Pre-Calculus

Contents

Polynomials

PART

I

1 Linear Functions

To investigate polynomials, we start with the simplest polynomials which are polynomials with degree 0 (constant functions), $f(x) = c$, or polynomials with degree one, $f(x) = ax + b$, which are also called **linear functions** (線性函數).

The graph of constant function $f(x) = c$ is just a horizontal line with y coordinate being fixed as c. On the other hand, the graph of $f(x) = ax + b$ is a line on the xy plane. Let's study how values of a (the coefficient before x^1 term) and b (the constant coefficient) determine properties of this line.

1.1 Geometrical Meaning of Coefficients

First observe that for $f(x) = ax + b$,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(ax_2 + b) - (ax_1 + b)}{x_2 - x_1} = a, \quad \text{for any } x_1, x_2. \tag{1.1}$$

Let us explain the formula above. Consider two points on the graph of $f(x)$, say, $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Then $x_2 - x_1$ is the difference of their x coordinates which is denoted by Δx. And, $f(x_2) - f(x_1)$ is the difference of their y coordinates which is denoted by Δy. Hence Equation (1.1) says that $\Delta y/\Delta x = a$ no matter where x_1, x_2 are. Imagine that we are moving along the graph $y = f(x) = ax + b$. When we have Δx horizontal displacement, the vertical displacement is always $\Delta y = a \cdot \Delta x$ which means that the "steepness" of the graph is the same everywhere. And, the vector going from the point $(x_1, f(x_1))$ to $(x_2, f(x_2))$ is

$$(x_2 - x_1, f(x_2) - f(x_1)) = (\Delta x, \Delta y) = (\Delta x, a\Delta x) \mathbin{/\!/} (1, a).$$

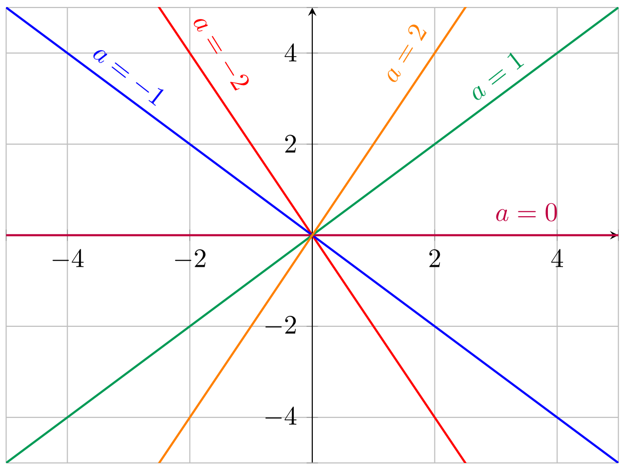

Figure 1.1. Graphs of $f(x) = ax$

Since the value $a = \Delta y/\Delta x$ describe the steepness of this line, we call it the **slope** (斜率) of the line $y = ax + b$. Moreover, vectors connecting any two points on the line $y = ax + b$ are parallel to $(1, a)$. Thus, we say that $(1, a)$ is a direction vector of the line.

If $a > 0$, $y = ax + b$ is "uphill" and y increases when x increases. If $a < 0$, $y = ax + b$ is "downhill" and y decreases when x increases. Moreover, larger $|a|$ gives steeper line. If $a = 0$, the function $f(x) = ax + b = b$ is a constant function and its graph, $y = ax + b = b$, is a horizontal line. **Figure 1.1** shows graphs of $f(x) = ax$ with different values of a.

Now we discuss the constant term b. Recall that the graph of $f(x) = ax+b$ is obtained by shifting the graph of $f(x) = ax$ vertically by b units. If $b > 0$, we shift the line $y = ax$ upward by b units and obtain the line $y = ax + b$. If $b < 0$, we shift the line $y = ax$ downward by $|b|$ units and obtain the line $y = ax + b$. Also for $f(x) = ax + b$, $b = f(0)$, i.e. the line $y = ax + b$ passes the point $(0, b)$. Hence b is called the **y intercept** (y 截距) of the line. **Figure 1.2** shows graphs of $f(x) = 2x + b$ with different values of b.

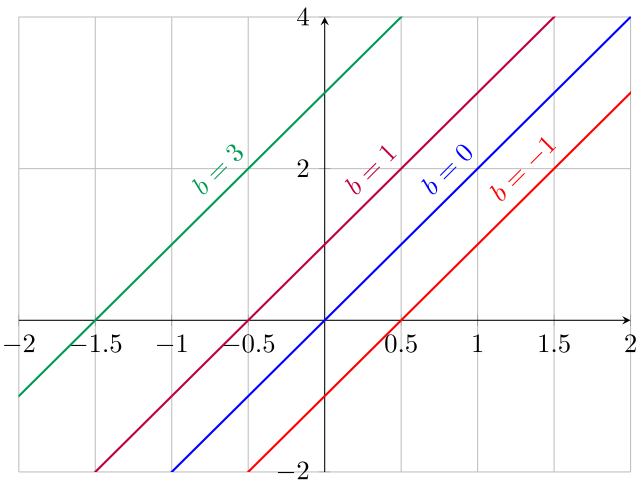

Figure 1.2. Graphs of $f(x) = 2x + b$

So far we have seen that the graph of a linear function $f(x) = ax + b$ is a line and constants a and b have geometrical meanings.

> **Proposition 1.1.** The graph of a linear function $f(x) = ax + b$ is a line with slope a and b as its y intercept.

Sometimes we may encounter lines given by equations of the form $ax + by = c$. How could we find its slope and the y intercept? Note that if $b \neq 0$, then

$$ax + by = c \iff by = -ax + c \iff y = -\frac{a}{b}x + \frac{c}{b}.$$

The last equation $y = -\frac{a}{b}x + \frac{c}{b}$ says that this line is the graph of the linear function $-\frac{a}{b}x + \frac{c}{b}$. Hence, from the above discussion we recognize that the coefficient $-\frac{a}{b}$ is the slope of the line and the constant term $\frac{c}{b}$ is the y intercept.

Exercise 1.1. Find the slope and the y intercept of the line $-2x + 5y = 1$.

1.2 Equations of Lines

Now, given a line on the xy plane, how could we find the equation that represents it? There are several ways to describe a line. For each description of a line, we want to derive the corresponding equation.

1. The Point-Slope Form (點斜式)

If the slope, m, and a point on the line, (x_0, y_0), are given, then the line is uniquely determined. For any other point (x, y) on the line, we compute the differences of the x and y coordinates of (x, y) and (x_0, y_0). Let $\Delta y = y - y_0$ and $\Delta x = x - x_0$. Then $\Delta y/\Delta x$ must be the slope, m. Hence,

$$\frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0} = m \implies y - y_0 = m(x - x_0) \implies y = y_0 + m(x - x_0).$$

Therefore the point (x, y) on the line satisfies the formula $y = y_0 + m(x - x_0)$ which is the equation of the line (point-slope form).

Example. Find the equation of the line with slope 3 passing the point $(2, 1)$.
Solution. By the point slope form, we know that the equation of the line is

$$y - 1 = 3(x - 2) \implies y = 1 + (3x - 6) \implies y = 3x - 5.$$

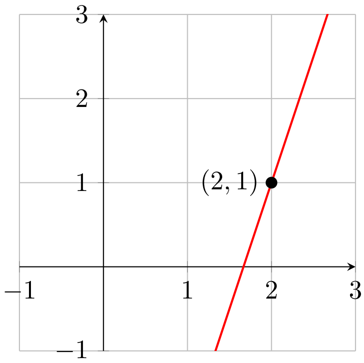

Figure 1.3. $y = 3x - 5$

Exercise 1.2. Find the equation of the line with slope -2 passing the point $(-3, 1)$.

Exercise 1.3. Find the equation of the line with slope m passing the point $(0, b)$ i.e. with b as the y intercept. (This is the "slope intercept form").

2. The Two-Point Form (兩點式)

A line can be determined by given any two points on it, say, (x_1, y_1) and (x_2, y_2). Because from these two points, we can derive the slope of the line which is $\Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$. Thus, the line passes the point (x_1, y_1) with slope $(y_2 - y_1)/(x_2 - x_1)$. Then by the point-slope form, the equation of the line is

$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{(Two-Point Form)}$$

Remark. Note that the line also passes through the point (x_2, y_2). Hence the equation of the line is

$$y = y_2 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_2).$$

You may want to show that

$$y = y_2 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_2) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Example. Find the equation of the line passing points $(2, 0)$ and $(1, 4)$.
Solution. Let $(x_1, y_1) = (2, 0)$, $(x_2, y_2) = (1, 4)$. The slope of is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{1 - 2} = -4.$$

And the equation of the line is

$$y = y_1 + m(x - x_1) = 0 + (-4)(x - 2) = -4x + 8.$$

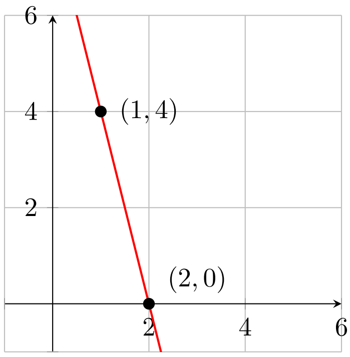

Figure 1.4. $y = -4x + 8$

You may use two-point form directly. The equation of the line is

$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) = 0 + \frac{4 - 0}{1 - 2}(x - 2) = -4x + 8.$$

Exercise 1.4. Find the equation of the line passing points $(x_0, 0)$ and $(0, y_0)$.

Exercise 1.5. Find the equation of the line passing points $(-2, 1)$ and $(-2, 3)$.

Did you encounter problem doing the above exercise? You may have observed that if a line passes two points (x_1, y_1) and (x_2, y_2) with the same x coordinates i.e., $x_1 = x_2$, then the two-point equation

$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

makes no sense since the denominator $x_2 - x_1$ is 0.

