1002微乙01-05班期中考解答和評分標準

1. (10%) 令 $f(x,y)=e^{2x^2+y^2}$, $x(u,v)=u\ln v$, $y(u,v)=ve^u$. 求 $\dfrac{\partial f}{\partial v}$。

Solution:

解法1

$$\begin{aligned}\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}\\ &= (4xe^{2x^2+y^2})\cdot\frac{u}{v}+(2ye^{2x^2+y^2})\cdot e^u\\ &= e^{2u^2\ln^2 v+v^2e^{2u}}(\frac{4u^2\ln v}{v}+2ve^{2u})\end{aligned}$$

解法2

$$f(x(u,v),y(u,v))=e^{2u^2\ln^2 v+v^2e^{2u}}$$

$$\frac{\partial f}{\partial v}=e^{2u^2\ln^2 v+v^2e^{2u}}\cdot(2u^2\cdot 2\ln v\cdot\frac{1}{v}+2ve^{2u})$$

★評分標準

(1)解法一第一步寫對得4分，第二步四項每項1.5分，x(u,v)、y(u,v)沒代回式子扣兩分

(2)解法二$f(x(u,v),y(u,v))$寫對得3分，微分第一項寫對得3分，後兩項每項2分

2. (10%) 設 $f(x,y)=(x^2+1)^2y-xy^3$。

(a) 求 $f(x,y)$ 在點 $(1,2)$ 沿 $(2,1)$ 的方向導數。

(b) 在點 $(1,2)$處，$f(x,y)$ 沿哪一方向的方向導數最大。

Solution:

(a) Since $\nabla f(1,2)=(4x(x^2+1)-y^3,(x^2+1)^2-3xy^2)|_{(1,2)}=(8,-8)$ (2%) and $\vec{u}=(\frac{2}{\sqrt{5}},\frac{-1}{\sqrt{5}})$ (1%),

$$D_{\vec{u}}=\nabla f(1,2)\cdot(\frac{2}{\sqrt{5}},\frac{-1}{\sqrt{5}})=\frac{8}{\sqrt{5}}\ (2\%)$$

(b) 由課本定理可知，沿$(1,-1)$方向$\nabla f(1,2)$會有最大值。 (5%)

3. (10%) 求 $e^{(x^2-y^2)}+x\sin z=1$ 在 $(1,1,0)$ 的切平面方程式。

Solution:

Find the tangent plane equation of

$$e^{(x^2-y^2)} + x\sin z = 1 \text{ at } (1,1,0)$$

Let $f = e^{(x^2-y^2)} + x\sin z - 1$
then

$$\nabla f = (2xe^{(x^2-y^2)} + \sin z, -2ye^{(x^2-y^2)}, x\cos z)$$

$$\nabla f|_{(1,1,0)} = (2,-2,1) \quad \text{(8 points)}$$

Hence the equation is

$$2x - 2y + z = 0 \quad \text{(2 points)}$$

4. (15%) 使用 Lagrange 乘子法求 $f(x,y) = x^2y$ 在 $x^2 + 2y^2 = 6$ 限制條件下的最大值與最小值。

Solution:

令

$$f(x,y) = x^2y \text{ 和 } g(x,y) = x^2 + 2y^2 - 6,$$

則由 Lagrange multiplier method:

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y), \\ g(x,y) = 0, \end{cases} \quad \text{(3 分)}$$

可得

$$\begin{cases} 2xy = \lambda \cdot 2x, \\ x^2 = \lambda \cdot 4y, \\ x^2 + 2y^2 - 6 = 0. \end{cases} \quad \text{(3 分)}$$

由 $2xy = \lambda \cdot 2x \Rightarrow x = 0$ or $\lambda = y$.

case 1: $x = 0$. 此時代入 $x^2 + 2y^2 - 6 = 0$ 得 $y = \pm\sqrt{3}$. (3 分)

case 2: $\lambda = y$. 此時代入 $x^2 = \lambda \cdot 4y$ 得 $x^2 = 4y^2$. 故解

$$\begin{cases} x^2 = 4y^2, \\ x^2 + 2y^2 - 6 = 0, \end{cases}$$

得 $(x,y) = (2,1), (2,-1), (-2,1), (-2,-1)$. (4 分)

綜合上述, 計算

$$\begin{aligned} f(0,\sqrt{3}) &= 0, \\ f(0,-\sqrt{3}) &= 0, \\ f(2,1) &= 4, \\ f(2,-1) &= -4, \\ f(-2,1) &= 4, \\ f(-2,-1) &= -4, \end{aligned}$$

