991微乙01-05班期末考解答和評分標準

1. (10%) 求 $\dfrac{d}{dx}\displaystyle\int_{\sin x}^{x^2}\sqrt{1+t^4}\,dt$ 。

Sol:

Set $F(x)=\displaystyle\int_a^x\sqrt{1+t^4}\,dt$, where a is a constant.

$\Rightarrow \displaystyle\int_{\sin(x)}^{x^2}\sqrt{1+t^4}\,dt = F(x^2)-F(\sin(x))$

By using the Fundamental Calculus Theorem and the Chain Rule, we can get:

$$\begin{aligned}\frac{d}{dx}\int_{\sin(x)}^{x^2}\sqrt{1+t^4}\,dt &= F'(x^2)2x-F'(\sin(x))\cos(x)\\ &= 2x\sqrt{1+x^8}-\cos(x)\sqrt{1+\sin^4(x)}\end{aligned}$$

Basically, if the answer is correct, students can get 10 points. And substitution error will lose 1∼4 points.

2. (15%) (a) 將 $\dfrac{1}{x^3+1}$ 化成部份分式。(7%)

(b) 求 $\displaystyle\int\frac{dx}{x^3+1}$ 。(8%)

Sol:

(a)

$$\frac{1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}. \quad \text{(1 pts)}$$

Then,

$$A(x^2-x+1)+(Bx+C)(x+1)=1.$$

Take $x=-1$, we obtain $3A=1$, so $A=\frac{1}{3}$. (2 pts)

Then,

$$\begin{aligned}(Bx+C)(x+1) &= 1-\frac{1}{3}(x^2-x+1)\\ &= -\frac{1}{3}(x^2-x-2)\\ &= -\frac{1}{3}(x+1)(x-2).\end{aligned}$$

We can get $B=-\frac{1}{3}$ (2 pts) and $C=\frac{2}{3}$ (2 pts). Hence,

$$\frac{1}{x^3+1}=\frac{1}{3(x+1)}+\frac{-x+2}{3(x^2-x+1)}.$$

(b)

$$\begin{aligned}\int \frac{1}{x^3+1}dx &= \int \frac{1}{3(x+1)} + \frac{-x+2}{3(x^2-x+1)}\,dx \\ &= \frac{1}{3}\left[\int \frac{1}{x+1}dx - \frac{1}{2}\int \frac{2x+1-3}{x^2-x+1}\,dx\right]\end{aligned}$$

and we have

$$\begin{aligned}\int \frac{1}{x+1}\,dx &= \ln\mid x+1\mid. \quad \text{(2 pts)} \\ \int \frac{2x+1}{x^2-x+1}\,dx &= \frac{1}{(x^2-x+1)^2}\,d(x^2-x+1) = \ln\mid x^2-x+1\mid. \quad \text{(2 pts)} \\ \int \frac{3}{(x^2-x+1)}\,dx &= \int \frac{3}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \\ &= 2\sqrt{3}\int \frac{1}{(\frac{2(x-\frac{1}{2})}{\sqrt{3}})^2+1}\,d(\frac{2(x-\frac{1}{2})}{\sqrt{3}}) \\ &= 2\sqrt{3}\arctan(\frac{2(x-\frac{1}{2})}{\sqrt{3}}). \quad \text{(2 pts)}\end{aligned}$$

Hence,

$$\int \frac{1}{x^3+1}\,dx = \frac{1}{3}\ln\mid x+1\mid - \frac{1}{6}\ln\mid x^2-x+1\mid + \frac{2}{\sqrt{3}}\arctan(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}) + C$$

Each coefficient of the first two terms worths 1 pts.

3. (10%) 求 $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$ 。

Sol:

Let $x = \tan(\theta)$ **(+4%)**

Since

$$\frac{dx}{d\theta} = \sec^2(\theta)\,d\theta,$$

we have

$$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx = \int \frac{\sec(\theta)}{\tan^2(\theta)\,d\theta} \quad \textbf{(+2\%)}.$$