This is the only case that a line is not the graph of a linear function which means that on the line y coordinates do not depend on x coordinates and its equation is be of the form $y = ax + b$. In fact, the line containing points (x_1, y_1) and (x_1, y_2) is "vertical" with fixed x coordinate and hence its equation is $x = x_1$ while y coordinates can be any real numbers. Also, it is impossible to compute the slope of a vertical line (since for any two points on the line, $\Delta x = 0$), and we say that its slope is "infinity" which is denoted by ∞.

3. The Point-Normal Form (點法式)

Sometimes we are given a normal vector, $\mathbf{n} = (a, b)$, and a point, (x_0, y_0), of a line. The normal vector $\mathbf{n}$ has the property that it is perpendicular to any vector connecting two points on the line. Hence if (x, y) is a point on the line, then $\mathbf{n}$ is normal to the vector $(x - x_0, y - y_0)$ which means that $\mathbf{n} \cdot (x - x_0, y - y_0) = a(x - x_0) + b(y - y_0) = 0$. Therefore the equation of the line is

$$a(x - x_0) + b(y - y_0) = 0. \qquad \text{(Point-Normal Form)}$$

See **Figure 1.5** for an illustration.

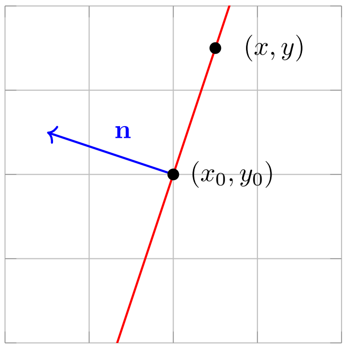

Figure 1.5.

Example. Find the equation of the line with normal vector $\mathbf{n} = (1, -2)$ passing the point $(2, 3)$.
Solution. Suppose that (x, y) is a point on the line. Then $\mathbf{n} = (1, -2)$ must be perpendicular to the vector $(x - 2, y - 3)$. Hence the equation of the line is

$$\begin{aligned}\mathbf{n} \cdot (x - 2, y - 3) &= (1, -2) \cdot (x - 2, y - 3) \\ &= 1 \cdot (x - 2) + (-2)(y - 3) = 0 \iff x - 2y = -4.\end{aligned}$$

We could further convert it to the standard form

$$2y = x + 4 \implies y = \frac{1}{2}x + 2.$$

See **Figure 1.6** for an illustration.

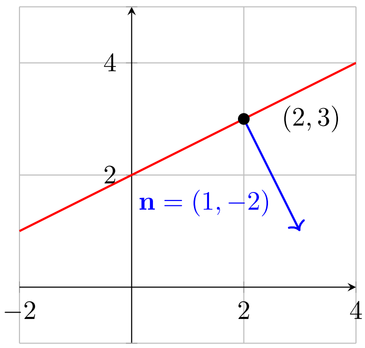

Figure 1.6. $x - 2y = -4$

Exercise 1.6. Find the equation of the line with normal vector $\mathbf{n} = (0, 1)$ passing the point $(3, 4)$.

Exercise 1.7. Suppose that a line has a normal vector, $\mathbf{n} = (a, b)$, passing a point, (x_0, y_0), where $b \neq 0$. Find the slope and the y intercept of the line.

At the end of this section, we discuss the relationship between slopes of two orthogonal (垂直的) lines.

Suppose that two lines L_1, L_2 with equations $y = m_1x + b_1$ and $y = m_2x + b_2$ respectively are orthogonal. (Here we don't discuss the vertical and horizontal lines.)

From **Figure 1.7** we know that one of the lines has positive slope and the other has negative slope. Say $m_1 > 0$ and $m_2 < 0$. Let $O = (x_0, y_0)$ be the point of intersection of the lines. Draw the vertical line $x = x_0 + 1$. Let A be the intersection of L_1 and the line $x = x_0 + 1$. Let B be the intersection of L_2 and the line $x = x_0 + 1$. Let C be the point $(x_0 + 1, y_0)$. Then $\overline{OC}$ has length 1 unit which is the horizontal displacement Δx. The length of $\overline{AC}$ is the vertical displacement of L_1, $\Delta y = m_1 \Delta x$, which is m_1 unit. Similarly, $\overline{CB}$ has length $-m_2 \cdot \Delta x = -m_2$ unit (since $m_2 < 0$).

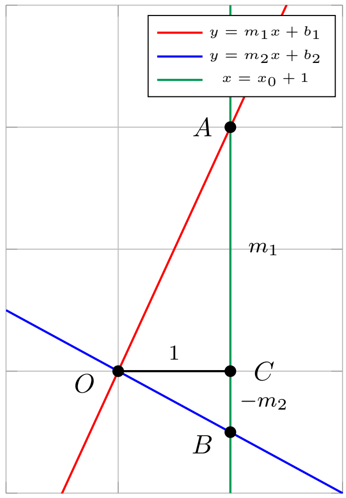

Figure 1.7.

Now observe that $\triangle OAB$, $\triangle CAO$, and $\triangle COB$ are right triangles and $\triangle OAB \sim \triangle CAO \sim \triangle COB$. Hence

$$\overline{AC} : \overline{OC} = \overline{OC} : \overline{BC} \implies m_1 : 1 = 1 : (-m_2) \implies m_1 \cdot (-m_2) = 1.$$

Therefore, we conclude that the product of the slopes of two orthogonal lines is -1. On the other hand, if lines L_1, L_2 have slopes m_1, m_2 respectively and $m_1 \cdot m_2 = -1$, then we can show that L_1, L_2 are orthogonal. Consider the direction vectors of the lines, $(1, m_1)$ and $(1, m_2)$. Because the inner product of $(1, m_1)$ and $(1, m_2)$ is $1 \cdot 1 + m_1 \cdot m_2 = 1 - 1 = 0$, we conclude that the direction vectors are orthogonal which implies that the lines are orthogonal.

Let's summarize above discussions into the following property.

Proposition 1.2. Lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are orthogonal if and only if

$$m_1 \cdot m_2 = -1.$$

Example. Find the equation of the line that is orthogonal to the line $2x - 3y = 6$ and passes the point $(-2, 1)$.
Solution. To know the slope of the line $2x - 3y = 6$, we rewrite the equation into the "standard form", $y = ax + b$.

$$2x - 3y = 6 \iff 3y = 2x - 6 \iff y = \frac{2}{3}x - 2.$$

Then from the equation $y = \frac{2}{3}x - 2$, we know that the slope of the line $2x - 3y = 6$ is $\frac{2}{3}$. Hence, the line orthogonal to $2x - 3y = 6$ has slope $\frac{-1}{\frac{2}{3}} = -\frac{3}{2}$. Moreover, the required line passes the point $(-2, 1)$. Therefore, by the point-slope form, the line orthogonal to $2x - 3y = 6$ passing $(-2, 1)$ is

$$y - 1 = -\frac{3}{2}(x - (-2)) \implies y = 1 - \frac{3}{2}x - 3 \implies y = \frac{-3}{2}x - 2.$$

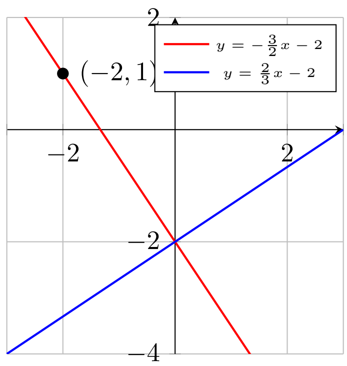

Figure 1.8. Graph of $y = \frac{-3}{2}x - 2$

Exercise 1.8. Show that lines $ax + by = c$ and $bx - ay = d$ are orthogonal for all $a \cdot b \neq 0$.

Exercise 1.9. Two lines L_1, L_2 are orthogonal and intersect at $(1, -1)$. If L_1 passes the point $(-1, 0)$, find the equation of L_2.

2 Quadratic Functions

In this section we study polynomials with degree 2, $f(x) = ax^2 + bx + c$, which are also called **quadratic functions** (二次函數). We will introduce the method of **Completing the Square** (配方法) to help us sketch graphs of quadratic functions and solve for their roots.

2.1 Graphs of Quadratic Functions

Since the graph of a function provides abundant information, we would like to sketch and investigate graphs of quadratic functions. Let's start with drawing graphs of the simplest quadratic functions $f(x) = ax^2$ where $a \neq 0$ is a constant.

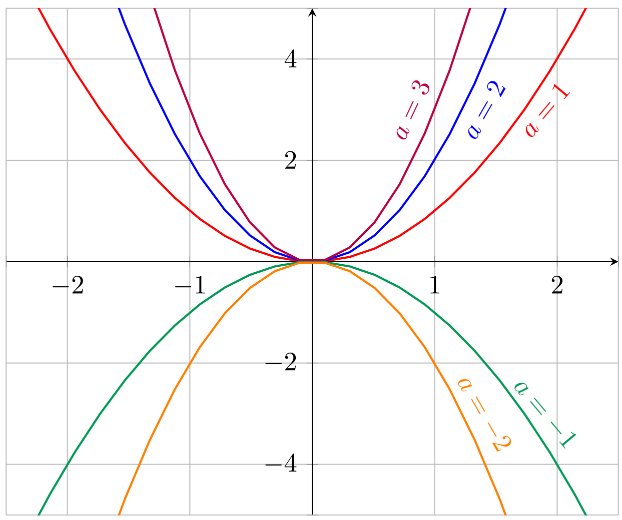

Figure 2.1. Graphs of $f(x) = ax^2$

From **Figure 2.1** and basic algebra, we know that for $a > 0$, $f(x) = ax^2 \geqslant 0$ and f obtains its minimum value 0 at $x = 0$. Thus $(0, 0)$ is the lowest point and the "vertex" of the graph. Moreover, the curve $y = f(x) = ax^2$ is symmetric with respect to the y-axis. (Check that $f(-x) = f(x)$.) This U-shaped graph is called an "open upward **parabola**" (開口向上的拋物線). On the other hand, for $a < 0$, the graph of $f(x) = ax^2$ is below the x-axis, symmetric with respect to the y-axis, and with the vertex (the highest point) at $(0, 0)$. This upside down U-shaped curve is called an "open downward parabola". We also observe that for larger $|a|$, the graph is steeper with narrower opening because $|f(x)|$ grows faster as $|x|$ increases.

