Pre-Calculus

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PART

I

Trigonometry

1 Trigonometric functions - episode I

1.1 Angles

The subject matter in trigonometry (三角學) is an angle. An angle is formed when two rays are emitted from a common point. Pictorially it looks like the following.

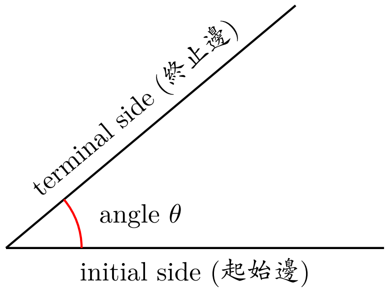


Figure 1. An angle θ

In mathematics, when we talk about angles, we usually refer to the measure (i.e. the size) of an angle. One should think of an angle as a 'oriented' measurement that we measure it from the initial side to the terminal side. Conventionally, if the angle is measured in a counter-clockwise direction, then the angle is positive. Otherwise, if the angle is measured in a clockwise direction, then the angle is negative.

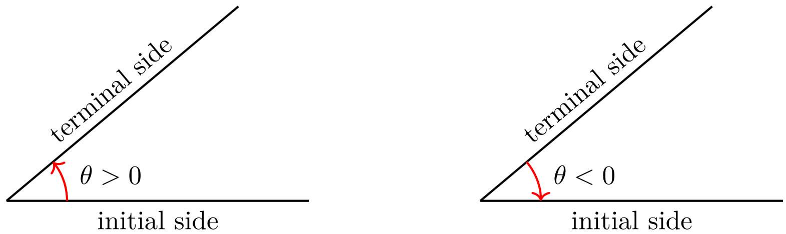


Figure 2. Orientation of an angle

Historically, mankind sets the angle around a circle (that the initial side and the terminal side coincide) to be 360° (degree) and measure the size of other angles proportionally. However, we shall see later in §2 that the degree measurement is not the most convenient one for mathematical purpose.

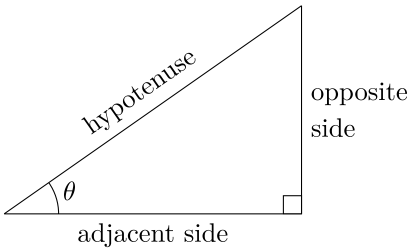
1.2 Trigonometric functions in a right-angled triangle

Trigonometric functions (三角函數) are used to convert between the lengths of the sides and the angles of a right-angled triangle.

Definition 1.1. Let θ be an acute angle inside a right-angled triangle. By labelling the sides of the triangle as on the right, define

$$\sin(\theta) = \frac{\text{opp.}}{\text{hyp.}}, \quad \cos(\theta) = \frac{\text{adj.}}{\text{hyp.}}, \quad \tan(\theta) = \frac{\text{opp.}}{\text{adj.}}$$

where we abbreviate the lengths of the adjacent side (鄰邊), the opposite side (對邊) and the hypotenuse (斜邊) by adj., opp. and hyp. respectively.



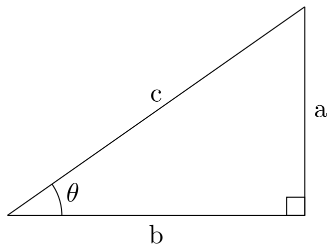
There are a few basic identities that fall out immediately from the above definition.

Theorem 1.1. The following are true.

1. If $\cos(\theta) \neq 0$, then $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.
2. (Co-angle identities)
$$\sin(90^\circ - \theta) = \cos(\theta) \text{ and } \cos(90^\circ - \theta) = \sin(\theta).$$
3. (Pythagorean identity)
$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

Remark. For a non-negative number k, we usually write $\sin^k(x)$ to denote $(\sin(x))^k$. For example, $\sin^2(x)$ means "*the square of* $\sin(x)$". Similar convention is adopted for other trigonometric functions.

Proof. In this proof, we let θ be the angle of the triangle on the right whose sides have lengths a, b and c respectively.



1. Note that the right hand side of the equality only makes sense when $\cos(\theta) \neq 0$ and in this case, we have
$$\text{RHS} = \frac{\sin(\theta)}{\cos(\theta)} = \frac{a/c}{b/c} = \frac{a}{c} \cdot \frac{c}{b} = \frac{a}{b} = \tan(\theta) = \text{LHS}.$$
2. Since the sum of angles of a triangle is 180°, the remaining angle of the triangle is $90^\circ - \theta$. By applying the definition of sine and cosine to this angle, we have
$$\sin(90^\circ - \theta) = \frac{b}{c} = \cos(\theta) \text{ and } \cos(90^\circ - \theta) = \frac{a}{c} = \sin(\theta).$$
3. By Pythagoras theorem, we have that $a^2 + b^2 = c^2$. Therefore,
$$\text{LHS} = \sin^2(\theta) + \cos^2(\theta) = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1.$$

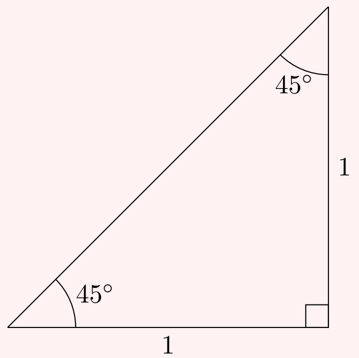
□

1.3 Special angles

While in general the exact values of trigonometric functions are uneasy to write down, there are a few 'special angles' whose values of sine and cosine can be written down very neatly.

Special angle : 45°

Consider the following triangle.



