992微乙01-05班期中考解答和評分標準

1. (15%) 設 $f(x,y) = x^3 + \frac{9}{2}xy + \frac{23}{8}y^2$ 。

   (a) 求 $\nabla f(1,2)$ 。(5%)

   (b) 求在點 $(1,2)$ 的方向導數之最大值。(10%)

Sol:

(a) **Get 2 points.**

$$\nabla f(x,y) = (3x^2 + \frac{9}{2}y, \frac{9}{2}x + \frac{23}{4}y) \tag{1}$$

Get full points.

$$\nabla f(1,2) = (12, 16) \tag{2}$$

(b) **Get 2 points.**

Let direction as $(\cos(\theta), \sin(\theta))$ or let direction as (a,b) and $a^2 + b^2 = 1$.

Get full points.

$$max_{\{\theta\}} 12\cos(\theta) + 6\sin(\theta) = 20 \tag{3}$$

Get full points.

$$max_{\{(a,b)|a^2+b^2=1\}} 12a + 6b = 20 \tag{4}$$

Get full points.

The direction to get max value is parallel to $\nabla f(1,2) = (12, 16)$.

$$\sqrt{12^2 + 16^2} = 20 \tag{5}$$

2. (10%) 求 $x + y^2 + z^3 - 4x^2yz + 1 = 0$ 在點 $(1,1,1)$ 處之切平面方程式 。

Sol:

Let $f(x,y,z) = x + y^2 + z^3 - 4x^2yz + 1$, then

$$\nabla f = (f_x, f_y, f_z) = (1 - 8xyz, 2y - 4x^2z, 3z^2 - 4x^2y).$$

So $\nabla f(1,1,1) = (-7, -2, -1)$.

The $\nabla f(1,1,1)$ is the normal vector of the tangent plane of $f(x,y,z) = 0$ at $(1,1,1)$.

Hence, the equation of tangent plane at $(1,1,1)$ is $-7(x-1)-2(y-1)-(z-1)=0$.

Or equivalently, $7x+2y+z=10$.

評分標準:

- ∇f : 每個分量1 分, 共3 分
- $\nabla f(1,1,1)$:1 分
- 寫出切平面方程式得6 分
- 最後的方程式化簡出錯不扣分
- 如果計算錯誤造成後面方程式寫錯, 只要錯得不多, 原則上不會扣到後面的6 分; 如果錯太多, 這6 分就拿不到

3. (15%) 令 $f(x,y)=e^{-(x^2+y^2)}xy$ 。

   (a) 求 f_x , f_y , f_{xx} , f_{yy} , f_{xy} 。(5%)

   (b) 求 $f(x,y)$ 之極值 。(10%)

Sol:

(a) $f_x = -(2x^2-1)ye^{-(x^2+y^2)}$ (1 %)

$f_y = -(2y^2-1)xe^{-(x^2+y^2)}$ (1 %)

$f_{xx} = 2xy(2x^2-3)e^{-(x^2+y^2)}$ (1 %)

$f_{yy} = 2xy(2y^2-3)e^{-(x^2+y^2)}$ (1 %)

$f_{xy} = (2x^2-1)(2y^2-1)e^{-(x^2+y^2)}$ (1 %)

(b) $f_x = f_y = 0 \Longrightarrow (x,y) = (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}),(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}),(0,0)$ (1 %)

Let $H(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = [4(2x^2-3)(2y^2-3)x^2y^2-(2x^2-1)^2(2y^2-1)^2]e^{-2(x^2+y^2)}$

(1) $H(0,0) = -1 < 0$.

By Second Derivative Test f has saddle point at $(0,0)$ (3 %)

(2) $f_{xx}(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) = f_{xx}(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}) = -e^{-1} < 0$,

and $H(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) = H(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}) = 4e^{-2} > 0$.

By Second Derivative Test

f has local maximum at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ (2 %)

The maximum extreme value is $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = \frac{1}{2}e^{-1}$ (1 %)

(3) $f_{xx}(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f_{xx}(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = e^{-1} > 0,$

and $H(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = H(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = 4e^{-2} > 0.$

By Second Derivative Test f has local minimum at $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ (2 %)

The minimum extreme value is $f(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = \frac{-1}{2}e^{-1}$ (1 %)

4. (10%) 求點 $(0,0)$ 到 $17x^2 + 12xy + 8y^2 = 100$ 的最長和最短距離 。

Sol:

In order to find the extreme value of $\sqrt{x^2+y^2}$ given $17x^2 + 12xy + 8y^2 = 100$.

Let $f(x,y) = x^2 + y^2$ and $g(x,y) = 17x^2 + 12xy + 8y^2 - 100$. From

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases} \text{(5 pts)} = \begin{cases} 2x = \lambda(34x + 12y) \\ 2y = \lambda(12x + 16y) \\ 17x^2 + 12xy + 8y^2 = 100 \end{cases}$$

we get $(x, y, \lambda) = (2, 1, \frac{1}{20})$ or $(2, -4, \frac{1}{5})$.

Therefore the maximum is $\sqrt{20}$ and the minimum is $\sqrt{5}$. (5 pts)

(exactly one correct answer is worth 4pt, and no square root is worth 3 pts.)