Next, we deal with more general quadratic functions. With the knowledge of translations of functions, we can easily obtain the graph of $f(x) = a(x - h)^2 + k$ by shifting the graph of ax^2. For example, if both h, k are positive, the graph of $f(x) = a(x - h)^2 + k$ is obtained by shifting the graph of ax^2 horizontally h units to the right and then vertically k units upward (**Figure 2.2**). Thus the curve $y = a(x - h)^2 + k$ has the vertex (h, k) and is symmetric with respect to the line $x = h$. Similarly, if $h < 0$ and $k > 0$, the graph of $f(x) = a(x - h)^2 + k$ is obtained by shifting the graph of ax^2 horizontally $|h|$ units to the left and then vertically k units upward (**Figure 2.2**). Hence the curve $y = a(x - h)^2 + k$ has vertex $(-|h|, k) = (h, k)$ and is symmetric with respect to the line $x = -|h| = h$. You can fill up the table for the rest two cases with the help of **Figure** 2.3. As a conclusion, graphs of $f(x) = a(x-h)^2 + k$ are just translated parabolas!

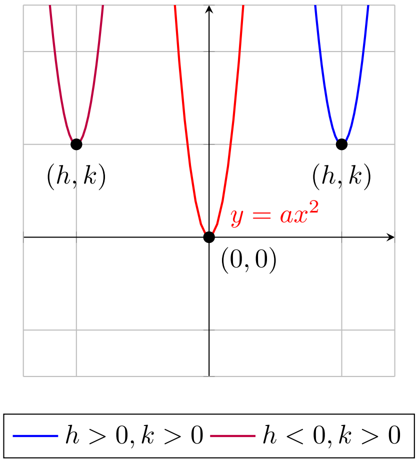

Figure 2.2. Shifting $f(x) = ax^2$

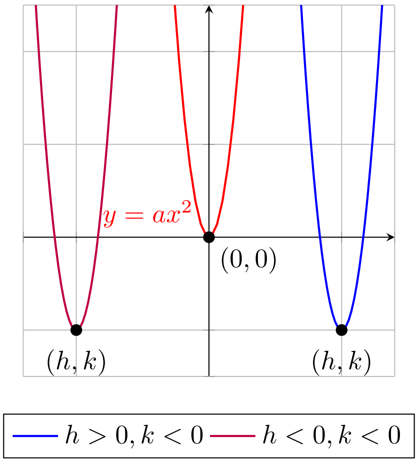

Figure 2.3. Shifting $f(x) = ax^2$

Properties of $y = a(x-h)^2 + k$			
Signs of h, k	How it is obtained by $y = ax^2$	Vertex	Axis of Symmetry
$h > 0,\ k > 0$	Shift $y = ax^2$ 1. h units to the right 2. k units upward	(h, k)	$x = h$
$h < 0,\ k > 0$	Shift $y = ax^2$ 1. $-\lvert h \rvert$ units to the left 2. k units upward	(h, k)	$x = h$
$h > 0,\ k < 0$			
$h < 0,\ k < 0$			

Hence if we can write general functions $f(x) = ax^2 + bx + c$ into the "vertex form", $a(x-h)^2 + k$, then we know the graph shape, the vertex, and the axis of symmetry of $y = f(x)$. The good news is every quadratic function can be converted into the vertex form by a method called **"Completing the Square"**. Note that the square term of the vertex form, $a(x-h)^2$, and the original polynomial $ax^2 + bx + c$ must have the same x^2 and x terms. Thus the idea of "completing the square" is to absorb the highest two terms into one square. The followings are the step by step process.

Completing the Square

1. Goal: Convert a quadratic polynomial $ax^2 + bx + c$ to the form $a(x-h)^2 + k$ where variable x only appears inside the square and h, k are constants.

2. Strategy for completing the square:

   (i) **Step 1.** Factor out the coefficient a of the quadratic function ax^2+bx+c.

   $$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right).$$

   (ii) **Step 2.** Write the monic quadratic function $x^2 + \frac{b}{a}x + \frac{c}{a}$ as the sum of $\left(x + \frac{b}{2a}\right)^2$ and the constant $\frac{c}{a} - \left(\frac{b}{2a}\right)^2$. It is convenient to memorize that the constant term inside the square, $\frac{b}{2a}$, is half of the coefficient of x, $\frac{b}{a}$. Then, because $\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$, the constant $\frac{c}{a} - \left(\frac{b}{2a}\right)^2$ is derived by basic algebra. Memorize this key step:

   $$x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2.$$

   (iii) **Step 3.** Multiply the result from **Step 2.** with a. Then we finish completing the square:

   $$\begin{aligned} ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] \\ &= a\left(x + \frac{b}{2a}x\right)^2 + c - \frac{b^2}{4a}. \end{aligned}$$

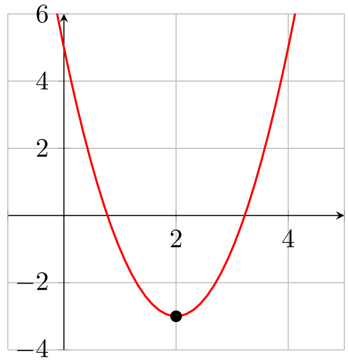

Figure 2.4. $y = 2x^2 - 8x + 5$

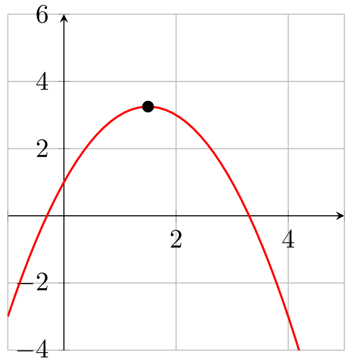

Figure 2.5. $y = -x^2 + 3x + 1$

Example. Complete the square and write the following quadratic functions in vertex form.

1. $2x^2 - 8x + 5$,
2. $-x^2 + 3x + 1$.

Solution.

1. We have

$$\begin{aligned} 2x^2 - 8x + 5 = 2\left(x^2 - 4x + \frac{5}{2}\right) &= 2\left[(x-2)^2 + \frac{5}{2} - (-2)^2\right] \\ &= 2(x-2)^2 + 5 - 2 \cdot 4 = 2(x-2)^2 - 3. \end{aligned}$$

2. We have

$$\begin{aligned} -x^2 + 3x + 1 = -(x^2 - 3x - 1) &= -\left[\left(x - \frac{3}{2}\right)^2 - 1 - \left(\frac{3}{2}\right)^2\right] \\ &= -\left(x - \frac{3}{2}\right)^2 + 1 + \frac{9}{4} = -\left(x - \frac{3}{2}\right)^2 + \frac{13}{4}. \end{aligned}$$

Exercise 2.1. Complete the square and write the following quadratic functions in vertex form.

(a) $x^2 - 4x + 1$ (b) $3x^2 + x + 2$ (c) $-2x^2 + 4x + 3$

Remark. In this section, we show that the method of completing the square can help us convert a quadratic function to the vertex form and solve the roots of a quadratic function. Later in Calculus course, completing the square is also needed for integrating rational functions.

After completing the square, we write general quadratic functions $ax^2 + bx + c$ in the form

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = a\left[x - \left(-\frac{b}{2a}\right)\right]^2 + \frac{4ac - b^2}{4a}.$$

With this vertex form, we can conclude that the graph of $ax^2 + bx + c$ is obtained by shifting the parabola $y = ax^2$. Moreover, it has vertex $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$ and is symmetric with respect to the line $x = -\frac{b}{2a}$. Let's summarize these results.

Proposition 2.1. The graph of $f(x) = ax^2+bx+c$, $a \neq 0$, has the following properties.

1. The graph is a translation of the parabola $y = ax^2$. If $a > 0$, it is an open upward parabola. If $a < 0$, it is open downward. For larger $|a|$, the opening of the graph is narrower.
2. The point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$ is the vertex of the graph. If $a > 0$, the vertex is the lowest point. If $a < 0$, the vertex is the highest point.
3. The graph is symmetric with respect to the line $x = -\frac{b}{2a}$.

From the graph of $f(x) = ax^2 + bx + c$, we know behaviors of the function.

Proposition 2.2. Consider a quadratic function $f(x) = ax^2 + bx + c$.

1. If $a > 0$ then $f(x)$ obtains minimum value $\frac{4ac-b^2}{4a}$ at $x = -\frac{b}{2a}$ and $f(x)$ tends to infinity as $|x|$ grows large.
2. If $a < 0$ then $f(x)$ obtains maximum value $\frac{4ac-b^2}{4a}$ at $x = -\frac{b}{2a}$ and $f(x)$ tends to negative infinity as $|x|$ grows large.
3. $f(x) = f(y)$ if x and y have mean value $-\frac{b}{2a}$ i.e.

$$f\left(x - \frac{b}{2a}\right) = f\left(-x - \frac{b}{2a}\right) \quad \text{or} \quad f(x) = f\left(-x - \frac{b}{a}\right)$$

for all x.

2.2 Roots of Quadratic Functions

We say that γ is a root of the quadratic function $f(x) = ax^2 + bx + c$ if $f(\gamma) = 0$. Therefore, if γ is a root of f then the graph of f intersects the x-axis at the point $(\gamma, 0)$. If we want to find roots of $f(x) = ax^2 + bx + c$, we need to solve the equation $f(x) = ax^2 + bx + c = 0$. With the vertex form, we have

$$ax^2 + bx + c = 0 \implies a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0 \implies \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

If the right hand side of the equation is positive i.e. $b^2 - 4ac > 0$, then we can apply square root to both sides of the equation, and derive that

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2|a|} \implies x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In this case, there are two distinct real roots of f, $x = \frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}$.

