991微乙 01-05班期中考解答和評分標準

1. (10%) 求 (a) $\lim_{x\to 1}\dfrac{\sqrt{x^3-1+x^2}-x}{x-1}$。(5%)　(b) $\lim_{n\to\infty}\dfrac{\sqrt{4n^4+3n+1}}{\sqrt[3]{8n^6+3n^5-2}}$。(5%)

Sol:

(a)

$$\begin{aligned}
\lim_{x\to 1}\frac{\sqrt{x^3-1+x^2}-x}{x-1} &= \lim_{x\to 1}\frac{x^3-1+x^2-x^2}{(x-1)(\sqrt{x^3-1+x^2}+x)}\\
&= \lim_{x\to 1}\frac{x^3-1}{(x-1)(\sqrt{x^3-1+x^2}+x)}\\
&= \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)(\sqrt{x^3-1+x^2}+x)}\\
&= \lim_{x\to 1}\frac{x^2+x+1}{\sqrt{x^3-1+x^2}+x}\\
&= \frac{1+1+1}{\sqrt{1-1+1}+1}\\
&= \frac{3}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sqrt{4n^4+3n+1}}{\sqrt[3]{8n^6+3n^5-2}} &= \lim_{n\to\infty}\frac{\sqrt{4n^4+3n+1}\times\frac{1}{n^2}}{\sqrt[3]{8n^6+3n^5-2}\times\frac{1}{n^2}}\\
&= \lim_{n\to\infty}\frac{\sqrt{4+\frac{3}{n^3}+\frac{1}{n^4}}}{\sqrt[3]{8+\frac{3}{n}-\frac{2}{n^6}}}\\
&= \frac{\sqrt{4}}{\sqrt[3]{8}}\\
&= 1
\end{aligned}$$

評分標準:

1. 使用羅必達者需先證明
2. 沒寫 limit 者扣 3分
3. (b) 使用估計者 0分

2. (15%) 給定 $x^3+xy+2y^3=4$

(a) 求過點 $P=(1,1)$ 之切線方程式。(10%)

(b) 求在點 $P=(1,1)$ 之 $\dfrac{d^2y}{dx^2}$。(5%)

Sol:

(a) $\dfrac{d}{dx}(x^3+xy+2y^3) = 3x^2 + y + xy' + 6y^2y' = 0$ —— (1)

$\Rightarrow\ 3+1+y'+6y'=0\ \Rightarrow\ y' = \dfrac{-4}{7}$

Therefore the equation of tangent line is:

$$\frac{y-1}{x-1} = \frac{-4}{7}$$

(b) $\dfrac{d}{dx}(\dfrac{d}{dx}) = 6x + y' + y' + xy'' + 12y(y')^2 + 6y^2y'' = 0$ —— (2)

$\Rightarrow\ 6 - \dfrac{8}{7} + y'' + 12\dfrac{16}{49} + 6y''\ \Rightarrow\ y'' = \dfrac{-430}{343}$

評分標準：

如果 (1) 錯了, 但是後面的邏輯正確有六分。

如果 (1) 寫對後面沒寫, 那有一分。

(2) 寫對有三分, 微分錯誤扣一分。底下計算錯誤扣一分。

(b) 小題如果是微分的觀念錯誤 (例如對 x^2 微分) 則不適用上述標準, 原則上只給一分。

3. (10%) 求 $\dfrac{d}{dx}(\cos x)^{\sin x}$ 。

Sol:

$\dfrac{d}{dx}(\cos x)^{\sin x}$

$= \dfrac{d}{dx}e^{\sin x \cdot \ln(\cos x)}$ (this for three points)

$= e^{\sin x \cdot \ln(\cos x)} \cdot \left(\cos x \cdot \ln(\cos x) + \sin x \cdot \left(\dfrac{-\sin x}{\cos x}\right)\right)$ (this for three points respectively)

$= \cos x^{\sin x} \cdot (\cos x \cdot \ln(\cos x) - \sin x \cdot \tan x)$ (all of the calculus are correct, get ten points)

4. (10%) 若 $y = \alpha x + 2$ 與 $y = 2\ln x$ 相切 , 求 α 之值以及切點的座標 。

Sol:

The tangent point has the following constraints.