Now let $u = \sin\theta$, then the original integral now become :

$$= \int \frac{1}{u^2}\,du. \quad \textbf{(+2\%)}$$

Hence we have the answer:,

$$\begin{aligned}&\frac{-1}{u} + c \quad \textbf{(+1\%)} \\ &= -\sin^{-1}(\theta) + C = \frac{-\sqrt{x^2+1}}{x} + C, \quad \textbf{(+1\%)}\end{aligned}$$

where C is a constant in $\mathbb{R}$

4. (10%) 求 $\int_1^e x(\ln x)^2\,dx$ 。

Sol:

$$\begin{aligned}
&\int_1^e x(\ln x)^2\,dx \\
&= \frac{1}{2}x^2(\ln x)^2\Big|_1^e - \int_1^e x\ln x\,dx \qquad \text{(by integration by parts)} \\
&= \frac{1}{2}e^2 - (\frac{1}{2}x^2\ln x - \frac{1}{4}x^2)\Big|_1^e \qquad \text{(by integration by parts)} \\
&= \frac{1}{2}e^2 - \frac{1}{2}e^2 + \frac{1}{4}e^2 - \frac{1}{4} \\
&= \frac{1}{4}(e^2-1)
\end{aligned}$$

評分標準:

1. 做對第一次分佈積分 5分
2. 做對第二次分佈積分 5分
3. 多常數 c 扣1分
4. 符號錯誤 扣2分
5. 沒代上下界 扣3分
6. 變數變換沒換上下界 0分

5. (10%) 求 $\begin{cases} y^2 = x^5 + x^2 \\ x = 0 \\ y = 6 \end{cases}$ 所圍區域對 y-軸旋轉的旋轉體體積 。

Sol:

Attention!! In this case , we can't use disk method , because we can't write down the explicit function x depend on y If use this method and explain why you can't solve it , you can get 5 %

Shell Method :

1.find the boundary of integration 2 %

$$y = x^5 + x^2$$

Let

$$y = 6$$

to find x

$$6 = x^5 + x^2$$

and x=2

2.wrute down the integration function 5 % + integration 3 %

$$
\begin{aligned}
V &= \int_0^2 2x\pi(6-\sqrt{x^5+x^2})\,\mathrm{d}x \\
&= \int_0^2 12\pi x\mathrm{d}x - 2\pi\int_0^2 x^2(\sqrt{x^3+1})\,\mathrm{d}x \\
&= 24\pi - \frac{2\pi}{3}\int_0^2 \sqrt{x^3+1}\,\mathrm{d}(x^3+1) \\
&= \frac{112\pi}{9}
\end{aligned}
$$

6. (15%) (a) 寫出 $\ln(1+x)$ 在 $x=0$ 的泰勒展式。(5%)

(b) 令 $f(x)=\ln\sqrt{\dfrac{1+x}{1-2x}}$，求 $f^{(6)}(0)$。(10%)

Sol:

(a) Let $f(x)=\ln(1+x)$,hence $f(0)=\ln(1)=0$

$\ln(1+x)^{(1)} = \dfrac{1}{1+x} \Longrightarrow f^{(1)}(0)=1$

$\ln(1+x)^{(2)} = -(1+x)^{(-2)} \Longrightarrow f^{(2)}(0)=-1$

$\ln(1+x)^{(3)} = 2(1+x)^{(-3)} \Longrightarrow f^{(3)}(0)=2$

$\ln(1+x)^{(4)} = -6(1+x)^{(-4)} \Longrightarrow f^{(4)}(0)=-6$

continue this way,we can get general items $\dfrac{(-1)^{(n-1)}}{n}x^n$.

Hence $\ln(1+x)=0+x+\dfrac{-1}{2}x^2+\dfrac{1}{3}x^3+\cdots+\dfrac{(-1)^{(n-1)}}{n}x^n+\cdots$ for the first some items,you get 3 points, for the general items,you get 2 points.

(b)

$$
\begin{aligned}
f(x) &= \ln\big(\frac{1+x}{1-2x}\big)^{\frac{1}{2}} = \frac{1}{2}[\ln(1+x)-\ln(1-2x)] \quad \text{(1 points)} \\
&= \frac{1}{2}[0+x+\frac{-1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-\frac{1}{6}x^6+\cdots] \\
&- \frac{1}{2}[-2x-2x^2-\frac{8}{3}x^3-4x^4-\frac{32}{5}x^5-\frac{32}{3}x^6-\cdots]
\end{aligned}
$$

(2 points and 4 points respectively)

By Taylor expression,we know that

$$
\frac{f^{(6)}(0)}{6!} = \frac{1}{2}[-\frac{1}{6}+\frac{32}{3}] = \frac{63}{12}
$$

hence $f^{(6)}(0)=3780$ (3 points).

