的位能補貼
- 若任意 n 個操作都有足夠位能，則從位能的變化算出個別操作的均攤費用

- 跟 accounting method 的主要差異：資料結構本身有 potential/credit，而不是每個物件都有 credit

Potential method: approach

1. Guess a potential function Φ that takes the current data structure state as input and outputs a potential level
2. Validity check: check if the potential level is never lower than the initial value after any sequence of n operations
   - Let D_i be the state of data structure after i^{th} operation
   - WLOG, check if $\Phi(D_0) = 0$ and $\forall\, i = 1 \dots n, \Phi(D_i) \geq 0$
   - If the check fails, go back to Step 1
3. Calculate the per-op amortized costs based on the potential function (see next slide)
4. Calculate the total cost based on per-op amortized costs $T(n) = \Sigma_{i=1}^{n} \widehat{c_i}$

Potential function

- Potential function Φ maps a data structure state to a real number
  - D_0 is the initial state of data structure
  - D_i is the state of data structure after i^{th} operation
  - c_i is the actual cost of i^{th} operation
  - $\widehat{c_i}$ is the amortized cost of i^{th} operation, defined as $\widehat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1})$

Show that $T(n) = \Sigma_{i=1}^{n} \widehat{c_i}$ is an upper bound for the actual cost $\Sigma_{i=1}^{n} c_i$.

- Hint: Show that for any sequence of n ops, if $\Phi(D_i) \geq \Phi(D_0)$ throughout the execution, then $\Sigma_{i=1}^{n}(\widehat{c_i} - c_i) \geq 0$.
- Based on the definition, we have
$$\sum_{i=1}^{n} \widehat{c_i} = \sum_{i=1}^{n}(c_i + \Phi(D_i) - \Phi(D_{i-1})) = \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$
- Passing the validity check ensures $\Phi(D_n) \geq \Phi(D_0)$
- Thus, $\Sigma_{i=1}^{n}(\widehat{c_i} - c_i) \geq 0$

Potential method for stack

1. Guess $\Phi(D_i)$ to be the # of objects in the stack after the i-th op
2. Validity check:
   - $\Phi(D_0) = \mathbf{0}$, because stack is initially empty
   - $\Phi(D_i) \geq \mathbf{0}$, because # of objects in stack is always $\geq \mathbf{0}$

$\Phi(D_i)$: the # of objects in the stack after the i-th op
c_i: the actual cost of the i-th op
$\hat{c_i}$: the amortized cost of the i-th op, defined as $\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1})$

3. Compute per-op amortized cost:
   - For `push(S,x)`: $\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + (|S| + 1) - |S| = 2$
   - For `pop(S)`: $\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + (|S| - 1) - |S| = 0$
   - For `multipop(S,k)`: $\hat{c_i} = 0$

4. All operations have $O(1)$ amortized cost, so total cost of n operations is $O(n)$

Q: justify why $\hat{c_i} = 0$ for `multipop(S,k)`

Potential method for k-bit counter

1. Guess $\Phi(D_i)$ to be the # of 1's in the counter after the i-th op
2. Validity check:
   - $\Phi(D_0) = 0$, because counter is initially all 0's
   - $\Phi(D_i) \geq 0$, because # of 1's cannot be negative

$\Phi(D\_i)$: the # of 1's in the counter after the i-th op
c_i: the actual cost of the i-th op
$\hat{c_i}$: the amortized cost of the i-th op, defined as $\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1})$

3. Compute the amortized cost of `INCREMENT`:
   - Let $LSB_0(x)$ be the index of the least significant 0 bit of x
   - For example, $LSB_0(01011011) = 2$, and $LSB_0(01011111) = 5$
   - $\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1})$
     $= (LSB_0(i-1) + 1) + (-LSB_0(i-1) + 1) = 2$

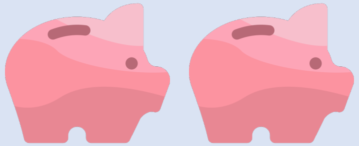
4. All operations have $O(1)$ amortized cost, so the total cost of n operations is $O(n)$

Amortized analysis: 3 common techniques

Aggregate method (聚集方法)

- Determine an upper bound on the cost over any sequence of n operations, $T(n)$
- The average cost per operation is then $T(n)/n$
- All operations have the same amortized cost

Accounting method (記帳方法)



- Each operation is assigned an amortized cost (may differ from the actual cost)
- Each object of the data structure is associated with a credit
- Need to ensure that every object has sufficient credit at any time

Potential method (位能方法)

- Similar to accounting method; each operation is assigned an amortized cost
- The data structure as a whole maintains a credit (i.e., potential)
- Need to ensure that the potential level is nonnegative at any time

* 三種方法一般都能獲得相同的分析結果，可依個人偏好採用

Example #3: dynamic table (insert only)

