98學年度第1學期 微積分乙統一教學期末考解答

1. (10%) 設 $f(x)=\displaystyle\int_x^{\sin x}\frac{dt}{1+t^7}$，求 $f'(x)$。(提示: 不要算 $\displaystyle\int\frac{dt}{1+t^7}$。)

   Sol:

   $f(x)=\displaystyle\int_x^{\sin x}\frac{1}{1+t^7}dt=\int_a^{\sin x}\frac{1}{1+t^7}dt-\int_a^{x}\frac{1}{1+t^7}\,dt$

$$
\begin{aligned}
f'(x)&=\frac{d}{dx}\int_a^{\sin x}\frac{1}{1+t^7}dt-\frac{d}{dx}\int_a^{x}\frac{1}{1+t^7}\,dt\\
&=\frac{1}{1+\sin^7 x}(\sin x)'-\frac{1}{1+x^7}(x)'\\
&=\frac{\cos x}{1+\sin^7 x}-\frac{1}{1+x^7}
\end{aligned}
$$

2. (10%) 求 $\displaystyle\int_0^{\frac{\pi}{3}}x\sec^2 x\,dx$。

   Sol:

$$
\begin{aligned}
\int_0^{\frac{\pi}{3}}x\sec^2 x\,dx&=\int_0^{\frac{\pi}{3}}x\,d\tan x\\
&=x\tan x\Big|_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}\tan x\,dx\\
&=\frac{\pi}{3}\tan\frac{\pi}{3}+\int_0^{\frac{\pi}{3}}\frac{1}{\cos x}\,d\cos x\\
&=\frac{\pi}{3}\sqrt{3}+\log(\cos x)\Big|_0^{\frac{\pi}{3}}\\
&=\frac{\pi}{\sqrt{3}}-\log 2
\end{aligned}
$$

3. (10%) 求 $\displaystyle\int\frac{x^2+1}{(x+1)(x-1)^3}\,dx$。

   Sol:

   The form of the partial fraction decomposition is
   $\dfrac{1+x^2}{(x+1)(x-1)^3}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}+\dfrac{D}{(x-1)^3}.$
   Multiplying by $(x+1)(x-1)^3$, we have $x^2+1=(A+B)x^3+(-3A-B+C)x^2+(3A-B+D)x+(-A+B-C+D)$.

Equating coefficients and solving, we obtain $A = \dfrac{-1}{4}$, $B = \dfrac{1}{4}$, $C = \dfrac{1}{2}$, and $D = 1$, so

$$\begin{aligned}
\int \frac{1+x^2}{(x+1)(x-1)^3}\,dx &= \int \left[\frac{\frac{-1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1} + \frac{\frac{1}{2}}{(x-1)^2} + \frac{1}{(x-1)^3}\right] dx \\
&= \frac{-1}{4}\log(x+1) + \frac{1}{4}\log(x-1) - \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{(x-1)^2} + C.
\end{aligned}$$

4. (10%) 求 $\displaystyle\int \frac{e^x\,dx}{e^{2x}+2e^x+2}$ 。

Sol:

Let $u = e^x \Rightarrow du = e^x\,dx$

Then

$$\begin{aligned}
\int \frac{e^x}{e^{2x}+2e^x+2}\,dx &= \int \frac{1}{u^2+2u+2}\,du \\
&= \int \frac{1}{(u+1)^2+1}\,d(u+1) \\
&= \tan^{-1}(u+1) + C \\
&= \tan^{-1}(e^x+1) + C
\end{aligned}$$

5. (10%) 由 $y = x^2+1$ 及 $y = x+1$ 所包圍之區域，繞 y 軸旋轉。求所得旋轉體之體積。

Sol:

$x^2+1 = x+1 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$

$$\begin{aligned}
V &= \int_0^1 2\pi x[(x+1)-(x^2+1)]\,dx \\
&= \int_0^1 2\pi(x^2-x^3)\,dx \\
&= 2\pi(\frac{1}{3}x^3 - \frac{1}{4}x^4)\Big|_0^1 \\
&= \frac{\pi}{6}
\end{aligned}$$