If $b^2 - 4ac = 0$, then $f(x) = 0$ is equivalent to $\left(x + \frac{b}{2a}\right)^2 = 0$ which has only one real solution, $x = -\frac{b}{2a}$. In addition, from the vertex from, $f(x) = a\left(x + \frac{b}{2a}\right)^2$, we know that the vertex of the graph is $\left(-\frac{b}{2a}, 0\right)$. Hence $f(x)$ obtains its maximum value (when $a < 0$) or the minimum value (when $a > 0$) at the root, $x = -\frac{b}{2a}$, with $f\left(-\frac{b}{2a}\right) = 0$. This means that the graph of f is **tangent** (相切) to the x-axis at the root.

In the last case $b^2 - 4ac < 0$, the equation $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$ has no real solutions because the left hand side is always non-negative but the right hand side is negative. Therefore, the the graph of f and the x-axis do not intersect.

Since the sign of $b^2 - 4ac$ determines the number of roots of $ax^2 + bx + c$, we give it a special name, the **discriminant** (判别式) of f, and denote it by D. We conclude the above discussion in terms of discriminant.

Proposition 2.3. For a quadratic function $f(x) = ax^2 + bx + c$, we call $D = b^2 - 4ac$ the discriminant of f.

1. If $D > 0$, then the equation $f(x) = 0$ has two distinct real roots, $x = \frac{-b\pm\sqrt{D}}{2a}$, and the graph of f meets the x-axis twice at points $\left(\frac{-b\pm\sqrt{D}}{2a}, 0\right)$.
2. If $D = 0$, then the equation $f(x) = 0$ has one real root, $x = \frac{-b}{2a}$. The graph of f is tangent to the x-axis at the only intersection point, $\left(-\frac{b}{2a}, 0\right)$.
3. If $D < 0$, then the equation $f(x) = 0$ has no real roots and the graph does not intersect the x-axis. If $a > 0$, the graph of f is above the x-axis. If $a < 0$, the graph is below the x-axis.

Example. Find roots of $f(x) = 2(x+1)^2 - 5$.
Solution. Since $f(x)$ is already in the vertex form, we don't need to expand it out and apply the formula for the roots. We directly start with the equation $f(x) = 0$.

$$f(x) = 2(x+1)^2 - 5 = 0 \implies 2(x+1)^2 = 5 \implies (x+1)^2 = \frac{5}{2} \implies x+1 = \pm\sqrt{\frac{5}{2}} \implies x = -1 \pm \sqrt{\frac{5}{2}}.$$

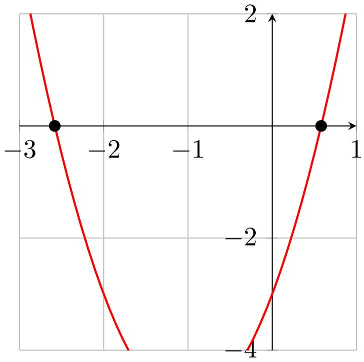

Figure 2.6. $y = 2(x+1)^2 - 5$

Example. Find roots of $f(x) = 2x^2 - \sqrt{12}x + 1$.
Solution. For this quadratic function, the coefficients are $a = 2$, $b = -\sqrt{12}$, $c = 1$. Then the discriminant $D = b^2 - 4ac = (-\sqrt{12})^2 - 4 \cdot 2 \cdot 1 = 12 - 8 = 4$. Then we apply the root formula. Roots of f are $\dfrac{-b \pm \sqrt{D}}{2a} = \dfrac{\sqrt{12} \pm \sqrt{4}}{2 \cdot 2} = \dfrac{\sqrt{12} \pm 2}{4}$.

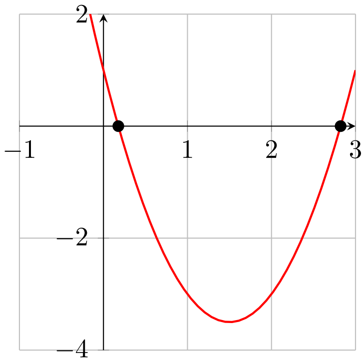

Figure 2.7. $y = 2x^2 - 6x + 1$

We recap methods for solving roots of a quadratic function from the above examples.

Methods for finding roots of a quadratic function $f(x)$

1. If $f(x)$ is in the vertex form $a(x-h)^2 + k$, then we solve the equation $f(x) = a(x-h)^2 + k = 0$ directly.

$$f(x) = a(x-h)^2 + k = 0 \implies (x-h)^2 = -\frac{k}{a} \implies x = h \pm \sqrt{\frac{-k}{a}} \quad \text{if} \quad \frac{-k}{a} \geqslant 0.$$

2. If $f(x)$ is in the general form $ax^2 + bx + c$, we can apply the root formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad \text{if} \quad b^2 - 4ac \geqslant 0.$$

This formula is very useful and worth memorizing !!

Exercise 2.2. Find roots of $f(x) = -2(x-2)^2 + 3$.

Exercise 2.3. Find roots of $f(x) = \frac{1}{\sqrt{3}}x^2 + 2x + \sqrt{3}$.

Exercise 2.4. Let $f(x) = x^2 + bx + 1$, where b is a constant. For what values of b does f have real root(s)? Find root(s) of f in terms of b. For what values of b does f have no real roots?

3 Factorization of Polynomials

Polynomials with higher degrees are much more complicated. Before analyzing them, we would want to first factorize them into a product of polynomials with lower degrees. You will learn that factoring polynomials is the preliminary step to solve for roots and sketch graphs. Factorization is also an essential tool for integrating rational functions. Therefore it is very important to master the skill.

3.1 Goal of Factorization

To factorize a polynomial, we need to find its **factors** (因式) which are polynomials with lower degrees that can **divide** (整除) the given one. Of course, constant functions $f(x) = c$ are factors of every polynomials. But we are interested in non-constant factors i.e. factors with degree $\geqslant 1$. After splitting a polynomial into a product of factors, how could we conclude that every factor can not be factorized anymore so that the factorization is completed? We say that a polynomial is **irreducible** (不可分解的，不可約的) if it cannot be factored into a product of two non-constant polynomials. By this definition, linear functions are irreducible. In general, whether a polynomial is irreducible depends on the type of polynomials we are looking for. The Fundamental Theorem of Algebra says that every degree n polynomial with **complex coefficients** (複係數多項式) has, counted with multiplicity, exactly n complex roots. This implies that each polynomial with complex coefficients can be factored into a product of linear functions with complex coefficients and only until this kind of product, the factorization is completed. (A number x_0 is a **root** of a polynomial $f(x)$ if $f(x_0) = 0$. On the marginal space we show that x_0 is a root of a polynomial $f(x)$ if and only if $x - x_0$ is a linear factor of f. Hence each root corresponds to a linear factor.)

However, we often deal with polynomials with **real coefficients** (實係數多項式). **In this section, we would like to factor polynomials with real coefficients and look for factors also with real coefficients**.

First recall that $f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$, $b^2 - 4ac < 0$ has no real roots. Therefore quadratic functions with negative discriminant have no linear factors with real coefficients and they are irreducible.

Now consider a general n-th degree polynomial with real coefficients, P. Because P is also a polynomial with complex coefficients, P has n complex roots, γ_k, $1 \leqslant k \leqslant n$. It can be shown that if $\gamma_k = a + bi$ is a root of P with $a, b \in \mathbb{R}$ and $b \neq 0$, then its complex conjugate $a - bi$ is also a root of P. Hence $(x - (a + bi))(x - (a - bi)) = a^2 - 2ax + a^2 + b^2$ is an irreducible factor of P (Check that the discriminant $(2a)^2 - 4(a^2 + b^2)$ is negative.). If γ_k is a real root of P, then the linear function $x - \gamma_k$ is an irreducible factor of P with real coefficients. As a conclusion, a polynomial with real coefficients, P, can be factorized as a product of linear functions and irreducible quadratic functions. And only until this type of factors the factorization is completed.

Remark. Note that a linear function $x - x_0$ is a factor of a polynomial $f(x)$ if and only if x_0 is a root of f.

Proof. Divide $f(x)$ by $x - x_0$. Let r be the remainder which is a constant. Then we write

$$f(x) = (x - x_0)Q(x) + r$$

where $Q(x)$ is a polynomial. Plug in $x = x_0$, we conclude that

$$f(x_0) = 0 \cdot Q(x_0) + r = r.$$

Hence x_0 is a root of f i.e. $f(x_0) = 0$ if and only if $r = 0$ which means that $x - x_0$ is a factor of f. □

Proposition 3.1. Polynomials with real coefficients can be factored as a product of factors with real coefficients which are linear functions, $ax+b$, or irreducible quadratic functions, $ax^2 + bx + c$ where $b^2 - 4ac < 0$.

3.2 Techniques of Factorization

Here we introduce some techniques of factoring polynomials. Let's start with quadratic polynomials. Then investigate polynomials with higher degrees.

1. Factorize quadratic polynomials.

A quadratic function, $f(x) = ax^2 + bx + c$ is irreducible if $D = b^2 - 4ac < 0$. Hence only quadratic polynomials with positive discriminant can be factorized as a product of two linear functions. There are two methods for factorizing quadratic functions.

Cross Method (十字交乘法).

Suppose that coefficients of $f(x) = ax^2 + bx + c$ are integers, and the linear factors of f are also with integer coefficients, say

$$ax^2 + bx + c = (p_1x + q_1)(p_2x + q_2) \quad \text{, where } a, b, c, p_i, q_i \text{ are integers.}$$

Then $p_1p_2 = a$ and $q_1q_2 = c$ which means that p_1, p_2 are factors of a and q_1, q_2 are factors of c. Moreover, the cross multiplication $p_1q_2 + p_2q_1$ gives b. Hence to find linear factors of f, we start with factorizing integers a and c as products of two integers, say $p_1p_2 = a$ and $q_1q_2 = c$. Then arrange factors of a (i.e. p_1 and p_2) in the left column and factors of c (i.e. q_1 and q_2) in the right column. Finally we draw two cross lines. One line is from left-up to right-down connecting p_1 and q_2. The other line connects p_2 and q_1. These lines remind us to do multiplication, $p_1 \cdot q_2$ and $p_2 \cdot q_1$. If the sum of these products is b then $(p_1x + q_1)$ and $(p_2x + q_2)$ are linear factors of f. Otherwise, $(p_1x + q_1)$ and $(p_2x + q_2)$ can not divide f and we should try different factors of a and c.