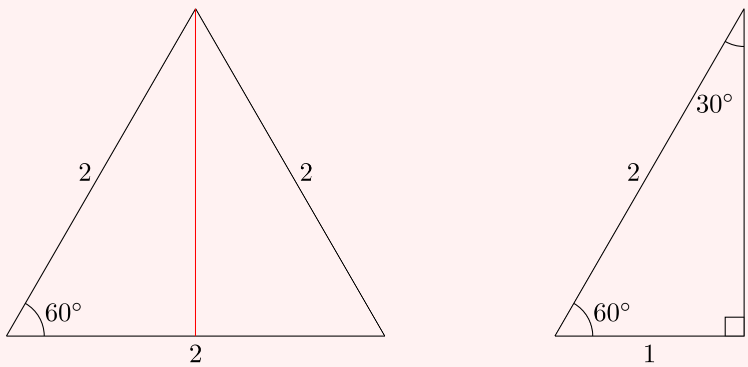
Isosceles right triangle (等腰直角三角形)

Since the sum of angle of a triangle is 180°, the remaining angle of this triangle is also 45° and hence the above is isosceles. Consequently, the length of the vertical edge of the triangle is also 1 and the Pythagoras Theorem implies that of the hypotenuse is $\sqrt{2}$. Therefore, we conclude that

$$\sin(45^\circ) = \cos(45^\circ) = \frac{1}{\sqrt{2}} \text{ and } \tan(45^\circ) = 1.$$

Special angle : 45°

Consider half of an equilateral triangle (正三角形) of length as below.



Equilateral triangle & $30 - 60 - 90$ triangle

The Pythagoras theorem implies that the length of the vertical edge of the triangle on the right equals to $\sqrt{2^2 - 1^2} = \sqrt{3}$. Therefore, we conclude that

$$\begin{aligned} &\sin(30^\circ) = \frac{1}{2}, \quad &&\cos(30^\circ) = \frac{\sqrt{3}}{2}, \quad &&\tan(30^\circ) = \frac{1}{\sqrt{3}}, \\ &\sin(60^\circ) = \frac{\sqrt{3}}{2}, \quad &&\cos(60^\circ) = \frac{1}{2}, \quad &&\tan(60^\circ) = \sqrt{3}. \end{aligned}$$

1.4 Reciprocal trigonometric functions

In calculus, one often take the reciprocal (倒數) of trigonometric functions and therefore it seems worthy to grant them with special names.

Definition 1.2. The secant, cosecant and cotangent functions are defined as

$$\sec(\theta) = \frac{1}{\cos(\theta)}, \quad \csc(\theta) = \frac{1}{\sin(\theta)}, \quad \cot(\theta) = \frac{1}{\tan(\theta)}$$

respectively.

Then we can rephrase the Pythagoras identity in Theorem 1.1 in terms of these reciprocal functions.

Theorem 1.2. The following are valid.

1. $1 + \tan^2(\theta) = \sec^2(\theta)$,
2. $1 + \cot^2(\theta) = \csc^2(\theta)$.

Proof. By the Pythagoras identity in Theorem 1.1, we have

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

Then the first claim follows from dividing both sides of the above identity by $\cos^2(\theta)$ whereas the second one follows from dividing both sides by $\sin^2(\theta)$. □

1.5 Exercise

1. Verify the identity $\dfrac{\sin(x)}{\cos^3(x)} = \tan(x)\sec^2(x)$ for all $0 < x < \dfrac{\pi}{2}$.

2. Find the relations between the integers a, b, c, d such that

$$\frac{\cos^a(x)}{\sin^b(x)} = \cot^c(x) \cdot \csc^d(x) \text{ for all } 0 < x < \frac{\pi}{2}.$$

2 Radian measure and arclength

In this section, we will introduce a different way to measure an angle. To begin with, we recall that the circumference of a circle equals to $2\pi r$. Therefore, in terms of degree measurement, the arc of a circular sector whose central angle equals to $x°$ has length

$$2\pi r \cdot \frac{x}{360} = r \cdot \frac{2\pi x}{360}.$$

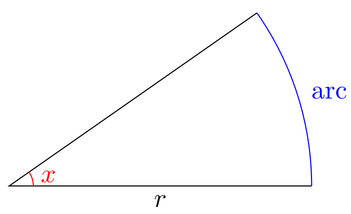


Figure 3. Arc of a circle

However, this formula is not very pretty - although it is very intuitive that the length of the arc is directly proportional to the central angle. The extra factors such as 2π and 360 have diluted such an intuition. To simplify the formula above, we are going to introduce the radian measure (弧度).

Definition 2.1. Radian measure is an alternative way to measure the size of an angle and 1 radian is defined to be equal to $\dfrac{360}{2\pi}$ degrees.

Proportionally, $x°$ equals to exactly $\dfrac{2\pi x}{360}$ radians. Therefore, we can immediately simplify the above formula for lengths of circular arcs in terms of radian measures.

Theorem 2.1. Let θ (in radians) be the central angle of an arc that belongs to a circle of radius r. The length of this arc equals to $r \cdot \theta$.

Indeed, the advantage of using the radian measures is beyond just simplifying the formula of arc-lengths. Having simplified this formula, we will, in MATH4006 Calculus 1, derive some formulas of limits (極限) and derivatives (導數) that involve trigonometric functions. And these formulas take a much prettier shape if we work in terms of radian measures rather than degrees.

For the convenience of readers, here we record the key identities and special angles discussed in §1 in terms of radian measures.