6. (10%) 介於 $y = \sec x$, $y = \tan x$ 之間，由 $x = 0$ 到 $x = \dfrac{\pi}{4}$ 之區域，繞 x 軸旋轉。求所得旋轉體之體積。

Sol:

$$\begin{aligned}
&\int_0^{\pi/4} \pi(\sec^2 x - \tan^2 x)\, dx \\
&= \pi \int_0^{\pi/4} 1\, dx \\
&= \frac{\pi^2}{4}
\end{aligned}$$

7. (10%) 求 $f(x) = \sqrt{1-x}$ 在 $x = 0$ 之泰勒展式，寫出一般項。

Sol:

Use the following formula:

$(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + \cdots$

Take $\alpha \to \frac{1}{2}$, $x \to -x$, then we have:

$$\begin{aligned}
(1-x)^{\frac{1}{2}} &= \binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{1}(-x) + \binom{\frac{1}{2}}{2}(-x)^2 + \cdots + \binom{\frac{1}{2}}{n}(-x)^n + \cdots \\
&= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}(-x)^k \\
&= \sum_{k=0}^{\infty} (-1)^k \binom{\frac{1}{2}}{k} x^k \\
&= 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k}(-x)^k
\end{aligned}$$

Note that, for $k \neq 0$:

$$\begin{aligned}
\binom{\frac{1}{2}}{k} &= \frac{(\frac{1}{2})(\frac{1}{2}-1)\cdots[\frac{1}{2}-(k-1)]}{k!} \\
&= (-1)^{k-1}\frac{(1)(3)\cdots(2k-3)}{2^k k!} \\
&= (-1)^{k-1}\frac{(1)(3)\cdots(2k-3)}{2^k k!} \\
&= (-1)^{k-1}\frac{(2k-2)!}{2^{2k-1}k!(k-1)!}
\end{aligned}$$

For $k = 0$, $\binom{\frac{1}{2}}{0} = 1$.

Finally we get the general n-th term($n > 1$) , which is

$$-\frac{(2n-4)!}{2^{2n-3}(n-1)!(n-2)!}x^{n-1}$$

8. (10%) 求 $f(x) = \tan^{-1} x$ 在 $x = 0$ 之泰勒展式，寫出一般項。並求出 $f^{(19)}(0)$ 及 $f^{(20)}(0)$。

Sol:

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} \Rightarrow \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

$\frac{f^{(19)}(0)}{19!} = -\frac{1}{19} \Rightarrow f^{(19)}(0) = -18!$

$f^{(20)}(0) = 0$

9. (10%) 求 $\lim_{x\to 0} \frac{7^x - 1}{2^x - 1}$。

Sol:

$$
\begin{aligned}
&\lim_{x\to 0} \frac{7^x - 1}{2^x - 1} \qquad (\tfrac{0}{0} \text{ form, by using L'Hôpital Theorem})\\
&= \lim_{x\to 0} \frac{7^x \ln 7}{2^x \ln 2}\\
&= \frac{\ln 7}{\ln 2}
\end{aligned}
$$

10. (10%) 求曲面 $z = e^{-xy} + y$ 在點 $(0, 1, 2)$ 之切平面方程式。

Sol:

Let $f(x, y) = e^{-xy} + y$

$$
\begin{aligned}
\frac{\partial f(x,y)}{\partial x} &= e^{-xy}(-y) + 0\\
\frac{\partial f(x,y)}{\partial y} &= e^{-xy}(-x) + 1\\
\frac{\partial f(x,y)}{\partial x}\Big|_{(0,1)} &= e^{-0}(-1) + 0 = -1\\
\frac{\partial f(x,y)}{\partial y}\Big|_{(0,1)} &= e^{-0}(-0) + 1 = 1
\end{aligned}
$$

The tangent plane at $(0, 1, 2)$ is ($f(0, 1) = 2$, the point is in the surface.)

$$
z = f(0,1) + (\frac{\partial f(x,y)}{\partial x}\Big|_{(0,1)})(x - 0) + (\frac{\partial f(x,y)}{\partial y})\Big|_{(0,1)}(y - 1)
$$

That is $z = -x + y + 1$.