(i) **Step 1.** Factorize coefficients a and c, say $\quad a = p_1p_2, \quad c = q_1q_2$.

(ii) **Step 2.** Arrange these factors in two columns. Draw cross lines and do multiplication, p_1q_2 and p_2q_1:

$$\begin{array}{ccc} ax^2 & +bx & +c \\ p_1 & \times & q_1 \\ p_2 & & q_2 \end{array}$$

(iii) **Step 3.** Check whether $p_1q_2 + p_2q_1 = b$. If $p_1q_2 + p_2q_1 = b$, then $ax^2 + bx + c = (p_1x + q_1)(p_2x + q_2)$. If $p_1q_2 + p_2q_1 \neq b$, then try other factors of a and c.

Example. Factorize $-2x^2 + 5x + 12$.

Solution. The coefficient of x^2 is -2, and the constant term is 12. First we factorize -2 and 12, say, $-2 = (-1) \cdot 2$ and $12 = 6 \cdot 2$. Then, arrange these factors in a matrix and do cross multiplication:

$$\begin{array}{ccc} -2x^2 & +5x & +12 \\ -1 & \times & 6 \\ 2 & & 2 \end{array}$$

We observe that $(-1) \cdot 2 + 2 \cdot 6 = 10 \neq 5$. Hence this factorization fails. We turn to try other factors of -2 and 12, say $-2 = (-1) \cdot 2$ and $12 = 4 \cdot 3$. Arrange them in a matrix and do cross products:

$$\begin{array}{ccc} -2x^2 & +5x & +12 \\ -1 & \times & 4 \\ 2 & & 3 \end{array}$$

The sum of cross multiplications is $(-1) \cdot 3 + 2 \cdot 4 = 5$. Hence

$$-2x^2 + 5x + 12 = (-x + 4)(2x + 3).$$

Example. Factorize $8x^2 - 14x + 3$.
Solution. In the first step, we should factorize numbers 8 and 3. If we choose positive factors of 8 and 3, the sum of cross products is positive and can never be -14. Hence we ought to split 8 or 3 into a product of two negative numbers, say $8 = 2 \cdot 4$ and $3 = (-1) \cdot (-3)$. Then, arrange these factors in a matrix and do cross multiplication:

$$\begin{array}{ccc} 8x^2 & -14x & +3 \\ 2 & & -3 \\ 4 & & -1 \end{array}$$

The sum of cross multiplications is $2 \cdot (-1) + 4 \cdot (-3) = -14$. Hence

$$8x^2 - 14x + 3 = (2x-3)(4x-1).$$

Exercise 3.1. Factorize $3x^2 + 5x - 2$, $6x^2 + x - 12$, and $3x^2 - 11x + 6$.

The cross method is convenient if you can guess the right factors of coefficients a and c. However, if the polynomial or its factors have non integer coefficients, the cross method fails. There is another approach that can help factorizing all quadratic polynomials.

Factoring by the root formula.

Consider a polynomial $f(x) = ax^2 + bx + c$. We have shown that x_0 is a root of f if and only if $x - x_0$ is a factor of $f(x)$. Moreover, by the root formula, f has roots $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Hence $x - \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x - \frac{-b-\sqrt{b^2-4ac}}{2a}$ are factors of f. Therefore

$$\frac{f(x)}{\left(x - \frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x - \frac{-b-\sqrt{b^2-4ac}}{2a}\right)} \quad \text{is a constant.}$$

By comparing the coefficient of x^2, we know that the constant is a. In conclusion,

$$f(x) = a\left(x - \frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x - \frac{-b-\sqrt{b^2-4ac}}{2a}\right).$$

Example. Factorize $f(x) = 2x^2 - \sqrt{3}x - 1$.
Solution. Observe that $f(x)$ has two real roots which are

$$\frac{\sqrt{3} \pm \sqrt{\left(-\sqrt{3}\right)^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{\sqrt{3} \pm \sqrt{11}}{4}$$

and the leading coefficient is 2. Hence

$$f(x) = 2\left(x - \frac{\sqrt{3}+\sqrt{11}}{4}\right)\left(x - \frac{\sqrt{3}-\sqrt{11}}{4}\right).$$

Exercise 3.2. Factorize $3x^2 - 2x - 2$ and $-x^2 + 5x - \frac{1}{4}$.

2. Factorize polynomials with higher degrees.

In general, it is difficult to factorize polynomials with large degrees. The best strategy is to find obvious factors first. Divide the polynomial by the obvious factor. Then turn to factorize the quotient. Here we provide some approaches to finding obvious factors.

(a) Checking special function values.

Recall that $x - x_0$ is a factor of the polynomial $f(x)$ if and only if $f(x_0) = 0$. Hence we can substitute different values and check whether they are roots of f. Computing values of the function may be complicated but $f(1)$ and $f(-1)$ are especially easy and we should check them in the first place. Suppose that $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then

$$f(1) = a_n + a_{n-1} + \cdots + a_1 + a_0 \quad \text{is the sum of coefficients.}$$

Hence if the sum of coefficients of f is 0, then $x - 1$ is a factor of f. Similarly,

$$f(-1) = (-1)^n a_n + (-1)^{n-1} a_{n-1} + \cdots - a_1 + a_0$$

is the "alternating sum" of coefficients. If $a_0 - a_1 + \cdots + (-1)^n a_n = 0$, then $x + 1$ is a factor of f. Sometimes there are obvious roots. For example, $f(x) = x^n - a^n$ has a root $x = a$. Therefore $x - a$ divides f.

Example. Factorize $f(x) = x^3 - 3x^2 + 2$.
Solution. Observe that $f(1) = 1 - 3 + 2 = 0$. Hence $x - 1$ is a factor of f. We divide f by $x - 1$ and obtain that

$$f(x) = \big(x - 1\big)\big(x^2 - 2x - 2\big).$$

Then we further factorize the quotient $x^2 - 2x - 2$ as $\left(x - 1 + \sqrt{3}\right)\left(x - 1 - \sqrt{3}\right)$ (by the root formula). Therefore

$$f(x) = \Big(x - 1\Big)\Big(x - 1 + \sqrt{3}\Big)\Big(x - 1 - \sqrt{3}\Big).$$

Example. Factorize $f(x) = x^3 + x^2 + x + 1$.
Solution. Observe that $f(-1) = (-1) + 1 + (-1) + 1 = 0$. Thus $x + 1$ is a factor of f. Divide f by $x + 1$ and obtain $f(x) = \big(x + 1\big)\big(x^2 + 1\big)$. Note that $x^2 + 1$ is irreducible. Hence we have completed the factorization.

Exercise 3.3. Factorize $x^3 - 4x^2 + 5x - 2$ and $x^3 + 2x + 3$.

(b) Factoring by grouping.

Try to find patterns and group similar terms so that we can find a factor.

Example. Factorize $f(x) = x^3 + 2x^2 + x + 2$.
Solution. We rewrite f as $\left(x^3 + 2x^2\right) + (x + 2)$ and pull out common factor from the first group. Hence

$$f(x) = \left(x^3 + 2x^2\right) + (2x + 4) = x^2(x + 2) + (x + 2) = \left(x^2 + 1\right)(x + 2).$$

We can also write f as $(x^3 + x) + (2x^2 + 2)$. Then

$$f(x) = \left(x^3 + x\right) + \left(2x^2 + 2\right) = x\left(x^2 + 1\right) + 2\left(x^2 + 1\right) = (x + 2)\left(x^2 + 1\right).$$

Example. Factorize $f(x) = x^4 + x^3 + 2x^2 + x + 1$.
Solution. We write f as $\left(x^4 + x^3 + x^2\right) + \left(x^2 + x + 1\right)$. Therefore

$$\begin{aligned} f(x) &= \left(x^4 + x^3 + x^2\right) + \left(x^2 + x + 1\right) \\ &= x^2\big(x^2 + x + 1\big) + \big(x^2 + x + 1\big) = \big(x^2 + 1\big)\big(x^2 + x + 1\big). \end{aligned}$$

Then we check that both x^2+1 and x^2+x+1 are irreducible. Thus we have complete the factorization.

Exercise 3.4. Factorize $x^3 - 2x^2 - 3x + 6$. Factorize $x^3 + x^2 + x + 1$ by grouping.

3.3 Special Identities

Some factorization appears frequently. Such formulas are collected and marked as special identities. We list them here and show their applications.

Proposition 3.2 (Special Identities (Part 1)).

1. $(x+a)^2 = x^2 + 2ax + a^2$.
2. $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$.
3. $x^2 - a^2 = (x-a)(x+a)$.
4. $x^3 - a^3 = \big(x-a\big)\big(x^2 + ax + a^2\big)$.
5. $x^3 + a^3 = \big(x+a\big)\big(x^2 - ax + a^2\big)$.
6. $x^n - a^n = \big(x-a\big)\big(x^{n-1} + ax^{n-2} + \cdots + a^k x^{n-k-1} + \cdots + a^{n-1}\big)$.

Sometimes these equations appear as algebraic identities instead of polynomial factorization.

Proposition 3.3 (Special Identities (Part 2)).

1. $(a+b)^2 = a^2 + 2ab + b^2$.
2. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
3. $a^2 - b^2 = (a-b)(a+b)$.
4. $a^3 - b^3 = \big(a-b\big)\big(a^2 + ab + b^2\big)$.
5. $a^3 + b^3 = \big(a+b\big)\big(a^2 - ab + b^2\big)$.
6. $a^n - b^n = \big(a-b\big)\big(a^{n-1} + a^{n-2}b + \cdots + a^{n-k-1}b^k + \cdots + b^{n-1}\big)$.

You are encouraged to verify each equation either by multiplication or factorization. Note that the last identity is not the completed factorization but it is already useful.