(the same slope & the same point)

$$\frac{2}{x} = \alpha$$

$$\alpha x + 2 = 2\ln x \qquad \text{(4 pts)}$$

Solve the equations, we get

$$(x, y) = (e^2, 4) \qquad \text{(3 pts)}$$

$$\alpha = 2e^{-2} \qquad \text{(3 pts)}$$

5. (10%) 估計 $\dfrac{1}{1+\ln 0.997}$ 。

Sol:

Let

$$f(x) = \frac{1}{1+\ln(1+x)}, \quad (2\%)$$

we need to evaluate $f(-0.003)$ by linear approximation near $x=0$.

$$f'(x) = \frac{-1/(1+x)}{(1+\ln(1+x))^2} \quad (2\%)$$

$$f'(0) = \frac{-1}{(1+\ln 1)^2} = -1 \quad (2\%)$$

$$f(a) = f(0) + f'(0)a + ag(a), \quad \text{with} \lim_{a\to 0} g(a) = 0 \quad (2\%)$$

$$f(-0.003) = 1 + (-1)(-0.003) + (-0.003)g(-0.003) \approx 1 + (-1)(-0.003) = 1.003 \quad (2\%)$$

6. (10%) 求函數 $f(x) = x^6 - 3x^2$ 在 $[-2, 3]$ 上的最大值與最小值 。

Sol:

(1) Differential (3%)

$$f(x) = x^6 - 3x^2$$

$$f'(x) = 6x^5 - 6x = 6x(x^2+1)(x+1)(x-1)$$

(2) find Candidate points (5%)

$$f'(x) = 0 \quad x = 0, 1, -1$$

the candidate points are

$$x = 0, 1, -1 x = -2, 3 \text{ (boundary points)}$$

(3) find the Max and min by first-order test or second-order test (2%)

(i) first-order test:

$$-2 < x < -1 \quad \to f'(x) < 0$$

$$-1 < x < 0 \quad \to f'(x) > 0$$

$$0 < x < 1 \quad \to f'(x) < 0$$

$$1 < x < 3 \quad \to f'(x) > 0$$

$f(1), f(-1)$ are local min ,and $f(0)$ is local Max

compare with the boundary points

f(1)=f(-1)=-2 are minimum, and f(3)=702 is maximum

(i) second-order test:

$$f''(x) = 30x^4 - 6$$

$$f''(-1) = 24 > 0$$

$$f''(0) = -6 < 0$$

$$f''(1) = 24 > 0$$

$f(1), f(-1)$ are local min ,and $f(0)$ is local Max

compare with the boundary points

f(1)=f(-1)=-2 are minimum, and f(3)=702 is maximum

7. (25%) 若 $y = f(x) = \dfrac{x(x+5)}{x-4}$

(a) $y = f(x)$ 在 ________________ (區間) 遞增。(3%)

$y = f(x)$ 在 ________________ (區間) 遞減。(3%)

(b) $y = f(x)$ 之極大值：________________ (座標)。(3%)

$y = f(x)$ 之極小值：________________ (座標)。(3%)

(c) $y = f(x)$ 在 ________________ (區間) 凹向上。(3%)

$y = f(x)$ 在 ________________ (區間) 凹向下。(3%)

(d) $y = f(x)$ 所有的漸近線爲 ________________

________________。(4%)

(e) 畫出 $y = f(x)$ 之圖形。(3%)

Sol: Let $y = f(x) = \dfrac{x(x+5)}{x-4}$.

(a) Write

$$f(x) = x + 9 + \frac{36}{x-4}, \tag{1}$$

and compute

$$\begin{aligned} f'(x) &= 1 + (-1)(36)(x-4)^{-2} \\ &= 1 - \frac{36}{(x-4)^2} = \frac{(x-10)(x+2)}{(x-4)^2}. \end{aligned}$$

So $f'(x) = 0$ if and only if $x = -2$ or $x = 10$. We see that

$$\begin{cases} f'(x) > 0 & \text{if } x < -2 \text{ or } x > 10, \\ f'(x) < 0 & \text{if } -2 < x < 10 \text{ and } x \neq 4. \end{cases}$$

Therefore, $f(x)$ is **increasing** on $(-\infty, -2) \cup (10, \infty)$ and **decreasing** on $(-2, 4) \cup (4, 10)$.