In radian measures

Identities.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta) \text{ and } \cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$
$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Special angles.

$$\sin\frac{\pi}{6} = \frac{1}{2}, \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}},$$
$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \tan\frac{\pi}{4} = 1,$$
$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{3} = \frac{1}{2}, \quad \tan\frac{\pi}{3} = \sqrt{3}.$$

3 Trigonometric functions - episode II

3.1 Unit circle

From now on, we will be using radians to measure angles. The disadvantage of Definition 1.1 is apparent - that we can only define sine and cosine of acute angles. To extend the definition to an arbitrary angle, we will employ the concept of a 'unit circle', which refers to a circle of radius 1.

Definition 3.1. Consider the Cartesian coordinate system (笛卡爾座標系統) and a unit circle centred at the origin (See Figure 4). Let θ radian be the angle oriented such that the initial side coincides with the positive x-axis. Let $\big(x(\theta), y(\theta)\big)$ be the coordinates of the point P at which the terminal side of the angle intersects with the unit circle. Cosine and sine of the angle is defined to be, respectively, the x and y-coordinates of this point. In other words,

$$\cos(\theta) = x(\theta) \text{ and } \sin(\theta) = y(\theta).$$

In the case when $\cos(\theta) \neq 0$, we define $\tan(\theta) = \dfrac{\sin\theta}{\cos\theta}$.

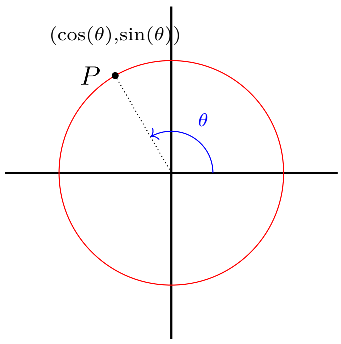


Figure 4. The unit circle

Example. Let's find out the value of $\cos\dfrac{2\pi}{3}$. To do this, we first draw the angle on a coordinate system with a unit circle (see below) and label the point of interest by P. By employing the second special triangle in §1.3, we can figure out that the coordinates of P equal to $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Therefore, by definition, we have

$$\cos\frac{2\pi}{3} = -\frac{1}{2} \text{ and } \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example. What are the values of $\sin 0$ and $\cos 0$? The angle 0 refers to the case when the initial side and the terminal side coincide. Therefore, the terminal side in this case intersects with the unit circle at $(1, 0)$. Hence, by definition,

$$\cos 0 = 1 \text{ and } \sin 0 = 0.$$

Example. Let's find out the value of $\tan\left(-\frac{\pi}{4}\right)$. First recall that a negative angle refers to an angle oriented in the clockwise direction. Having drawn the correct angle and label the point of interest by P. By employing the first special triangle in §1.3, the coordinates of P equal to $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Therefore, by definition, we have

$$\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \text{ and } \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

and hence

$$\tan\left(-\frac{\pi}{4}\right) = \frac{\sin\left(-\frac{\pi}{4}\right)}{\cos\left(-\frac{\pi}{4}\right)} = \frac{-1/\sqrt{2}}{1/\sqrt{2}} = -1.$$

3.2 Reference angle and CAST criterion

From the examples in §3.1, you probably realise that computing the values of sine and cosine really blow down to, up to a sign, computing those of a suitably chosen acute angle. For example to compute sine or cosine of $\frac{2\pi}{3}$, what we really have to do is to compute those of $\frac{\pi}{3}$ and decide their positiveness. The angle $\frac{\pi}{3}$ is called a *reference angle* (參考角) of $\frac{2\pi}{3}$ and in general it is defined as follows.

Definition 3.2. The reference angle $\overline{\theta}$ of an angle θ is the acute angle formed by the terminal side and the x-axis.

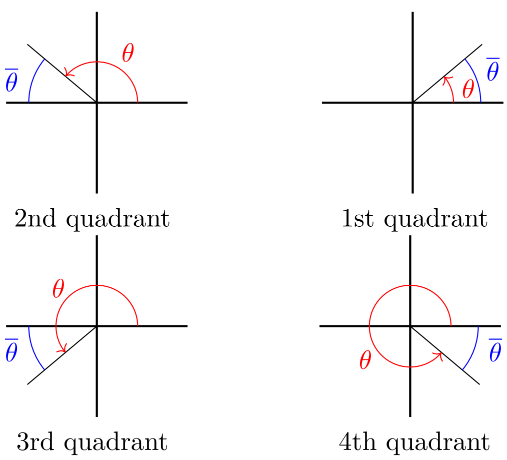


Figure 5. Reference angle in each quadrant (象限)

Example. The reference angle of $\frac{5\pi}{4}$ is $\frac{\pi}{4}$. The reference angle of $\frac{2\pi}{3}$ is $\frac{\pi}{3}$. The reference angle of any acute angle is itself.

By definition, cosine and sine is defined to be the x and y-coordinates of the intersection point of the terminal side with the unit circle. Therefore, by looking at which quadrant the terminal side belongs to, we can decide the sign (i.e. positiveness) of sine and cosine (and hence tangent).

Example. If the terminal side lies in the second quadrant (for example, when $\frac{\pi}{2} < \theta < \pi$), the x-coordinate of the point of interest is negative whereas its y-coordinate is positive. Therefore, we have

$$\cos(\theta) < 0 \text{ and } \sin(\theta) > 0 \text{ in this case.}$$

Of course, this implies that $\tan(\theta) < 0$ in this case.

If we conduct an identical analysis to each quadrant, then we can summarise of the signs of the trigonometric functions as follows.