Identities regarding $(x+a)^2$ and $(x+a)^3$ are called **binomial expansions** (二項式展開). In general, for any positive integer n, we have

$$(x+a)^n = \sum_{k=0}^{n} C^n_k a^k x^{n-k} \quad \text{where} \quad C^n_k = \frac{n!}{k!(n-k)!}.$$

These expansions are heavily used in Probability and Statistics. Here we present some computations based on binomial expansions which will occur in Calculus course.

Example. Simplify $\frac{(x+h)^2-x^2}{h}$ for $h \neq 0$.
Solution.
$$\frac{(x+h)^2-x^2}{h} = \frac{(x^2+2hx+h^2)-x^2}{h} = \frac{2xh+h^2}{h}.$$
For $h \neq 0$, we can cancel the common factor h of numerator and denominator, and derive that
$$\frac{(x+h)^2-x^2}{h} = \frac{2xh+h^2}{h} = 2x+h.$$

Example. Simplify $\frac{(x-h)^3-x^3}{h}$ for $h \neq 0$.
Solution. First observe that
$$\begin{aligned}(x-a)^3 = [x+(-a)]^3 &= x^3+3x^2(-a)+3x(-a)^2+(-a)^3 \\ &= x^3-3ax^2+3a^2x-a^3.\end{aligned}$$
Therefore, we may simplify the quantity by
$$\frac{(x-h)^3-x^3}{h} = \frac{(x^3-3hx^2+3h^2x-h^3)-x^3}{h} = \frac{-3hx^2+3h^2x-h^3}{h}.$$
For $h \neq 0$, we can cancel the common factor h of numerator and denominator, and derive that
$$\frac{(x-h)^3-x^3}{h} = \frac{-3hx^2+3h^2x-h^3}{h} = -3x^2+3hx-h^2.$$

Exercise 3.5. Expand $(x-a)^2$ and $(x-a)^n = [x+(-a)]^n$ where n is a positive integer.

Exercise 3.6. Simplify $\frac{(x+h)^n-x^n}{h}$ for $h \neq 0$, where n is a positive integer.

With identities about x^2-a^2 and x^3-a^3 we can derive more factorization.

Example. Factorize x^4-4.
Solution. Observe that $x^4-4=(x^2)^2-2^2$ and we apply the identity $a^2-b^2=(a-b)(a+b)$ with $a=x^2$ and $b=2$. Hence $x^4-4=\left(x^2\right)^2-2^2=\left(x^2-2\right)\left(x^2+2\right)$. Again $x^2-2=x^2-\left(\sqrt{2}\right)^2=\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)$. Therefore
$$x^4-4=\left(x^2\right)^2-2^2=\left(x^2-2\right)\left(x^2+2\right)=\Big(x-\sqrt{2}\Big)\Big(x+\sqrt{2}\Big)\Big(x^2+2\Big).$$

Example. Factorize x^4+1.
Solution. Observe that $x^4+1 \geqslant 1$ and has no real roots. Hence it must be a product of two irreducible quadratic functions. Now we try a different type of completing the square. Instead of combing the highest two terms in a square, here we absorb the highest and the constant term in a square. Since $\left(x^2+1\right)^2$ and x^4+1 has the same leading terms and constant terms, we write $x^4+1=\left(x^2+1\right)^2-2x^2$ which is of the form a^2-b^2. Hence
$$\begin{aligned}x^4+1 &= \left(x^2+1\right)^2-2x^2 \\ &= \left[\left(x^2+1\right)-\sqrt{2}x\right]\left[\left(x^2+1\right)+\sqrt{2}x\right] = \left(x^2-\sqrt{2}x+1\right)\left(x^2+\sqrt{2}x+1\right).\end{aligned}$$

Example. Factorize $1 - x^6$.
Solution. Observe that $1 - x^6 = 1 - \left(x^3\right)^2 = \left(1 - x^3\right)\left(1 + x^3\right)$. By the factorization of $a^3 - b^3$ and $a^3 + b^3$, we obtain that

$$1 - x^6 = \left(1 - x^3\right)\left(1 + x^3\right) = \left(1 - x\right)\left(1 + x + x^2\right)\left(1 + x\right)\left(1 - x + x^2\right).$$

There is another way of factoring $1 - x^6$. Write $1 - x^6 = 1 - \left(x^2\right)^3$. Then

$$1 - x^6 = 1 - \left(x^2\right)^3 = \left(1 - x^2\right)\left(1 + x^2 + x^4\right) = \left(1 - x\right)\left(1 + x\right)\left(1 + x^2 + x^4\right).$$

Then try to further factorize $x^4 + x^2 + 1$.

Exercise 3.7. Factorize $x^4 - x^2 + 1$, and $x^6 + 1$.

4 Roots of Polynomials

In this section, our goal is to look for all real roots of a polynomials (i.e. determine where the graph intersects the x axis) and solve for equations of polynomials.

4.1 Roots of Polynomials

A polynomial with real coefficients, $f(x)$, can be factorized as a product of linear functions, $a_i x + b_i$, and irreducible quadratic polynomials $\alpha_j x^2 + \beta_j x + \gamma_j$, i.e.

$$f(x) = \left(a_1 x + b_1\right) \cdots \left(a_m x + b_m\right)\left(\alpha_1 x^2 + \beta_1 x + \gamma_1\right) \cdots \left(\alpha_n x^2 + \beta_n x + \gamma_n\right).$$

Thus we have

$$f(x) = 0 \Longleftrightarrow \left(a_1 x + b_1\right) \cdots \left(a_m x + b_m\right)\left(\alpha_1 x^2 + \beta_1 x + \gamma_1\right) \cdots \left(\alpha_n x^2 + \beta_n x + \gamma_n\right) = 0.$$

Since irreducible polynomials $\alpha_j x^2 + \beta_j x + \gamma_j \neq 0$ for all x, $f(x) = 0$ is equivalent to $(a_1 x + b_1) \cdots (a_m x + b_m) = 0$ which means that one of $a_i x + b_i$, $1 \leqslant i \leqslant m$ is zero. Therefore, real roots of f are

$$-\frac{b_i}{a_i}, \quad 1 \leqslant i \leqslant m.$$

If linear factor $a_i x + b_i$ is repeated k_i times (i.e. $\left(a_i x + b_i\right)^{k_i}$ is a factor of f but $\left(a_i x + b_i\right)^{k_i + 1}$ can not divide f), then the real root $-\frac{b_i}{a_i}$ is said to be with **multiplicity** k_i.

Example. Find all real roots of $f(x) = x^3 - 3x^2 + 2$.
Solution. Factorize $x^3 - 3x^2 + 2$ as $\left(x - 1\right)\left(x - 1 + \sqrt{3}\right)\left(x - 1 - \sqrt{3}\right)$. Then

$$\begin{aligned} f(x) = 0 \quad &\Longleftrightarrow \quad \left(x - 1\right)\left(x - 1 + \sqrt{3}\right)\left(x - 1 - \sqrt{3}\right) = 0 \\ &\Longleftrightarrow \quad x = 1, 1 - \sqrt{3}, \text{ or } 1 + \sqrt{3}. \end{aligned}$$

Example. Find all real roots of $f(x) = 1 - x^6$.
Solution. After factorization,

$$1 - x^6 = \left(1 - x\right)\left(1 + x\right)\left(x^2 + x + 1\right)\left(x^2 - x + 1\right).$$

Since x^2+x+1, x^2-x+1 are irreducible, we have

$$\begin{aligned} f(x)=0 &\iff (1-x)(1+x)(x^2+x+1)(x^2-x+1)=0 \\ &\iff (1-x)(1+x)=0 \\ &\iff x=1 \text{ or } -1. \end{aligned}$$

Example. Find all real roots of $f(x)=x^4+x^3+2x^2+x+1$.
Solution. After factorization, $f(x)=\left(x^2+1\right)\left(x^2+x+1\right)$. Then

$$f(x)=0 \iff \left(x^2+1\right)\left(x^2+x+1\right)=0.$$

However, x^2+1 and x^2+x+1 are irreducible and can not be zero. Hence f has no real roots.

Example. Find all real roots of $f(x)=x^5+4x^4+4x^3$.
Solution. First factorize $x^5+4x^4+4x^3$. Observe that x^3 is a factor of f, and $f(x)=x^3\left(x^2+4x+4\right)$. Then by the binomial expansion, $x^2+4x+4=(x+2)^2$. Thus $f(x)=x^3(x+2)^2$. Therefore,

$$f(x)=0 \iff x^3(x+2)^2=0 \iff x=0 \text{ or } -2.$$

The factor x is repeated three times and the factor $x+2$ two times. Hence the root 0 is with multiplicity 3 and the root -2 is with multiplicity 2.

Exercise 4.1. Find all real roots together with multiplicities of following polynomials.

(a) x^4-2x^2+1. (b) x^3-2x^2-3x+6. (c) x^6+2x^3+1.

4.2 Solving equations of polynomials

Now we are ready to solve equations of polynomials.
The easiest equation is $p(x)=a$ where p is a polynomial and a is a constant. Note that

$$p(x)=a \iff p(x)-a=0.$$

Hence solutions of the equation $p(x)=a$ are just roots of the polynomial $p(x)-a$. Thus we can apply techniques about solving roots of $p(x)-a$ and find all solutions of $p(x)=a$.

Example. Solve the equation $p(x)=-2$, where $p(x)=x^3-3x^2+x-1$.
Solution. Since $p(x)=-2$ is equivalent to $p(x)+2=0$, we solve roots of $p(x)+2=x^3-3x^2+x-1+2=x^3-3x^2+x+1$. Note that $p(1)+2=0$. Thus $x-1$ is a factor of x^3-3x^2+x+1 and

$$x^3-3x^2+x+1=(x-1)(x^2-2x-1)=(x-1)(x-1-\sqrt{2})(x-1+\sqrt{2}).$$

Therefore, roots of $p(x)+2$ are $1, 1\pm\sqrt{2}$. And solutions of $p(x)=-2$ are $1, 1\pm\sqrt{2}$.

