1002微乙01-05班期末考解答和評分標準

1. (15%) 解微分方程 $x\frac{dy}{dx} = -2y + \sin x$, $y(\frac{\pi}{2}) = 0$.

Solution:

Since $xy' + 2y = \sin x$, then $y' + \frac{2}{x}y = \frac{\sin x}{x}$

The integral factor is $e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

Thus, $x^2y' + 2xy = x\sin x$ (6 points)

So, $x^2y = \int x\sin x dx = -x\cos x + \sin x + c$ (6 points)

Because $y(\frac{\pi}{2}) = 0$, $0 = 0 + 1 + c$. Therefore, $c = -1$ (3 points)

$x^2y = \int x\sin x dx = -x\cos x + \sin x - 1$ ($y = \frac{-x\cos x + \sin x - 1}{x^2}$)

2. 設某地區人口的生長率和死亡率分別為α, β ($\alpha > \beta$) 且外移人口數為 $m(> 0)$, 假設 α, β, m 皆為常數。若整個人口數在時間 t 的變化率滿足 $\frac{dP}{dt} = (\alpha - \beta)P - m$

(a) (12%) 求滿足初始條件 $P(0) = P_0$ 的 $P(t)$

(b) (6%) 問什麼時候人口會增加、停滯或減少? (Hint: 利用(a)所求出的 $P(t)$)

(c) (2%) 若在1847年時, 該地區有人口 800 萬, $\alpha - \beta = 1.6\%$, $m = 210,000$。問在 1850 年時的人口是增加、停滯或減少?

Solution:

(a)

$$\frac{dP}{dt} = (\alpha - \beta)P - m = ((\alpha - \beta)(P - \frac{n}{\alpha - \beta}) \tag{1}$$

You may get 2 pts for writing down Eq (1).

$$\int \frac{dP/dt}{P - \frac{m}{\alpha-\beta}}dt = \int (\alpha - \beta)dt \tag{2}$$

$$\ln\left|P - \frac{n}{\alpha - \beta}\right| = (\alpha - \beta)t + c \tag{3}$$

$$P - \frac{m}{\alpha - \beta} = c^* e^{(\alpha-\beta)t} \tag{4}$$

You may get 6 pts for writing down all Eqs (2), (3), (4). Additional 3 pts for the plugging in initial condition. At $t = 0$, $P - \frac{m}{\alpha - \beta} = c^*$. And the final 1 pt for

$$P = \frac{m}{\alpha - \beta} + (P_0 - \frac{m}{\alpha - \beta})e^{(\alpha-\beta)t} \tag{5}$$

(b) Take differentiation on both side of Eq (5) can get 3 pts.

$$\frac{dP(t)}{dt} = (\alpha - \beta)(P_0 - \frac{m}{\alpha - \beta})e^{(\alpha-\beta)t} \tag{6}$$

Because both $(\alpha - \beta)$ and $e^{(\alpha-\beta)t}$ are greater than zero. Thus

population increases while $P_0 > \frac{m}{\alpha - \beta}$.

population maintains while $P_0 = \frac{m}{\alpha - \beta}$.

population decreases while $P_0 < \frac{m}{\alpha - \beta}$.

You may get 1 pt for each description.

(c) You should write down all the things to get 2 pts.

$$\frac{m}{\alpha - \beta} = \frac{210000}{1.6/100} = 13126000 > 8000000. \tag{7}$$

Thus, population decreases.

3. (15%) 若 X_1, X_2 為白努利單次試驗, X_1, X_2 取值 1 的機率為 $\frac{1}{4}$; X_1, X_2 取值 0 的機率 $\frac{3}{4}$ 且 X_1, X_2 獨立。令 $Y = X_1 + X_2$,

(1) $P(Y=1)$, $P(Y=2)$

(2) Y 和 X_1 是否獨立?

Solution:

(1)

$$\begin{aligned} P(Y=1) &= P(X_1+X_2=1) = P(X_1=0,\, X_2=1) + P(X_1=1,\, X_2=0) \\ &\text{since } X_1 \text{and } X_2 \text{are independent} \\ &= P(X_1=0)P(X_2=1) + P(X_1=1)P(X_2=0) \qquad \text{(2 pts)} \\ &= \frac{3}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{3}{4} \\ &= \frac{3}{8} \qquad \text{(2 pts)} \end{aligned}$$

$$\begin{aligned} P(Y=2) &= P(X_1+X_2=2) = P(X_1=1,\, X_2=1) \\ &\text{since } X_1 \text{and } X_2 \text{are independent} \\ &= P(X_1=1)P(X_2=1) \qquad \text{(2 pts)} \\ &= \frac{1}{4}\cdot\frac{1}{4} \\ &= \frac{1}{16} \qquad \text{(2 pts)} \end{aligned}$$

(2) No, because it is not ture that $P(Y=a,\, X_1=b) = P(Y=a)P(X_1=b)$ for all a, b. For example,

$$P(Y=2,\, X_1=0) = P(X_1=0, X_2=2) = 0 \quad \text{(2 pts)}$$

but

$$P(Y=2)P(X_1=0) = \frac{1}{16}\cdot\frac{3}{4} = \frac{3}{64}\cdot \quad \text{(2 pts)}$$

They are not equal. Hence Y and X_1 are not independent. (3 pts)

4. (10%) 已知 $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx$ 瑕積分存在, 求其值。(假設 $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ 。)