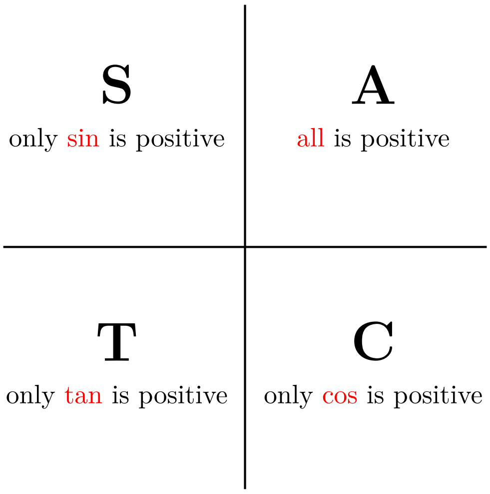


Figure 6. The CAST criterion

The word 'CAST' is useful for us to remember which trigonometric function is positive in each quadrant (here C, A, S, T refers to cosine, all, sine and tangent respectively). This is usually referred as the *CAST criterion*.

The upshot of the discussion here is to provide an alternative (and perhaps faster) way to determine the value of a trigonometric function.

Theorem 3.1. Let θ be an angle and $\overline{\theta}$ be its reference angle.

Then we have $\begin{cases} \sin(\theta) &= \pm\sin(\overline{\theta}) \\ \cos(\theta) &= \pm\cos(\overline{\theta}) \\ \tan(\theta) &= \pm\tan(\overline{\theta}) \end{cases}$ where the signs in each case can be determined by using the 'CAST criterion'.

Example. Let's compute the value of $\tan\frac{2\pi}{3}$. To use Theorem 3.1, we have to determine

- the relevant reference angle,

- the sign by CAST criterion.

The reference angle of $\frac{2\pi}{3}$ is $\frac{\pi}{3}$ and tangent is negative in the second quadrant. Therefore, Theorem 3.1 implies that

$$\tan\frac{2\pi}{3} = -\tan\frac{\pi}{3} = -\sqrt{3}.$$

Example. Let θ be an acute angle. We can simplify $\sin(\pi+\theta)$ as follows. Note that in this case

- the reference angle of $\pi+\theta$ is θ,
- sine is negative in the third quadrant.

Therefore, Theorem 3.1 implies that $\sin(\pi+\theta) = -\sin(\theta)$.

Remark. In the derivation of the above identity, we have assumed that (crucially) θ is acute. It turns out that the above identity is always valid for any θ - we will prove this in §5.

Example. Let θ be an acute angle. Let's simplify the expression $\sin\left(\frac{\pi}{2}+\theta\right)$. Note that in this case

- the reference angle of $\frac{\pi}{2}+\theta$ is $\frac{\pi}{2}-\theta$,
- sine is positive in the second quadrant.

Therefore, Theorem 3.1 implies that $\sin\left(\frac{\pi}{2}+\theta\right) = \sin\left(\frac{\pi}{2}-\theta\right) = \cos(\theta)$.

Remark. Again, this identity turns out to be valid for an arbitrary angle θ - see §5.

3.3 Exercise

Find the exact values of the following (without a calculator).

1. Find the exact values of the following (without a calculator).

$$\cos\frac{3\pi}{4}, \quad \tan\frac{17\pi}{6}, \quad \csc\frac{5\pi}{3}, \quad \sec\left(-\frac{4\pi}{3}\right).$$

2. Find all solutions to the equation $\sin(\theta) = 0$.
3. Suppose θ satisfies $\sin(\theta) = -\frac{1}{3}$.
   (a) What are the possible values of $\cos(\theta)$ and $\tan(\theta)$?
   (b) What are the possible values of $\sin(-\theta)$?
   (c) What are the possible quadrants that θ lie in ? To each of the quadrant that you specified, write down the exact values of $\cos(\theta)$.

4 Graphs of trigonometric functions

4.1 Graph of sin, cos and tan

As discussed in Chapter 1, an important way to understand a function is via studying its graphs. In this section, we will sketch the graphs of the trigonometric functions that we defined in §4.

Sine.

Recall that sine function $\sin(\theta)$ is defined to be the y-coordinate of the point of interest (see Definition 3.1). There are two key observations.

- As θ increases from 0 to $\pi/2$, this coordinate first increases from 0 to 1. Then from $\theta = \pi/2$ to $\theta = 3\pi/2$, it decreases from 1 to -1. Finally from $\theta = 3\pi/2$ to $\theta = 2\pi$, it increases again, from -1 to 0.
- The above propagation will repeat every time when θ has traversed for 2π radians.

These two observations will become visible if we input the function into Desmos.

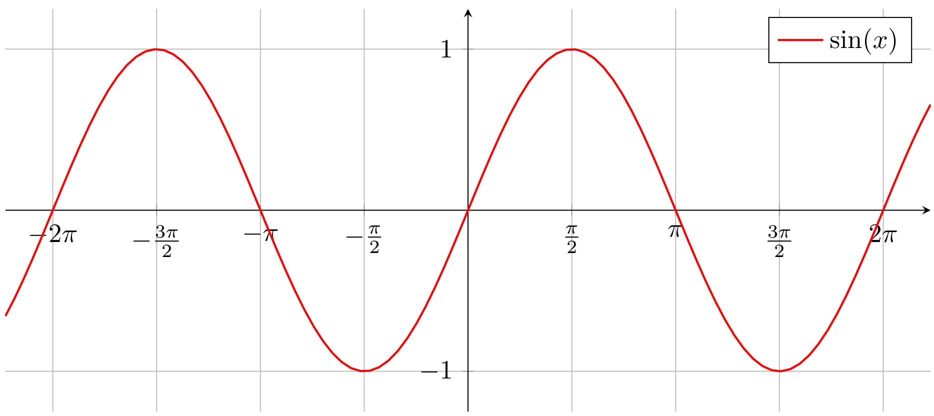


Figure 7. Sketch of sine curve

From the above, we see that the graph of $y = \sin(\theta)$ takes the shape of a 'wave'. In particular,

- the graph of $y = \sin(\theta)$ is squeezed between the horizontal lines $y = 1$ and $y = -1$. We say that $\sin(\theta)$ is *bounded from above* (向上有界) by 1 and *bounded from below* (向下有界) by -1.
- the shape of the graph will repeat itself in every interval of 2π radians. We say that $\sin(\theta)$ is periodic and the number 2π is called its *period* (週期).