Exercise 4.2. Find constants a such that $p(x)=a$ has no real solutions, where $p(x)=2x^2+3x+1$.

The most general equation of polynomials is of the form $p(x)=q(x)$ where $p(x)$ and $q(x)$ are polynomials. You may already have experience about solving this kind of problem. Let's see how two students solve the following equation.

Example. Solve the equation $p(x) = q(x)$, where $p(x) = -x^3 + 2x^2$ and $q(x) = 3x^2$.

- **The solution of student A:**

$$\begin{aligned} p(x) = q(x) \quad &\implies \quad -x^3 + 2x^2 = 3x^2 \\ &\implies \quad x^2(-x+2) = 3x^2 \quad \implies \quad -x + 2 = 3 \quad \implies \quad x = -1. \end{aligned}$$

- **The solution of student B:**

$$\begin{aligned} p(x) = q(x) \quad &\iff \quad -x^3 + 2x^2 = 3x^2 \\ &\iff \quad -x^3 - x^2 = 0 \\ &\iff \quad -x^2(x+1) = 0 \quad \iff \quad x = -1, 0. \end{aligned}$$

Whose answer is correct? If we plug in $x = 0$ to both $p(x)$ and $q(x)$, we see that at $x = 0$, $p(0) = q(0) = 0$. Hence $x = 0$ is also a solution to the equation $p(x) = q(x)$. But how does student A miss this solution? Because he cancels the common factor x^2 of both sides, he can't see that the root of x^2, $x = 0$, is also a solution. Hence when you solve an equation $p(x) = q(x)$, be aware that you shall not carelessly cancel non-constant common factors of both sides although you might be tempted to "simplify" the equation! The correct method of solving $p(x) = q(x)$ is rearranging the equation

$$p(x) = q(x) \quad \iff \quad p(x) - q(x) = 0.$$

Hence we conclude that finding solutions of the equation $p(x) = q(x)$ is equivalent to finding roots of the polynomial $p(x) - q(x)$.

Exercise 4.3. Suppose that $p(x), q(x), r(x)$ are non-constant polynomials. Show that solutions of $p(x)q(x) = p(x)r(x)$ are exactly solutions of $q(x) = r(x)$ if $p(x)$ is irreducible.

5 Signs of Polynomials

After factorization, not only all real roots are found, we can also determine intervals on which a polynomial is positive or negative. Roots of a polynomials divide the real line into several intervals. Factors of the polynomial may change signs at these roots but they remain same signs on each of the intervals. By listing signs of each factor we can determine whether the polynomial is positive or negative.

In particular, a polynomial $f(x)$ can be factorized into a product of linear factors and irreducible quadratic factors. We could further make leading coefficients of every factors positive before discussing their signs. If $ax + b$ is a linear factor with $a > 0$, then

$$\begin{aligned} ax + b = a\left(x + \frac{b}{a}\right) &< 0 \text{ for } x < -\frac{b}{a}; \\ &> 0 \text{ for } x > -\frac{b}{a}. \end{aligned}$$

If $ax^2 + bx + c$ is an irreducible factor (i.e. $b^2 - 4ac < 0$) and $a > 0$, then

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] > 0 \quad \text{for all } x.$$

Since the irreducible factor ax^2+bx+c is always positive, it doesn't affect the sign of $f(x)$ and we can ignore it. In conclusion, signs of each factor are determined and we can know where $f(x)$ is positive or negative.

Example. Find the intervals on which $f(x)=x^3+2x^2-x-2$ is positive or negative.
Solution. By grouping x^3+2x^2 and $-x-2$, we observe that

$$\begin{aligned} f(x) &= \left(x^3+2x^2\right)-\left(x+2\right) \\ &= x^2(x+2)-(x+2) \\ &= \left(x^2-1\right)\left(x+2\right) \\ &= (x-1)(x+1)(x+2). \end{aligned}$$

Roots of $f(x)$ are $1,-1$ and -2. They divide the real line into intervals

$$(-\infty,-2), \quad (-2,-1), \quad (-1,1) \quad \text{and} \quad (1,\infty).$$

The linear factor $x-1$ changes signs at $x=1$. The factor $x+1$ changes signs at $x=-1$. And $x+2$ changes signs at $x=-2$. All these linear polynomials are with positive leading coefficients, and we list their signs on each interval

	$(-\infty,-2)$	-2	$(-2,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$x-1$:	$-$		$-$		$-$		$+$
$x+1$:	$-$		$-$		$+$		$+$
$x+2$:	$-$		$+$		$+$		$+$

Since $f(x)=(x-1)(x+1)(x+2)$, signs of $f(x)$ are obtained by multiplying signs of the factors. Foe example, on the interval $(-\infty,-2)$, $x-1$, $x+1$ and $x+2$ are all negative and the sign of $f(x)$ is $(-)\cdot(-)\cdot(-)$ which is $(-)$.

	$(-\infty,-2)$	-2	$(-2,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$x-1$:	$-$		$-$		$-$		$+$
$x+1$:	$-$		$-$		$+$		$+$
$x+2$:	$-$		$+$		$+$		$+$
$f(x)$:	$-$		$+$		$-$		$+$

Hence $f(x)$ is negative on $(-\infty,-2)$ and $(-1,1)$. $f(x)$ is positive on $(-2,-1)$ and $(1,\infty)$.

Example. Find the intervals on which $f(x)=2x^3+7x^2-15x$ is positive or negative.
Solution. By factoring out x and cross method, we write

$$f(x)=x\left(2x^2+7x-15\right)=x(2x-3)(x+5).$$

Roots of $f(x)$ are $-5,0$ and $3/2$. They divide the real line into intervals

$$(-\infty,-5), \quad (-5,0), \quad \left(0,\frac{3}{2}\right) \quad \text{and} \quad \left(\frac{3}{2},\infty\right).$$

The linear factor x changes signs at $x = 0$. The factor $2x - 3$ changes signs at $x = \frac{3}{2}$. And $x + 5$ changes signs at $x = -5$. All the linear factors are with positive leading coefficients, and we list their signs on each interval.

		-5		0		$\frac{3}{2}$	
x:	$-$		$-$		$+$		$+$
$2x-3$:	$-$		$-$		$-$		$+$
$x+5$:	$-$		$+$		$+$		$+$
$f(x)$:	$-$		$+$		$-$		$+$

Hence $f(x)$ is negative on $(-\infty, -5)$ and $\left(0, \frac{3}{2}\right)$. $f(x)$ is positive on $(-5, 0)$ and $\left(\frac{3}{2}, \infty\right)$.

Example. Find the intervals on which $f(x) = 1 - x^6$ is positive or negative.
Solution. By factorization, $1 - x^6 = \big(1 - x\big)\big(1 + x\big)\big(x^2 + x + 1\big)\big(x^2 - x + 1\big)$. Note that $1 - x$ has negative leading coefficient. We factor out the number -1 and write

$$f(x) = -(x - 1)(x + 1)(x^2 + x + 1)(x^2 - x + 1).$$

Since $x^2 + x + 1$, $x^2 - x + 1$ are irreducible with positive leading coefficients, we have $x^2 + x + 1 > 0$ and $x^2 - x + 1 > 0$ for all x. Hence they don't contribute to signs of $f(x)$. Roots of $f(x)$ are 1 and -1. They divide the real line into intervals $(-\infty, -1), (-1, 1)$ and $(1, \infty)$. The linear factor $x - 1$ changes signs at $x = 1$. The linear factor $x + 1$ changes signs at $x = -1$. Finally, don't forget the constant factor -1 which is negative for all x. Then we list signs of factors on each interval.

		-1		1	
$x-1$:	$-$		$-$		$+$
$x+1$:	$-$		$+$		$+$
-1:	$-$		$-$		$-$
$f(x)$:	$-$		$+$		$-$

Therefore, $f(x)$ is negative on $(-\infty, -1)$ and $(1, \infty)$. $f(x)$ is positive on $(-1, 1)$.

Exercise 5.1. Find the intervals on which $f(x) = 6x^3 + 13x^2 - 5x$ is positive or negative.

Exercise 5.2. Find the intervals on which $f(x) = -3x^3 + x^2 - 3x + 1$ is positive or negative.

A linear factor $ax + b$ with $a > 0$ may repeat n times. If n is even, then $(ax + b)^n > 0$ for all $x \neq -\frac{b}{a}$. Note that in this case, $(ax + b)^n$ does **NOT** change signs at $-\frac{b}{a}$. If n is odd, then

$$(ax+b)^n \begin{cases} < 0 & \text{for } x < -\frac{b}{a}; \\ > 0 & \text{for } x > -\frac{b}{a}. \end{cases}$$

An irreducible factor $ax^2 + bx + c$ with $a > 0$ is always positive. Hence $(ax^2 + bx + c)^m$ is always positive no matter m is even or odd.

Example. Find the intervals on which $f(x) = x^5 + 4x^4 + 4x^3$ is positive or negative.
Solution. By factoring out x^3 and following the binomial formula, we write

$$f(x) = x^3\left(x^2 + 4x + 4\right) = x^3(x+2)^2.$$

Roots of $f(x)$ are 0 and -2. They divide the real line into intervals

$$(-\infty, -2), \quad (-2, 0) \quad \text{and} \quad (0, \infty).$$

The factor x^3 changes signs at $x = 0$. The factor $(x+2)^2$ does **NOT** change signs at $x = -2$. And we list their signs on each interval.

	$(-\infty,-2)$	$(-2,0)$	$(0,\infty)$
x^3:	$-$	$-$	$+$
$(x+2)^2$:	$+$	$+$	$+$
$f(x)$:	$-$	$-$	$+$

Hence $f(x)$ is negative on $(-\infty, -2)$ and $(-2, 0)$. $f(x)$ is positive on $(0, \infty)$.