(b) Since $f(x)$ is increasing for $x < -2$ and decreasing for $-2 < x < 4$, $f(x)$ must have a local maximum at $x = -2$ with value $f(-2) = 1$. Similarly, $f(x)$ is decreasing for $4 < x < 10$ and increasing for $x > 10$, $f(x)$ must have a local minimum at $x = 10$ with value $f(10) = 25$.

Consequently, $f(x)$ has a **local maximum** at $(-2, 1)$ and a **local minimum** at $(10, 25)$.

(c) To find concavities of $f(x)$, we compute

$$f''(x) = -(36)(-2)(x-4)^{-3} = \frac{72}{(x-4)^3}.$$

Since $f''(x)$ never vanish for $x \neq 4$, so $f(x)$ has no point of inflection. Moveover, $f''(x) > 0$ for $x > 4$ and $f''(x) < 0$ for $x < 4$. Hence, $f(x)$ is **concave up** on $(4, \infty)$ and **concave down** on $(-\infty, 4)$.

(d) $f(x)$ is undefined at $x = 4$ and $|f(x)| \to \infty$ as $x \to 4$, so that $y = f(x)$ has a **vertical asymptote** $x = 4$.

By (1), observe that for large $|x|$, $f(x)$ is close to $x + 9$. In other words,

$$\lim_{|x| \to \infty} [(f(x) - (x + 9)] = \lim_{|x| \to \infty} \frac{36}{x - 4} = 0.$$

Thus, $y = x + 9$ is an **oblique asymptote** for $y = f(x)$.

(e) The figure shown below is the graph of $y = f(x)$. The blue lines are asymptotes. Note that the unit length of x-axis and y-axis are different.

第7 題評分標準如下:

(a) (遞增部份) 算出 $f'(x)$ 得 1 分，兩個區間 $(-\infty,-2)$、$(10,\infty)$ 各 1 分。

(遞減部份) 寫成$(-2,4)\cup(4,10)$ 或 $(-2,10)$ 都算對，得 3 分。

(b) x 座標和 y 座標寫出其中一個就給分。若極大極小寫顛倒，就都不給分。

(c) 算出 $f''(x)$ 得 2 分, 兩個區間 $(4,\infty)$、$(-\infty,4)$ 各 2 分，寫顛倒不給分。

(d) 寫出 $x=4$ 得 2 分; 寫出 $y=x+9$，或計算過程寫出 $f(x)=x+9+36(x-4)^{-1}$ 式子者，得 2 分。

(e) 分成 3 種程度給分：

(得 3 分) 位置、趨勢非常接近正確圖形，有標出兩條漸近線，兩個極值標示正確。

(得 2 分) 與正確圖形相去不遠，但有小錯誤，未標示兩條漸近線或標示錯誤。

(得 0 分) 與正確圖形的趨勢相差太大, 錯誤太多。

(其他) 答案是區間的項目，寫成不等式也可以。另外，有沒有包含端點都無所謂。

8. (10%) 甲車由原點 $(0,0)$ 以每小時 30 公里的速度往 x 軸正向行駛，乙車由點 $(0,13)$ (以公里爲單位) 以每小時 20 公里的速度往 y 軸負向行駛。求兩車最近之距離爲多少公里？

Sol:

At time t, the position of car A is $(30t,0)$, and the position of car B is $(0,13-20t)$. Then, the distance of A and B is $\sqrt{(30t-0)^2+[0-(13-20t)]^2}$ (3pts)

$$
\begin{aligned}
\Rightarrow \overline{AB} &= \sqrt{(30t-0)^2+[0-(13-20t)]^2} \\
&= \sqrt{1300t^2-520t+169} \\
&= \sqrt{1300(t^2-\frac{2}{5}t+\frac{1}{25})+169-\frac{1300}{25}} \\
&= \sqrt{1300(t-\frac{1}{5})^2+117} \\
&\geq \sqrt{117} \quad \text{(7pts)}
\end{aligned}
$$

Hence, we have the minimal distance of A and B is $\sqrt{117}$. And it happens at $\frac{1}{5}$ hour, which is 12 minutes after start.