More generally, we have the following.

Theorem 4.1. Let a, b be two non-zero real numbers.
Then the function $f(\theta) = a \cdot \sin(b \cdot \theta)$ is

- bounded from above by $|a|$ and bounded from below by $-|a|$,
- periodic with period $\dfrac{2\pi}{|b|}$.

Cosine.

While we can repeat an identical analysis to sketch the graph of $y = \cos(\theta)$, we can take a simpler way : observe from Example 3.2 that

$$\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right).$$

Therefore it suffices for us to sketch the graph of $y = \sin\left(\theta + \frac{\pi}{2}\right)$ which, by using the technique that we have seen in Chapter 1, is simply the translation (平移) of the graph of $y = \sin(\theta)$ to the left by $\frac{\pi}{2}$ radians. From this we obtain the graph of $y = \cos(\theta)$.

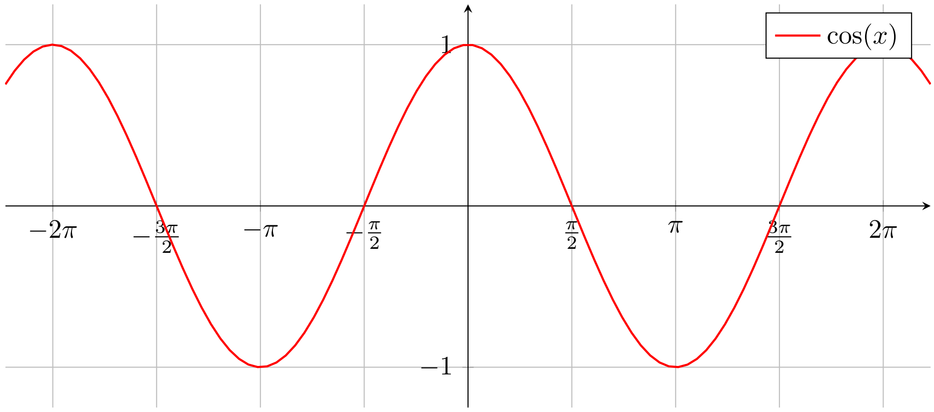


Figure 8. Sketch of cosine curve

In particular, since the graph of cosine is simply a translation of that of sine, an analogous result described in Theorem 4.2 is also true for cosine.

Theorem 4.2. Let a, b be two non-zero real numbers.
Then the function $f(\theta) = a \cdot \cos(b \cdot \theta)$ is

- bounded from above by $|a|$ and bounded from below by $-|a|$,
- periodic with period $\frac{2\pi}{|b|}$.

Tangent.

The story for tangent is a little different. To begin with, we let θ be an acute angle. Then observe that

- by Theorem 3.1, we have that $\tan(-\theta) = -\tan(\theta)$. Therefore, the graph of the function $y = \tan(\theta)$ has a rotational symmetry about the origin : to be specific, the graph is unchanged after rotation by π about the origin;
- by Theorem 3.1 again, we have that $\tan(\pi + \theta) = \tan(\theta)$. Therefore, $y = \tan(\theta)$ is periodic with period π.

Let's make these observations become visible with the help of Desmos.

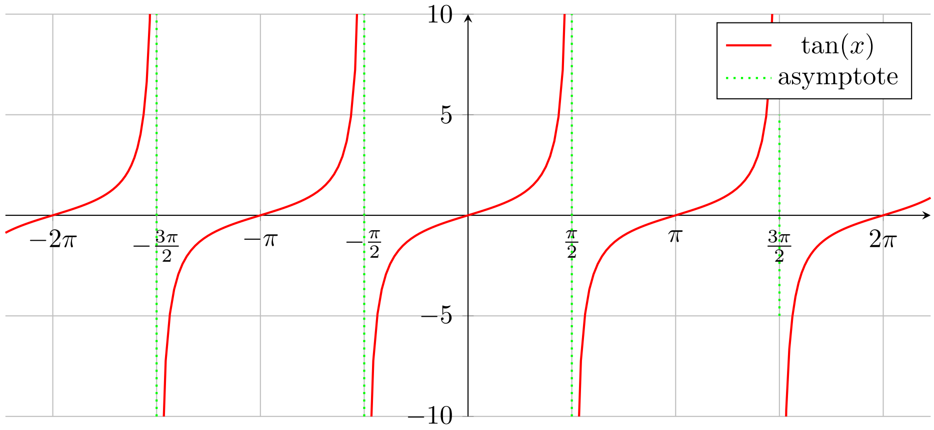


Figure 9. Sketch of tangent curve

The key differences of the graph of tangent from that of sine or cosine are :

1. $\tan(\theta)$ is 'unbounded' in the sense that it can get arbitrary large positively when θ is getting close to $\pi/2$ from the left, and negatively when θ is getting close to $-\pi/2$ from the right.
2. $\tan(\theta)$ is undefined at $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ ... because at those points $\cos(\theta) = 0$.
3. $\tan(\theta)$ has a shorter period of length π.