7. (15%)

(a) 求以下函數在 $x = 0$ 之泰勒展式非零的前三項 。

(i) $\sin x$ 。(5%)

(ii) $\tan^{-1} x$ 。(5%)

(b) 求 $\lim_{x\to 0} \dfrac{2\sin x - \tan^{-1} x - x}{x^5}$ 。(5%)

Sol:

(a) (i) Let $f(x) := \sin x$,

$$f^{(n)}(x) = \begin{cases} \cos x & \text{if } n \in 4\mathbb{N}+1 \\ -\sin x & \text{if } n \in 4\mathbb{N}+2 \\ -\cos x & \text{if } n \in 4\mathbb{N}+3 \\ \sin x & \text{if } n \in 4\mathbb{N}+4 \end{cases}$$

Then at $x = 0$,

$$f^{(n)}(0) = \begin{cases} 1 & \text{if } n \in 4\mathbb{N}+1 \\ 0 & \text{if } n \in 4\mathbb{N}+2 \\ -1 & \text{if } n \in 4\mathbb{N}+3 \\ 0 & \text{if } n \in 4\mathbb{N}+4 \end{cases}$$

The Taylor series of $f(x)$ at $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

(ii) Let $f(x) := \tan^{-1} x$,

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$\begin{aligned} f(x) &= \int f'(x) + C = \int \sum_{n=0}^{\infty} (-x^2)^n + C \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + \tan^{-1}(0) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \end{aligned}$$

$(\tan^{-1}(0) = 0)$

(b)

$$
\begin{aligned}
&\lim_{x\to 0}\frac{2\sin x-\tan^{-1}x-x}{x^5}\\
&=\lim_{x\to 0}\frac{1}{x^5}\left(2\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m+1}x-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}-x\right)\\
&=\lim_{x\to 0}\frac{1}{x^5}\left(2(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots)-(x-\frac{x^3}{3}+\frac{x^5}{5}+\cdots)-x\right)\\
&=\lim_{x\to 0}\frac{1}{x^5}\left(\frac{2x^5}{5!}-\frac{x^5}{5}+\cdots\right)\\
&=\lim_{x\to 0}\left(\frac{-11}{60}+\text{higher order terms}\right)\\
&=\frac{-11}{60}
\end{aligned}
$$

評分標準:

(a) (i) 正確寫出前三項即得五分, 寫錯若有計算過程則部分給分, 寫錯又無計算過程者零分.

(ii) 有計算過程算對者不論方法好壞皆有五分, 無過程背對答案者兩分, 背錯零分. 有計算過程算錯者, 好方法得三至四分, 方法不好者兩分至一分.

(b) 有計算過程正確算對者不論方法好壞皆得五分, 好方法算錯者三分至四分, 方法不好算錯者零分至三分.

8. (15%) 令 $f(x,y)=x^y$ 。求

(a) $\dfrac{\partial f}{\partial x}$ 。(5%)

(b) $\dfrac{\partial f}{\partial y}$ 。(5%)

(c) $z=f(x,y)$ 在點 $(2,2,4)$ 之切平面方程式 。(5%)

Sol:

$$f(x,y)=x^y=e^{ylnx}$$

(a)

$$\frac{\partial f}{\partial x}=\frac{y}{x}e^{y\ln x}=yx^{y-1} \qquad \text{(5 pts)}$$

(b)

$$\frac{\partial f}{\partial y}=(\ln x)e^{y\ln x}=(\ln x)x^y \qquad \text{(5 pts)}$$

(c) Tangent Plane at $(2, 2, 4)$:

$$z - 4 = \frac{\partial f}{\partial x}\bigg|_{(x,y)=(2,2)} (x-2) + \frac{\partial f}{\partial y}\bigg|_{(x,y)=(2,2)} (y-2) \qquad \text{(3 pts)}$$

$\Rightarrow$

$$z - 4 = 4(x-2) + 4\ln 2(y-2)$$

or

$$4x + 4(\ln 2)y - z - 4(2\ln 2 + 1) = 0$$

(2 pts)