Example. Find the intervals on which $f(x) = (-x+4)^2(-2x+3)^3\left(x^2+x+1\right)^5$ is positive or negative.
Solution. Note that $(-x+4)^2$ and $(-2x+3)^3$ are repeated linear factors and $\left(x^2+x+1\right)^5$ is a repeated irreducible factor. By factoring out $(-1)^2(-1)^3 = -1$, we can make leading coefficients of every factors positive.

$$f(x) = -\left(x-4\right)^2\left(2x-3\right)^3\left(x^2+x+1\right)^5.$$

Roots of $f(x)$ are 4 and $3/2$. They divide the real line into intervals $\left(-\infty, \frac{3}{2}\right)$, $\left(\frac{3}{2}, 4\right)$ and $(4, \infty)$. The factor $(x-4)^2$ does **NOT** change sign at $x = 4$. The factor $(2x-3)^3$ changes signs at $x = 3/2$. $\left(x^2+x+1\right)^5$ is always positive and we could ignore it. The constant factor -1 is always negative. Now we list signs of factors on each interval.

	$(-\infty,\frac{3}{2})$	$(\frac{3}{2},4)$	$(4,\infty)$
$(x-4)^2$:	$+$	$+$	$+$
$(2x-3)^3$:	$-$	$+$	$+$
-1:	$-$	$-$	$-$
$f(x)$:	$+$	$-$	$-$

Hence $f(x)$ is positive on $\left(-\infty, \frac{3}{2}\right)$. $f(x)$ is negative on $\left(\frac{3}{2}, 4\right)$ and $(4, \infty)$.

Exercise 5.3. Find the intervals on which $f(x) = (1-x)(2-x)^2(3-x)^3(4-x)^4(5-x)^5$ is positive or negative.

Exercise 5.4. Construct a polynomial $f(x)$ with roots $x = 0, -1, -2, -3$ such that $f(x)$ is positive on $(-\infty, -3)$, $(-1, 0)$, $(0, \infty)$ and $f(x)$ is negative on $(-3, -2)$, $(-2, -1)$.

6 Graphs of Polynomials

After solving real roots of a polynomial $f(x)$ and determining intervals on which $f(x)$ is positive or negative, we may have a rough idea of the graph of the polynomial.

Since $f(\gamma) = 0$ if and only if γ is a root of f, the graph $y = f(x)$ intersects the x-axis at roots of f. Assume that $\gamma_1 < \gamma_2 < \cdots < \gamma_k$ are real roots of f. Then the sign of $f(x)$ doesn't change on each interval (γ_i, γ_{i+1}). Suppose that $f(x) > 0$ on (γ_i, γ_{i+1}). We naively suspect that $f(x)$ shall increase when x is slightly to the right of γ_i so that $f(x)$ would be positive. But to some point f must begin to decrease in order that $f(\gamma_{i+1})$ is zero again. Similarly, if $f(x) < 0$ on (γ_i, γ_{i+1}), we guess that $f(x)$ would first decreases and then increases on (γ_i, γ_{i+1}).

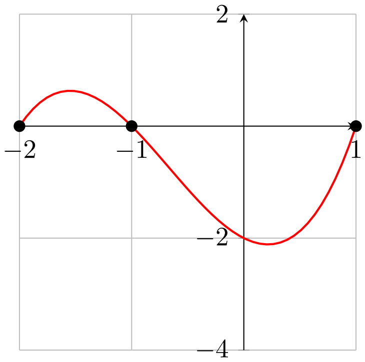

Figure 6.1. $y = x^3 + 2x^2 - x - 2$

Example. Roughly sketch the graph of $f(x) = x^3 + 2x^2 - x - 2$ for $-2 \leqslant x \leqslant 1$.
Solution. $f(x) = (x+2)(x+1)(x-1)$ and f has three real roots $-2, -1, 1$. Moreover, f is positive on the interval $(-2, -1)$ and negative on $(-1, 1)$. Therefore we guess that $f(x)$ first increases then decreases on $(-2, -1)$. On the other hand, f should first decrease then increase on $(-1, 1)$. Indeed, using a graphing calculator we can plot the graph (**Figure 6.1**) of f on $(-2, 1)$ which coincides with our intuition.

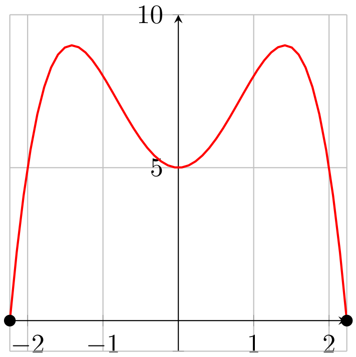

Figure 6.2. $y = -x^4 + 4x^2 + 5$

Example. Roughly sketch the graph of $f(x) = -x^4 + 4x^2 + 5$ for $-\sqrt{5} \leqslant x \leqslant \sqrt{5}$.
Solution. By cross method,

$$f(x) = -\left(x^2+1\right)\left(x^2-5\right) = -\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)\left(x^2+1\right).$$

f has two real roots $-\sqrt{5}, \sqrt{5}$. Moreover, f is positive on the interval $\left(-\sqrt{5}, \sqrt{5}\right)$. Therefore we guess that $f(x)$ first increases then decreases on $\left(-\sqrt{5}, \sqrt{5}\right)$. However, with a graphing calculator we can plot the graph (**Figure 6.2**) of f which shows that the curve $y = f(x)$ goes up and down **TWICE** on the interval out of our expectation.

From the last example, we learn that a polynomial may increases and decreases several times between two adjacent roots. In addition, we can not tell where a polynomial obtains its maximum (minimum) value and changes its trend of increasing or decreasing. Later in Calculus course, we will answer these questions.

Now, we discuss the graph of a polynomial when $|x|$ is large. Suppose that

$$f(x) = a_n x^n + a_{n-1} x^{-a} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$ and n is a positive integer. Note that

$$\frac{f(x)}{a_n x^n} = 1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \frac{a_{n-2}}{a_n}\frac{1}{x^2} + \cdots + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\frac{1}{x^n}.$$

If $|x|$ is large, then $\frac{1}{x}, \frac{1}{x^2}, \ldots, \frac{1}{x^n}$ are close to zero and $\frac{f(x)}{a_n x^n}$ is close to 1. Hence $f(x)$ behaves like $a_n x^n$ when $|x|$ is very large.

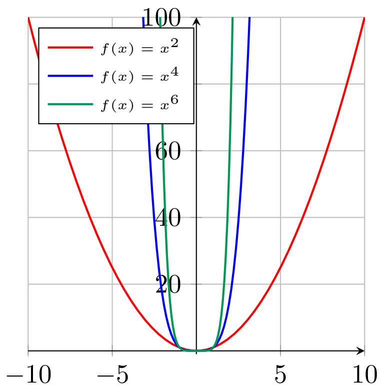

Figure 6.3. $y = x^{2n}, n = 1, 2, 3$

If $n > 1$ is an even integer, x^n is positive and becomes large without upper bounds as $|x|$ grows large. (**Figure 6.3**)

Suppose that n is an odd integer. Then for $x > 0$, $x^n > 0$ and x^n increases without upper bounds as x grows large. On the other hand, for $x < 0$, $x^n < 0$ and x^n becomes large negative without lower bounds as $|x|$ becomes large. (**Figure 6.4**)

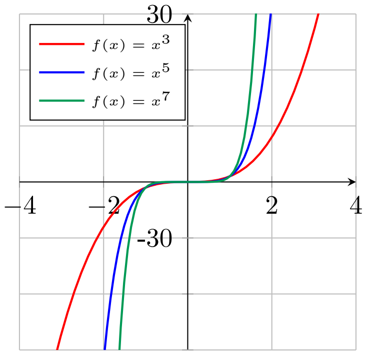

Figure 6.4. $y = x^{2n+1}, n = 1, 2, 3$

Therefore a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, with $a_n \neq 0$ and $n \geq 1$, is either large (positive) or large negative when $|x|$ is large depending on the sign of a_n and whether n is even or odd.

Example. Roughly sketch the graph of $f(x) = x^3 + 2x^2 - x - 2$ for $x \leqslant -2$ or $x > 1$.
Solution. $f(x) = (x + 2)(x + 1)(x - 1)$ and f is positive on the interval $(1, \infty)$ and negative on $(-\infty, -2)$. Moreover, when $|x|$ is large, $f(x)$ is close to x^3. Thus, we know that $f(x)$ grows large without upper bounds as x becomes large. If $x < -2$ and $|x|$ becomes large, then $f(x)$ becomes large negative without lower bounds. (**Figure 6.5**)

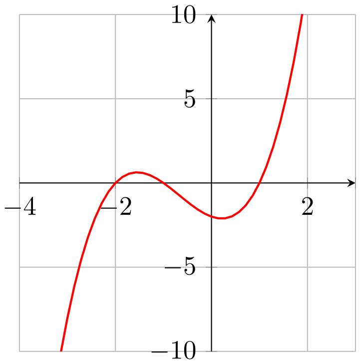

Figure 6.5. $f(x) = x^3 + 2x^2 - x - 2$

Example. Roughly sketch the graph of $f(x) = -x^4 + 4x^2 + 5$ for $x \leqslant -\sqrt{5}$ or $\sqrt{5} \leqslant x$.
Solution. By cross method,

$$f(x) = -\left(x^2 + 1\right)\left(x^2 - 5\right) = -\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)\left(x^2 + 1\right).$$

f has two real roots, $\sqrt{5}$ and $\sqrt{5}$. Moreover, f is negative on intervals $\left(-\infty, -\sqrt{5}\right)$ and $\left(\sqrt{5}, \infty\right)$. Also, $f(x)$ is close to $-x^4$. Hence, as $|x|$ grows large, $f(x)$ becomes large negative with lower bounds. (**Figure 6.6**)

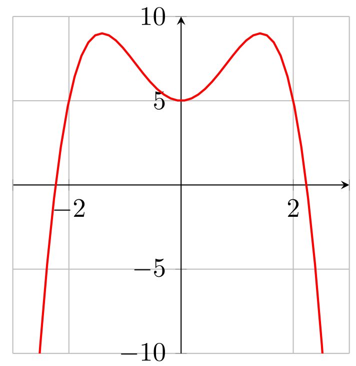

Figure 6.6. $f(x) = -x^4 + 4x^2 + 5$