4.2 Exercise

1. Sketch the graphs of the following functions.
   (a) $f(x) = 3\sin(4x)$.
   (b) $f(x) = -4\cos\left(3x - \frac{\pi}{4}\right)$
   (c) $f(x) = \tan(2x+1) - 3$
2. Use Desmos to sketch the graphs of $y = \sec(x)$, $y = \csc(x)$ and $y = \cot(x)$. Are these functions (i) bounded from above and/or from below (ii) periodic ?

5 Compound angle formulas

It should be clear to the readers that $\sin(a+b) \neq \sin(a) + \sin(b)$ (try $a = b = \frac{\pi}{6}$ to see it). Nevertheless, we are going to derive a formula that expresses $\sin(a+b)$ in terms of just trigonometric functions in a or b (analogously for cosine and tangent).

Theorem 5.1. [Compound angle formulas] Let a and b be two real numbers.

1. $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$.
2. $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$.
3. $\tan(a \pm b) = \dfrac{\tan(a) \pm \tan(b)}{1 \mp \tan(a)\tan(b)}$.

Proof. We will give a geometric proof to the above identities in the Appendix of this section. We would, nevertheless, recommend the readers to learn these identities by heart. □

We will give a few applications of the compound angle formulas. First of all, we can use them to compute the exact values of trigonometric functions for a wider range of angles.

Example. Using Theorem 5.1, we can compute the value of $\sin \frac{\pi}{12}$. To do this, we first observe that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ (i.e. $15° = 45° - 30°$).

$$\begin{aligned} \sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}. \end{aligned}$$

As promised we will give a proof to the two identities in §3.2 in full generality.

Example. Let's prove that $\sin(\pi + \theta) = -\sin(\theta)$ for all θ (not necessarily when θ is acute). To do this, we are going to spell out the LHS by the compound angle formula.

$$\begin{aligned} \text{LHS} = \sin(\pi + \theta) &= \sin(\pi)\cos(\theta) + \cos(\pi)\sin(\theta) \\ &= 0 \cdot \cos(\theta) + (-1) \cdot \sin(\theta) \\ &= -\sin(\theta) = \text{RHS}. \end{aligned}$$

Example. Let's prove that $\sin\left(\frac{\pi}{2} + \theta\right) = \cos(\theta)$ for all θ (not necessarily when θ is acute).

$$\begin{aligned} \text{LHS} = \sin\left(\frac{\pi}{2} + \theta\right) &= \sin \frac{\pi}{2} \cos(\theta) + \cos \frac{\pi}{2} \sin(\theta) \\ &= 1 \cdot \cos(\theta) + 0 \cdot \sin(\theta) \\ &= \cos(\theta) = \text{RHS}. \end{aligned}$$

5.1 Double angle formulas

Here we record a special case of Theorem 5.1 that evaluates trigonometric functions at 2θ. These are known as the double angle formulas.

Theorem 5.2 (Double angle formulas). Let θ be a real number. Then we have

1. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
2. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
3. $\tan(2\theta) = \dfrac{2\tan(\theta)}{1 - \tan^2(\theta)}$

Proof. These are derived by putting $a = b = \theta$ into the formulas for $\sin(a + b)$, $\cos(a + b)$ and $\tan(a + b)$ respectively in Theorem 5.1. □

Remark. There are alternative ways in writing down the double angle formula for cosine. Since we know that $\sin^2(\theta) + \cos^2(\theta) = 1$, we can rewrite

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \begin{cases} 1 - 2\sin^2(\theta) \\ 2\cos^2(\theta) - 1 \end{cases}.$$

All versions of double angle formula for cosine are used in various contexts.

Example. Here we give the an alternative way to compute the exact value of $\sin \frac{\pi}{12}$. Using the double angle formula for cosine, we have

$$\cos\left(2 \cdot \frac{\pi}{12}\right) = 1 - 2\sin^2\left(\frac{\pi}{12}\right).$$

Since $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, rearranging yields

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{2 - \sqrt{3}}{4} = \frac{4 - 2\sqrt{3}}{8}.$$

By the CAST criterion, we know that $\sin \frac{\pi}{12}$ is positive. Combining this with the fact that $4 - 2\sqrt{3} = (\sqrt{3} - 1)^2$, we conclude that

$$\sin \frac{\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

5.2 Product-to-sum formulas

Another useful corollary to the compound angle formulas are the product-to-sum formulas, which, as the name suggests, relate products of trigonometric functions with their sums.

Theorem 5.3. Let a and b be two real numbers. Then

1. $\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$,
2. $\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$,
3. $\sin(a)\sin(b) = -\frac{1}{2}(\cos(a+b) - \cos(a-b))$.