5. (10%) 計算 $\int_{y=0}^{y=1} \int_{x=\sqrt{y}}^{x=1} \frac{e^{x^2}}{x}\, dxdy$ 。

Sol:

$$\begin{cases} 0 \le y \le 1, \\ \sqrt{y} \le x \le 1. \end{cases} \Rightarrow \begin{cases} 0 \le x \le 1, \\ 0 \le y \le x^2. \end{cases}$$

$$
\begin{aligned}
&\int_0^1 \int_0^{x^2} \frac{e^{x^2}}{x} dy dx \\
&= \int_0^1 (\frac{e^{x^2}}{x} y\Big|_0^{x^2}) dx \\
&= \int_0^1 x e^{x^2} dx \\
&= \frac{1}{2} e^{x^2}\Big|_0^1 \\
&= \frac{1}{2}(e-1)
\end{aligned}
$$

The standard of answer:

Change coordinate 5 points

The answer 5 points

6. (10%) 利用極座標求 $\iint\limits_{\Omega} x^2 y\, dA$ ，其中 $\Omega = \{(x, y) :\ (x - \frac{1}{2})^2 + y^2 \le \frac{1}{4},\ y \ge 0\}$ 。

Sol:

Let $r\cos\theta := x, \quad r\sin\theta := y,$

then the domain

$$\frac{1}{4} \ge (r\cos\theta - \frac{1}{2})^2 + (r\sin\theta)^2 = r^2\cos^2\theta - r\cos\theta + \frac{1}{4} + r^2\sin^2\theta$$

which imply $0 \ge r^2 - r\cos\theta = r(r - \cos\theta)$,

that is

$$0 \le r \le \cos\theta. \tag{6}$$

And we still have $y \ge 0$, that is

$$r\sin\theta \ge 0.$$

And by (6),

$$\cos\theta \ge 0,$$

so we have

$$0 \le \theta \le \frac{\pi}{2} \tag{7}$$

Our integral

$$\begin{aligned}
\int_{\Omega} x^2 y \ \mathrm{d}x\mathrm{d}y &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\cos\theta} (r\cos\theta)^2 r\sin\theta \ r \ \mathrm{d}r\mathrm{d}\theta && (8)\\
&= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\cos\theta} r^4\cos^2\theta\sin\theta \ \mathrm{d}r\mathrm{d}\theta && (9)\\
&= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^5}{5}\right]_0^{\cos\theta} \cos^2\theta \ \mathrm{d}(-\cos\theta) && (10)\\
&= -\frac{1}{5}\int_{\theta=0}^{\frac{\pi}{2}} \cos\theta^5\cos^2\theta \ \mathrm{d}(\cos\theta) && (11)\\
&= -\frac{1}{5}\left[\frac{\cos^8\theta}{8}\right]_0^{\frac{\pi}{2}} = \frac{1}{40} && (12)
\end{aligned}$$

- If you write down equation (8) correctly, you can get 4%.
- If you compute (9) to (12) correctly, you get 6%; if not totally correct you can get partial grades.
- If you write equation (8) in a wrong expression (e.g. $\int_{r=0}^{\frac{\pi}{2}}$), the remaining computation may get no grade.
- you can also compute $\int_{\theta=0}^{\pi}\int_{r=0}^{\frac{1}{2}}(r\cos\theta+\frac{1}{2})^2(r\sin\theta)^2 r dr d\theta$ to solve this.

7. (15%) 利用變數變換 $u=\frac{y}{x^2}$、$v=\frac{x}{y^2}$，計算 $\iint\limits_R \frac{1}{x^2y^2}\,dA$，其中 R 爲 $y=x^2$，$y=2x^2$，$x=3y^2$ 及 $x=y^2$ 所圍成的區域。

Sol:

$u=x^{-2}y,\ v=xy^{-2} \quad\Rightarrow\quad x=u^{-\frac{2}{3}}v^{-\frac{1}{3}},\ x=u^{-\frac{1}{3}}v^{-\frac{2}{3}}$

Jacobian (5 %)

$$J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix} = \frac{1}{3}u^{-2}v^{-2}$$

Variable transformation (10 %)

$$\begin{aligned}
\iint\limits_R f(x,y)\,dxdy &= \int_1^3\int_1^2 u^2v^2 J(u,v)\,dudv \\
&= \int_1^3\int_1^2 u^2v^2\frac{1}{3}u^{-2}v^{-2}\,dudv \\
&= \int_1^3\int_1^2 \frac{1}{3}\,dudv = \frac{2}{3}
\end{aligned}$$

8. (15%) 計算三重積分 $\iiint\limits_{\Omega} ydV$，其中 Ω 爲 $x=0$，$y=0$，$z=0$，$x+y+2z=1$ 所圍成的區域。

Sol:

$$
\begin{aligned}
\iiint_{\Omega} y dV &= \underbrace{\int_0^1}_{1\text{ pt}} \underbrace{\int_0^{\frac{1-y}{2}}}_{2\text{ pts}} \underbrace{\int_0^{1-y-2z}}_{2\text{ pts}} y\, dxdzdy \\
&= \int_0^1 \int_0^{\frac{1-y}{2}} xy\Big|_{x=0}^{x=1-y-2z} dzdy \quad (3\text{ pts}) \\
&= \int_0^1 \int_0^{\frac{1-y}{2}} (1-y-2z)y\, dzdy \\
&= \int_0^1 y(1-y)z - z^2y\Big|_{z=0}^{z=\frac{1-y}{2}} dy \quad (3\text{ pts}) \\
&= \int_0^1 \frac{1}{4}y(1-y)^2\, dy \\
&= \frac{1}{4}\left(\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right)\Big|_{y=0}^{y=1} \quad (3\text{ pts}) \\
&= \frac{1}{4}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{48} \quad (1\text{ pt}).
\end{aligned}
$$

只要範圍寫錯, 接下來的運算皆不給分。