- A table stores objects and supports `insertion`
  - Initially the table T is empty
  - Each operation inserts an object to T
  - Double its size when out of slots (suppose this takes time linear to the orignial size); create a table of size 1 first if it's empty

i^{th} operation	Table	Actual cost (add + resize)
	`[]`	-
1: insertion	`[1]`	1
2: insertion	`[1,2]`	1+1=2
3: insertion	`[1,2,3,_]`	1+2=3
4: insertion	`[1,2,3,4]`	1
5: insertion	`[1,2,3,4,5,_,_,_]`	1+4=5

Example #3: dynamic table (insert only)

- Aggregate method
  - The actual cost $c_i = \begin{cases} i, & \text{if } i-1 \text{ is 2's power} \\ 1, & \text{otherwise} \end{cases}$
  - => $T(n) = \Sigma_{i=1}^{n} c_i \leq n + \Sigma_{i=1}^{\lfloor \lg(n-1) \rfloor} 2^i \leq 3n$
  - => The amortized cost of insertion is $O(1)$

i^{th} operation	Table	Actual cost (add + resize)
	`[]`	-
1: insertion	`[1]`	1
2: insertion	`[1,2]`	1+1=2
3: insertion	`[1,2,3,_]`	1+2=3
4: insertion	`[1,2,3,4]`	1
5: insertion	`[1,2,3,4,5,_,_,_]`	1+4=5

Example #3: dynamic table (insert only)

Use the accounting method

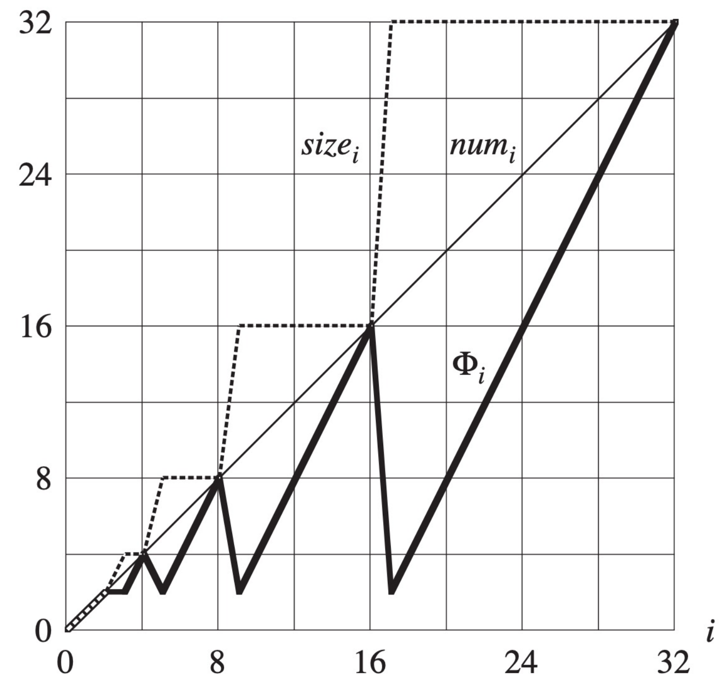
Hint: choose an amortized cost of 3

i^{th} operation	Table	Actual cost (add + resize)
	`[]`	-
1: insertion	`[1]`	1
2: insertion	`[1,2]`	1+1=2
3: insertion	`[1,2,3,_]`	1+2=3
4: insertion	`[1,2,3,4]`	1
5: insertion	`[1,2,3,4,5,_,_,_]`	1+4=5

Example #3: dynamic table (insert only)

Use the potential method

Hint: Consider a potential function $\Phi(T) = 2 * T.num - T.size$, where $T.num$ is the number of inserted objects and $T.size$ is the table size



Example #4: dynamic table (insert & deletion)

- A table stores objects and supports `insertion` and `deletion` of objects
  - Initially the table T is empty
  - Each operation inserts or deletes an object
  - Double its size when out of slots (i.e., load factor $= 1$)
  - Halve its size when the load factor is $\leq 1/4$
  - Suppose resizing takes time linear to the orignial size

Example #4: dynamic table (insert & deletion)

i^{th} operation	Table	Actual cost (add/delete + resize)
	[ ]	-
1: insertion	[1]	1
2: insertion	[1,2]	1+1=2
3: insertion	[1,2,3,_]	1+2=3
4: insertion	[1,2,3,4]	1
5: insertion	[1,2,3,4,5,_,_,_]	1+4=5
6: deletion	[1,2,3,4,_,_,_,_]	1
7: deletion	[1,2,3,_,_,_,_,_]	1
8: deletion	[1,2,_,_,_,_,_,_]	1
9: deletion	[1,_,_,_]	1+1=2
10: insertion	[1,2,_,_]	1

Example #4: dynamic table (insert & deletion)

Use the aggregate method

Example #4: dynamic table (insert & deletion)

Use the accounting method

Hint: insertion's amortized cost = 3, deletion's amortized cost = 2

Example #4: dynamic table (insert & deletion)

Use the potential method

Hint: Consider a potential function $\Phi(T) = 2 * num - size$ when $\alpha \geq \frac{1}{2}$ and $\Phi(T) = size/2 - num$ when $\alpha < \frac{1}{2}$, where num is the number of inserted objects, $size$ is the table size, $\alpha = \frac{num}{size}$ is the load factor.