Proof. We will prove the first identity and leave the proofs of the other two as an exercise to readers. According to Theorem 5.1, we have that

$$\begin{cases} \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\ \sin(a-b) &= \sin(a)\cos(b) - \cos(a)\sin(b) \end{cases}.$$

If we add these two identities, then we have $\sin(a+b) + \sin(a-b) = 2\sin(a)\cos(b)$ and hence

$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b)) \text{ as claimed.}$$

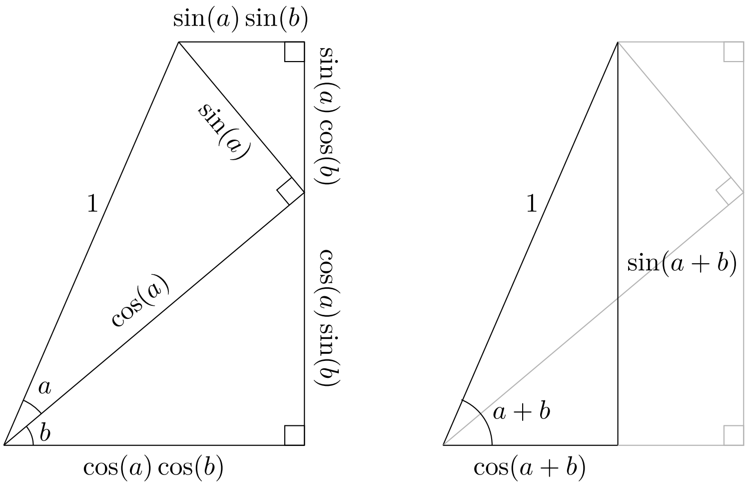
□

Appendix. Proof of the compound angle formulas

Here we will give a proof to the identities

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \text{ and } \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

Using Theorem 3.1, it suffices to prove the identity in the case when all of a and b, $0 < a + b < \frac{\pi}{2}$ are acute. The proof relies on fitting in a right-angled triangle with an acute angle a into a trapezium as shown in the figure on the left below.



Then we write down the lengths of all sides in the diagram in terms of trigonometric functions in a and b (also shown in the above figure). Both identities follow by comparing the lengths of the sub-triangle with an acute angle $a + b$ (see the figure on the right).

5.3 Exercise

1. Find the exact values of $\sin(75°)$ and $\cot\left(\frac{11\pi}{12}\right)$.

2. Prove that for any θ, we have $\sin(2\pi - \theta) = -\sin(\theta)$ and $\cos(2\pi - \theta) = \cos(\theta)$.

3. Derive the triple angle formula that

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta \text{ and } \cos(3\theta) = 4\cos^3\theta - 3\cos\theta.$$

4. Find the exact value of $\sin(52.5°)\cos(7.5°)$.

5. Complete the proof of Theorem 5.3.

6. Using the product-to-sum formulas, derive the sum-to-product formula that

$$\cos(u) + \cos(v) = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right).$$

6 Applications of trigonometry

In this section, we will give a few simple applications of trigonometry in the geometry of triangles. To be specific, we will derive a formula for the area of a triangle in terms of its sides and an angle and also prove the cosine law which can be regarded as an extension of the Pythagoras Theorem to general triangles.

6.1 Area of a triangle

Since trigonometric functions can be used to relate sides and angles of a triangle, we can rewrite the formula for the area of a triangle in terms of a relevant trigonometric function as follows.

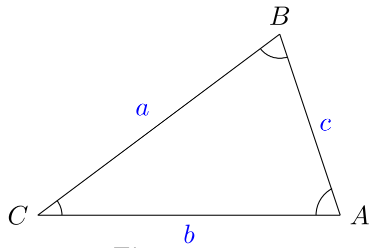
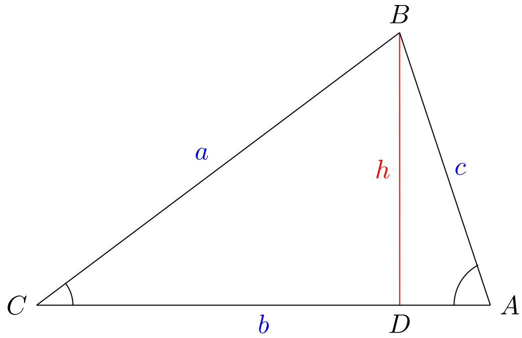


Figure 10.

Theorem 6.1. In a triangle labelled as in Figure 10, we have that

$$\text{Area of the triangle} = \frac{1}{2} \cdot ab \cdot \sin C.$$

Proof. We drop a perpendicular from B to the edge AC (as shown below).



Consider the right-angled triangle BCD, we have that $\sin C = \dfrac{h}{a}$ and therefore $h = a \sin C$. Now it suffices to note that

$$\text{Area of the triangle} = \frac{1}{2} \cdot (\text{Base}) \cdot (\text{Height}) = \frac{1}{2} \cdot b \cdot (a \sin C).$$

This completes the proof. □

6.2 Cosine law

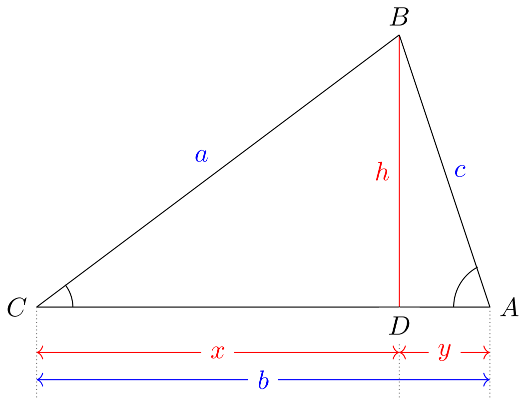
By the celebrated Pythagoras Theorem, we have '$c^2 = a^2 + b^2$' in a right-angled triangle. Certainly this equality is invalid for general triangles so one may wonder if the Pythagoras Theorem can be generalised in certain way - this has been addressed by the cosine law below.

Theorem 6.2 (Cosine law)**.** In a triangle labelled as in Figure 10.1, we have that

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Remark. In the special case when $C = \dfrac{\pi}{2}$, we have that $c^2 = a^2 + b^2$ which is essentially the statement of the Pythagoras Theorem. In this way, one can regard the cosine law as an extension of the Pythagoras Theorem to general triangles.