得最大值為 4, 最小值為 -4. (2 分) □

5. (15%) 求$f(x,y)=x^4+y^4-4xy+1$的極大值、極小值與鞍點。(假如存在的話)

Solution:

1° 找出候選點。(5%)

令 $\bigtriangledown f(x,y)=(f_x,f_y)=(4x^3-4y,4y^3-4x)=(0,0)$, 則得到

$$\begin{cases} 4x^3-4y=0 \\ 4y^3-4x=0 \end{cases} \Longrightarrow \begin{cases} x^3=y \\ y^3=x \end{cases} \Longrightarrow x^9=x$$

若 $x=0$,則 $(0,0)$ 為候選點。

若 $x\neq 0$,則 $x^8=1\Rightarrow x^8-1=0$

$\Rightarrow (x^4+1)(x^2+1)(x+1)(x-1)=0\Rightarrow x=1$ or $x=-1$ (只需要實數解)

故 $(1,1),(-1,-1)$ 為候選點。

2° 計算 $D(x,y)$。(4%)

$$D(x,y)=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}=\begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix}=144x^2y^2-16$$

3° 使用二階測試判斷各個候選點。(6%)

1. 在點$(0,0)$
   $D(0,0)<0\Longrightarrow (0,0)$ 為鞍點。

2. 在點$(1,1)$
   $D(1,1)>0$ & $f_{xx}(1,1)>0\Longrightarrow (1,1)$ 為極小點,極小值為 $f(1,1)=-1$ 。

3. 在點$(-1,-1)$
   $D(-1,-1)>0$ & $f_{xx}(-1,-1)>0\Longrightarrow (-1,-1)$ 為極小點,極小值為 $f(-1,-1)=-1$ 。

6. (10%) 求 $\int_0^1\int_x^1\cos(y^2)dydx$。

Solution:

$$\begin{aligned}
&\int_0^1 \int_x^1 \cos(y^2)\cdot dydx \\
&= \int_0^1 \int_0^y \cos(y^2)\cdot dxdy \quad \text{(Fubini's : + 4pts)} \\
&= \int_0^1 \cos(y^2) \int_0^y 1dx \cdot dy \\
&= \int_0^1 \cos(y^2)\cdot y \cdot dy \quad \text{(Integration of Monomials : + 2pts)} \\
&= \int_0^1 \frac{\cos(y^2)}{2} \cdot d(y^2) \\
&= \left.\frac{\sin(y^2)}{2}\right|_{y=0}^{y=1} \quad \text{(Integration of } \cos(y^2)y \text{ : +2 pts)} \\
&= \frac{1}{2}\left(\sin(1^2)-\sin(0^2)\right) = \frac{1}{2}\sin(1) \quad \text{(Computation of } \sin(x) \text{ : +2 pts)}
\end{aligned}$$

7. (15%) 求 $\displaystyle\iint_\Omega e^{\frac{(x^2+y^2)}{2}}dA$, 其中 Ω : $1 \le x^2+y^2 \le 2$ 且 $y \ge 0$。

Solution:

$\displaystyle\int_0^\pi \int_1^{\sqrt{2}} re^{\frac{r^2}{2}}drd\theta$ (5 points) $= \pi(e - e^{\frac{1}{2}})$ (10 points)

8. (15%) 利用變數變換計算 $\displaystyle\int_0^1 \int_0^{1-x} (x+y)^{\frac{3}{2}}(y-x)^2 dydx$。

Solution:

Assume $u = x+y$ and $v = y-x \Rightarrow x = \frac{u-v}{2} \quad y = \frac{u+v}{2}$ (4%)

and compute the Jacobian $J = \frac{1}{2}$ (1%)

Now we have to consider the range of integration. In x-y coordinate the range is an area which is enclosed by $x=0$, $y=0$, and $y=1-x$. Because we use the change of variables, we should find the corresponding integral range.

$x = 0 \Rightarrow \frac{u-v}{2} = 0 \qquad \Rightarrow v = u$

$y = 0 \Rightarrow \frac{u+v}{2} = 0 \qquad \Rightarrow v = -u$

$y = 1-x \qquad \Rightarrow u = 1$ (3%)

So by above argument we obtain the range of integration in u-v coordinate, we rewrite the integral

$$\int_0^1 \int_{-u}^u u^{\frac{3}{2}}v^2\frac{1}{2}dvdu = \int_0^1 \frac{1}{6}u^{\frac{3}{2}}v^3|_{-u}^u du = \frac{1}{3}\int_0^1 u^{\frac{9}{2}}du = \frac{2}{33}$$

(7%)