992微乙01-05班期末考解答和評分標準

1. (10%) 解微分方程 $y = y(t)$ 滿足 $y' = y(1-y)$, $y(0) = \frac{1}{2}$ 。

   Sol:

   By assumption,

$$\frac{dy}{dt} = y(1-y) \ \Rightarrow \ \frac{dy}{y(1-y)} = dt \ \Rightarrow \ \int dt = \int \frac{dy}{y(1-y)} \quad (1\%)$$
$$= \int \frac{1}{y} + \frac{1}{1-y} dy \quad (2\%)$$

$$\Rightarrow t + c = \ln|y| - \ln|1-y| \quad \text{for some constant } c \quad (2\%)$$

   By assumption $y(0) = \frac{1}{2}$, $\Rightarrow 0 + c = \ln|\frac{1}{2}| - \ln|1 - \frac{1}{2}| = 0 \ \Rightarrow \ c = 0 \quad (2\%)$

   and $y(0) = \frac{1}{2} > 0,\ 1 - y(0) = \frac{1}{2} > 0, \Rightarrow |y| = y, |1-y| = 1-y$ for t close to 0 s.t. $0 < y(t) < 1$.

$$\Rightarrow \ e^t = e^{\ln y - \ln(1-y)} = \frac{y}{1-y} \quad (1\%)$$
$$\Rightarrow \ 0 = (1-y)e^t - y = y(-e^t - 1) + e^t$$
$$\Rightarrow \ y(t) = \frac{e^t}{1+e^t} \quad (2\%)$$

   But then $0 < \frac{e^t}{1+e^t} < 1$ for all t, i.e. $y(t) = \frac{e^t}{1+e^t}$ is a solution defined for all $t \in \mathbb{R}$.

2. (15%) 解微分方程 $x\frac{dy}{dx} = y + x^2 \sin x$, $y(\frac{\pi}{2}) = \frac{\pi}{2}$ 。

   Sol:

   Solve the equation

$$xy' = y + x^2 \sin x \implies y' - \frac{1}{x}y = x \sin x$$

   Conside the integral factor

$$e^{\int -\frac{1}{x}\,dx} = \frac{1}{x} \quad (5\%)$$

   Then

$$\frac{d}{dx}(y\frac{1}{x}) = \sin x \quad (3\%)$$
$$\implies \ y\frac{1}{x} = -\cos x + c$$
$$\implies \ y = x(-\cos x + c) \quad (4\%)$$

By

$$y(\frac{\pi}{2}) = \frac{\pi}{2} \implies y(\frac{\pi}{2}) = \frac{\pi}{2}(0+c) = \frac{\pi}{2}$$

so

$$c = 1 \quad (3\%)$$

Hence

$$y = x(1-\cos x)$$

3. (15%) 令 $g(x,y) = \mathrm{P}(X=x, Y=y)$, X 取值 1 或 2 , Y 取值 1 或 2 或 3 。若已知

$$g(1,1) = \frac{2}{11}\ ,\ g(1,2) = \frac{3}{11}\ ,\ g(1,3) = \frac{1}{11}\ ,$$

$$g(2,1) = \frac{1}{11}\ ,\ g(2,2) = \frac{3}{11}\ ,\ g(2,3) = \frac{1}{11}\ ,$$

求 (a) $\mathrm{E}(X)$, (b) $\mathrm{Var}(X)$ 。

Sol:

(a)

$$\begin{aligned}\mathrm{E}(X) &= 1\cdot \mathrm{P}(X=1) + 2\cdot \mathrm{P}(X=2)\\ &= \underbrace{1\cdot(\frac{2}{11}+\frac{3}{11}+\frac{1}{11})}_{(3\%)} + \underbrace{2\cdot(\frac{1}{11}+\frac{3}{11}+\frac{1}{11})}_{(3\%)}\\ &= \frac{16}{11} \quad (1\%)\end{aligned}$$

(b)

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}^2(X)\\ &= \underbrace{1^2\cdot(\frac{2}{11}+\frac{3}{11}+\frac{1}{11})}_{(3\%)} + \underbrace{2^2\cdot(\frac{1}{11}+\frac{3}{11}+\frac{1}{11})}_{(3\%)} - \underbrace{\frac{256}{121}}_{(2\%)}\\ &= \frac{30}{121}\end{aligned}$$

4. (15%) 若 X_1 , X_2 爲白努利單次試驗 , X_1 、X_2 取值 1 的機率爲 $\frac{2}{5}$, X_1 、X_2 取值 0 的機率爲 $\frac{3}{5}$ 且 X_1 , X_2 獨立 。令 $Y = X_1 + X_2$, $Z = X_1 - X_2$, 求

(a) $\mathrm{P}(Y=1)$, $\mathrm{P}(Z=1)$ 及 $\mathrm{P}(Y=1, Z=1)$ 之值 。

(b) 試說明 Y 與 Z 是否獨立。

Sol:

(a)

$$\begin{aligned}
\mathrm{P}(Y=1) &= \mathrm{P}(\{X_1=1, X_2=0\} \cup \{X_1=0, X_2=1\}) \\
&= \mathrm{P}(X_1=1, X_2=0) + \mathrm{P}(X_1=0, X_2=1) \\
&= \mathrm{P}(X_1=1)\mathrm{P}(X_2=0) + \mathrm{P}(X_1=0)\mathrm{P}(X_2=1) \text{ (because } X_1, X_2 \text{ are indep.)} \\
&= (\frac{2}{5})(\frac{3}{5}) + (\frac{3}{5})(\frac{2}{5}) = \frac{12}{25}
\end{aligned}$$

(3%)

$$\begin{aligned}
\mathrm{P}(Z=1) &= \mathrm{P}(X_1=1, X_2=0) \\
&= \mathrm{P}(X_1=1)\mathrm{P}(X_2=0) \text{ (because } X_1, X_2 \text{ are indep.)} \\
&= (\frac{2}{5})(\frac{3}{5}) = \frac{6}{25}
\end{aligned}$$

(3%)

$$\begin{aligned}
\mathrm{P}(Y=1, Z=1) &= \mathrm{P}(X_1=1, X_2=0) \\
&= \mathrm{P}(X_1=1)\mathrm{P}(X_2=0) \text{ (because } X_1, X_2 \text{ are indep.)} \\
&= (\frac{2}{5})(\frac{3}{5}) = \frac{6}{25}
\end{aligned}$$

(4%)

(b) Y and Z are not independent, because

$$\mathrm{P}(Y=1, Z=1) = \frac{6}{25} \neq (\frac{12}{25})(\frac{6}{25}) = \mathrm{P}(Y=1)\mathrm{P}(Z=1)$$

(5%)

5. (15%) 設 $f_X(x) = \frac{2x}{k^2}$，$0 \le x \le k$，是 X 的機率密度函數；X 取值在 $[0, k]$。
(a) 求 $\mathrm{E}(X)$ (以 k 表示)。(b) 求 $\mathrm{Var}(X)$ (以 k 表示)。(c) 若 $\mathrm{Var}(X) = 2$，求 k 之值。

Sol:

(a)

$$\mathrm{E}(X) = \int_0^k x\frac{2x}{k^2}\,dx = \frac{1}{k^2}\int_0^k 2x^2\,dx = \frac{2}{3}k. \quad (5\%)$$

(b)

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \quad (4\%) \\
\mathrm{E}(X^2) &= \int_0^k x^2 \frac{2x}{k^2}\,dx = \frac{2}{k^2}\int_0^k x^3\,dx = \frac{k^2}{2} \\
\mathrm{Var}(X) &= \frac{k^2}{2} - \frac{4}{9}k^2 = \frac{1}{18}k^2. \quad (4\%)
\end{aligned}$$

(c)

$$\mathrm{Var}(X) = \frac{1}{18}k^2 = 2 \;\Rightarrow\; k = 6. \quad (2\%)$$

If the answer of (b) is wrong but the answer of (c) is correct corresponding to it, then you may get 1pt in (c).

6. (15%) 設 X , Y 爲獨立之隨機變數且其機率密度函數爲

$$f_X(x) = \begin{cases} 2^{-1}x^2e^{-x} & ,x \geq 0, \\ 0 & ,x < 0. \end{cases} \quad ; \quad f_Y(y) = \begin{cases} e^{-y} & ,y \geq 0, \\ 0 & ,y < 0. \end{cases}$$

若 $Z = X + Y$, 求 $f_Z(z)$ 。

Sol:

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\,dx \quad (4\%) \\
&= \int_0^z f_X(x)f_Y(z-x)\,dx \\
&= \int_0^z \frac{1}{2}x^2e^{-x}e^{-(z-x)}\,dx \quad (5\%) \\
&= \int_0^z \frac{1}{2}x^2e^{-z}\,dx \\
&= e^{-z}\int_0^z \frac{1}{2}x^2\,dx \quad (4\%) \\
&= e^{-z}\frac{z^3}{6} \;\; (z > 0) \quad (2\%)
\end{aligned}$$

7. (15%) 一本 300 頁的書有 15 個錯誤且其分佈近似於一個 Poisson 過程 。求

(a) 100 頁中恰有 5 個錯誤之機率 。

(b) 10 頁中至少有 1 個錯誤之機率 。

Sol:

(a)

$$\text{Poisson}(k, \lambda, T) = \frac{(\lambda T)^k}{k!} e^{-\lambda T} \quad (2\%) \tag{1}$$

$$\lambda = \frac{15}{300},\ m = 100 \times \lambda = 5. \quad (2\%) \tag{2}$$

$$\text{Poisson}(5, \frac{15}{300}, 100) = \quad (3\%) \tag{3}$$

$$= \frac{5^5}{5!} e^{-5}. \quad (1\%) \tag{4}$$

You may only get points if you have get points in previous equation. (i.e. If you have a mistake in line 2, you may not get points in line 3).

(b)

$$1 - \text{Poisson}(0, 0.05, 10) = 1 - \frac{0.5^0 e^{-0.5}}{0!} \tag{5}$$

$$= 1 - e^{-0.5} \tag{6}$$

Total 7 points. If you have already write down Eq (5) but you have a wrong answer. Then you may get 6 points. Other mistake will depend on your work to give your corresponded grade.