Proof. We drop a perpendicular from B to the edge AC (as shown below).



Consider the right-angled triangle BCD, we have that

$$\sin C = \frac{h}{a} \text{ and } \cos C = \frac{x}{a}.$$

Therefore, we have $h = a \sin C$ and $x = a \cos C$.
Then we apply the Pythagoras Theorem to the other right-angled triangle ABD and use the fact that $x + y = b$, we have

$$h^2 + (b - x)^2 = c^2.$$

Therefore, combining these we have

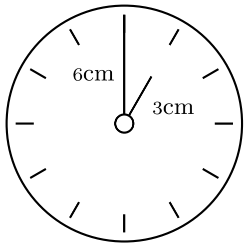
$$(a \sin C)^2 + (b - a \cos C)^2 = c^2.$$

By spelling out the LHS of the above equality, we have

$$\begin{aligned} c^2 &= a^2 \sin^2 C + (b^2 - 2ab \cos C + a^2 \cos^2 C) \\ &= a^2 (\sin^2 C + \cos^2 C) + b^2 - 2ab \cos C \\ &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

This completes the proof. □

Example. The hour arm and minute arm of a clock has lengths 6cm and 3cm respectively. Find the distance between the tips of the arms at 1 O'clock.



Solution. Let d be the distance between the tips of the arms. At 1 O'clock, the angle between the arms equals to $30° = \frac{\pi}{6}$. Therefore, the cosine law implies that

$$\begin{aligned} d^2 &= 3^2 + 6^2 - 2(3)(6)\cos\frac{\pi}{6} \\ &= 45 - 18\sqrt{3} \end{aligned}$$

Hence, $d = \sqrt{45 - 18\sqrt{3}}$.

6.3 Exercise

1. The lengths of the three sides of a triangle are $4, 5$ and 6 respectively. Find the angle opposite to the side of length 5 and find the area of the triangle.
2. Suppose you are walking towards a mountain one day. Initially you have to look up at an angle of $30°$ to see the top of the mountain. After walking for 5 km towards the mountain, you now look up at an angle of $45°$ to see the top of the mountain. How tall is the mountain?

7 Solving equations in trigonometric functions

In this section, we will discuss how to solve simple equations in trigonometric functions. The key to solve any such equation is to

- determine the relevant *reference angle*,
- determine *which quadrant(s)* that the solutions are expected to belong to.

We will demonstrate these technique with the following three examples.

Example. Solve $\sin x = \frac{1}{2}$.

Solution.

- Firstly we find the relevant reference angle : an acute angle $\overline{x}$ such that $\sin \overline{x} = \frac{1}{2}$. By using the table of special angles in §2, we have that $\overline{x} = \frac{\pi}{6}$.
- Since we are looking for x such that $\sin(x) = \frac{1}{2}$ which is positive, any such x should lie in either first or second quadrants by the CAST criterion.

Combining these we have that $x = \frac{\pi}{6}$, $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ are solutions. However, taken into account the fact that sine is periodic with period 2π, we deduce that the general solutions are

$$x = \frac{\pi}{6} + 2n\pi \text{ or } \frac{5\pi}{6} + 2n\pi$$

where n is an arbitrary integer.

Example. Solve $\tan x = -1$.

Solution. Note the tangent is a periodic function of period π. It suffices for us to solve the equation for x in the first or second quadrant.

- Firstly we find the relevant reference angle : an acute angle $\overline{x}$ such that $\tan \overline{x} = 1$. By using the table of special angles in §2, we have that $\overline{x} = \frac{\pi}{4}$.
- Since we are looking for x such that $\tan(x) = -1$ which is negative, any such x should lie in the second quadrant.

Therefore, we have that $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ is a solution.
Combining with the periodicity of tangent, the general solutions are given by

$$x = \frac{3\pi}{4} + n\pi$$

where n is an arbitrary integer.

We finish with a slightly complicated example.

Example. Solve $\sin(2x) = \cos(x)$.

Solution. The idea is to first convert $\sin(2x)$ in terms of $\sin(x)$ and $\cos(x)$ by the double angle formula. Using this, we have

$$2\sin(x)\cos(x) = \cos(x).$$

Rearranging yields $\cos(x)(2\sin(x) - 1) = 0$. Since we have a product of two terms equal to zero, at least one of them equals to zero. Hence we have

$$\cos(x) = 0 \text{ or } \sin(x) = \frac{1}{2}.$$

Then we solve these two equations independently.

- To solve $\cos(x) = 0$, we look at the graph of $y = \cos(x)$ and realise that it cuts the x-axis at
$$x = \frac{\pi}{2} + 2n\pi \text{ or } \frac{3\pi}{2} + 2n\pi$$
where n is an arbitrary integer.
- We have solved $\sin(x) = \frac{1}{2}$ in an earlier example : we had
$$x = \frac{\pi}{6} + 2n\pi \text{ or } \frac{5\pi}{6} + 2n\pi$$
where n is an arbitrary integer.

Hence, the solutions of the given equation are

$$x = \frac{\pi}{2} + 2n\pi \text{ or } \frac{3\pi}{2} + 2n\pi \text{ or } \frac{\pi}{6} + 2n\pi \text{ or } \frac{5\pi}{6} + 2n\pi$$

where n is an arbitrary integer.

7.1 Exercise

Solve the following equations.

1. $\sin(x) = -\frac{\sqrt{3}}{2}$.
2. $\sec(x) = 2$.
3. $\tan(3x) = 1$.

4. $\cos(2x) = -\cos(x)$.

5. $\sec^2(x) = \tan(x) - 1$.