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx &= \int_{-\infty}^{0} x^2 e^{-x^2}\, dx + \int_0^{\infty} x^2 e^{-x^2}\, dx \qquad (\spadesuit) \\ &= 2\int_0^{\infty} x^2 e^{-x^2}\, dx \\ &= \lim_{b\to\infty} \int_0^b x^2 e^{-x^2}\, dx \end{aligned}$$

解法1

$$\begin{aligned} \lim_{b\to\infty} \int_0^b x^2 e^{-x^2}\, dx &= \lim_{b\to\infty} \left[x\cdot(\frac{-1}{2}e^{-x^2})\big|_0^b + \frac{1}{2}\int_0^b e^{-x^2}\, dx \right] \\ &= \lim_{b\to\infty} (\frac{-1}{2}be^{-b^2}) + \int_0^{\infty} e^{-x^2}\, dx \\ &= 0 + \frac{1}{2}\cdot\frac{\pi}{2} \end{aligned}$$

解法2

$$\begin{aligned}\lim_{b\to\infty}\int_0^b x^2e^{-x^2}\ dx &= \lim_{b\to\infty}\int_0^{b^2}\frac{y^{\frac12}e^{-y}}{2}\ dy\ \ (\text{let y} = \text{x}^2)\\ &= \frac12\int_0^\infty y^{\frac12}e^{-y}\ dy\\ &= \frac12\Gamma(\frac32)\left(\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-\alpha}\ dx\right)\\ &= \frac12\cdot\frac{\pi}{2}\end{aligned}$$

所以

$$\int_{-\infty}^{\infty}x^2e^{-x^2}\ dx = 2\cdot\frac{\pi}{4}=\frac{\pi}{2}.$$

★評分標準

(1)(♠)有寫出來可得3分

(2)解法一依照寫對完整性得1到4分，解法二以此類推且沒說明積分為$\Gamma(\frac32)$ 扣兩分

(3)以上都寫對答案算錯或寫錯扣2到3分

5. (15%) 血管阻塞的時間 T 為一隨機變數, 取值在 1 秒 到 30 秒之間, 設其機率密度函數為 $f_T(t)=\frac{c}{t}$, $1\le t\le 30$, 其中 c 為一常數

(1) 求 c 之值

(2) 求 T 的期望值 $E(T)$

(3) 求 $T\ge 10$ 秒的機率

Solution:

(1) $1=\int_1^{30}f(t)dt\ (3\%)=\int_1^{30}\frac{c}{t}dt=c\ln t\Big|_1^{30}=c\ln 30\Rightarrow c=\frac{1}{\ln 30}\ (2\%)$

(2) $E(T)=\int_1^{30}t\cdot\frac{c}{t}dt\ (3\%)=\frac{1}{\ln 30}t\Big|_1^{30}=\frac{29}{\ln 30}\ (2\%)$

(3) $P(T\ge 10)=\int_{10}^{30}\frac{c}{t}dt\ (3\%)=\frac{1}{\ln 30}\ln t\Big|_{10}^{30}=\frac{\ln 3}{\ln 30}\ (2\%)$

6. (10%) 假設 X 為一隨機變數, 其機率密度函數為

$$f_X(x)=\begin{cases}\lambda e^{-\lambda x} & x\ge 0\\ 0 & x<0\end{cases}$$

其中 $\lambda>0$。令 $Y=cX$, $c>0$。求 Y 的機率密度函數, $f_Y(y)$。

Solution:

$$f_X(x)=\begin{cases}\lambda e^{-\lambda x} & x\ge 0\\ 0 & x<0\end{cases}$$

Let $Y=cX, c>0$. Find $f_Y(y)$.

sol.

$$
\begin{aligned}
F_Y(y) &= p\{Y \le y\} \\
&= p\{cx \le y\} \\
&= p\{x \le y/c\} \\
&= \int_0^{\frac{y}{c}} \lambda e^{-\lambda x} dx \\
&= 1 - e^{-\frac{\lambda}{c} y} \quad \text{(5 points)}
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) = F_Y'(y) &= (1 - e^{-\frac{\lambda}{c} y})_y \\
&= \frac{\lambda}{c} e^{-\frac{\lambda}{c} y}
\end{aligned}
$$

Then

$$
f_Y(y) = \begin{cases} \dfrac{\lambda}{c} e^{-\frac{\lambda}{c} y} & y \ge 0 \\ 0 & y < 0 \quad \text{(5 points)} \end{cases}
$$

7. (15%) 工廠生產某種繩子, 假設每 100 尺的平均瑕疵數為200個。 每尺的瑕疵數 Y 遵循著 Poisson 分佈。設出售每一尺的利潤是 $X = 50 - 2Y$。 求每售出一尺的平均利潤。

Solution:

$$
\begin{aligned}
&m = \frac{200}{100} = 2 \qquad \text{(The average : + 3pts)} \\
&\mathbb{P}_Y(k) = \frac{2^k}{k!} e^{-2} \qquad \text{(The p.m.f of the Poisson r.v Y : + 3pts)} \\
&\mathbb{E}(Y) = \sum_{k=0}^{\infty} k \frac{2^k}{k!} e^{-2} = 2e^{-2} \sum_{k=0}^{\infty} \frac{2^{k-1}}{(k-1)!} = 2e^{-2}e^{2} = 2 \qquad \text{(The mean of the Poisson : + 5pts)} \\
&\mathbb{E}(X) = 50 - 2\mathbb{E}(Y) = 46 \qquad \text{(The answer : + 2pts + 2pts)}
\end{aligned}
$$