1011微乙01-05班期末考解答和評分標準

1. (10%) 求 $\lim_{x\to 0} \dfrac{\sin x - x\cos x}{\tan^{-1} x - \sin x}$.

Solution:
由定理4.3(l'Hopital 法則):

$$\lim_{x\to 0} \frac{\sin x - x\cos x}{\tan^{-1} x - \sin x} = \lim_{x\to 0} \frac{\cos x - \cos x + x\sin x}{(\frac{1}{1+x^2}) - \cos x} = \lim_{x\to 0} \frac{\sin x + x\cos x}{\frac{-2x}{(1+x^2)^2} + \sin x} = \lim_{x\to 0} \frac{(\frac{\sin x}{x}) + \cos x}{\frac{-2}{(1+x^2)^2} + (\frac{\sin x}{x})} = \frac{1+1}{-2+1} = -2$$

評分標準:
(1)使用l'Hopital法則:第一次微對3分，第二次對3分，接下來的運算及答案4分。
(2)使用泰勒展式:一個展式對3分共9分，接下來的運算及答案1分。

2. (10%) 計算 $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$.

Solution:

2. $\displaystyle\int \frac{dx}{x^2+3x+2} = \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx[5\%] = \ln\left|\frac{x+1}{x+2}\right| + C[5\%] = \ln|x+1| - \ln|x+2| + C.$

Note.

1. No constant C or absolute value for ln $\Rightarrow -2\%$
2. Other methods: partial grades, 3% or 5%.

3. (10%) 計算 $\displaystyle\int x^2 \sin x \ dx$.

Solution:

$$\begin{aligned}\int x^2 \sin x dx &= -\int x^2 d(\cos x) = -[x^2 \cos x - 2\int \cos x (dx^2)] \text{.......................}(6\ pts)\\ &= -x^2 \cos x + 2\int x \cos x dx = -x^2 \cos x + 2\int x (d \sin x)\\ &= -x^2 \cos x + 2[x \sin x - \int \sin x dx] \text{...}(3\ pts)\\ &= -x^2 \cos x + 2x \sin x + 2\cos x + C,\ C \in \mathbb{R} \text{...............................}(1\ pt)\end{aligned}$$

4. (10%) 求函數 $y = \sin x$, $y = \cos x$ 的曲線在 $x = 0$ 到 $x = \pi$ 之間所夾的面積大小.

Solution:

The area is equal to the following integration.

$$\int_0^{\pi} |\sin x - \cos x| dx$$

Because $\cos x \geq \sin x$ if $0 \leq x \leq \frac{\pi}{4}$ and $\sin x \geq \cos x$ if $\frac{\pi}{4} \leq x \leq \pi$, the area is equal to

$$\int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x) dx \tag{1}$$

Then

$$\begin{aligned}
\int_0^{\pi} |\sin x - \cos x| dx &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x) dx \\
&= (\sin x + \cos x) \Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x) \Big|_{\frac{\pi}{4}}^{\pi} \\
&= (\sqrt{2} - 1) + (1 + \sqrt{2}) \\
&= 2\sqrt{2}
\end{aligned}$$

Grading evaluation:

1. If you write down the equation (1), you will get 2 points.
2. If one of the two integrations in the equation (1) is right, you will get 4 points, respectively.

5. (15%) 求函數 $f(x)=\frac{1}{2}x^2$ 由 $x=0$ 到 $x=1$ 的曲線長度.

Solution:

$f'(x)=x$

$$\begin{aligned}
\text{length } l &= \int_0^1 \sqrt{1+x^2}dx \text{ (7 pts)}\\
&= \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2\theta}\sec^2\theta d\theta \text{ (if we let } x=\tan\theta, \theta\in(-\frac{\pi}{2},\frac{\pi}{2})\text{), then } dx=\sec^2\theta d\theta)\\
&= \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta \text{ (3 pts)}\\
&= \int_0^{\frac{\pi}{4}} \sec\theta d\tan\theta\\
&= \sec\theta\tan\theta|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec\theta\tan^2\theta d\theta\\
&= \sec\theta\tan\theta|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec\theta(\sec^2\theta-1)d\theta\\
&= \sec\theta\tan\theta|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta + \int_0^{\frac{\pi}{4}} \sec\theta d\theta\\
&= (\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta
\end{aligned}$$

$$\begin{aligned}
\Rightarrow l = \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta &= \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)|_0^{\frac{\pi}{4}}\\
&= \frac{\sqrt{2}}{2} + \frac{\ln(\sqrt{2}+1)}{2} \text{ (5 pts)}
\end{aligned}$$

6. (15%) 求 $x = \dfrac{\sqrt{2y}}{1+y^2}$, $y = 1$ 及 $x = 0$ 所圍成的區域繞 y 軸旋轉之體積.

Solution:

$$
\begin{aligned}
\text{The volume} &= \pi \int_{y=0}^{y=1} x^2\,dy \\
&= \pi \int_0^1 \left(\frac{\sqrt{2y}}{1+y^2}\right)^2 dy \quad \text{(10 points)} \\
&= \pi \int_0^1 \frac{2y}{(1+y^2)^2}\,dy \\
&= \pi \int_0^1 \frac{du}{(1+u)^2} \qquad\qquad \text{(3 points)} \\
&= -\frac{\pi}{1+u}\bigg|_0^1 \\
&= \frac{\pi}{2} \qquad\qquad\qquad\qquad \text{(2 points)}
\end{aligned}
$$

7. (10%) 求 $y = \dfrac{\tan x}{x}$, $y = \dfrac{4}{\pi}$ 及 $x = 0$ 所圍成的區域繞 y 軸旋轉之體積.

Solution:

We have the following simple observations:

(i) $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$

(ii) $\left(\dfrac{\tan x}{x}\right)' = \dfrac{2x - \sin(2x)}{2x^2\cos^2 x} > 0, \quad \forall x \in (0, \frac{\pi}{2}) \implies \dfrac{\tan x}{x} \nearrow$ on $(0, \frac{\pi}{2})$

(iii) From (ii), $\dfrac{\tan x}{x} = \dfrac{4}{\pi}$ has a unique solution which is $x = \dfrac{\pi}{4}$ clearly.

(iv) $\dfrac{\tan x}{x} \leq \dfrac{4}{\pi}$ on $(0, \frac{\pi}{4}]$

Hence,

$$\begin{aligned}
Area &= 2\pi \int_0^{\pi/4} x\left(\frac{4}{\pi} - \frac{\tan x}{x}\right) dx \\
&= 2\pi \int_0^{\pi/4} \left(\frac{4}{\pi}x - \tan x\right) dx \\
&= 2\pi\frac{2}{\pi}\Big[x^2\Big]_0^{\pi/4} + 2\pi\Big[\ln|\cos x|\Big]_0^{\pi/4} \\
&= \frac{\pi^2}{4} - \pi\ln 2 && (2) \\
&= \frac{\pi^2}{4} + 2\pi\ln\frac{1}{\sqrt{2}} && (3)
\end{aligned}$$

Grade:

- **10** points if you have the right answer like (2) or (3) for example.
- **5** points if you have a mistake(s) with +/− in the coefficients.
- **0** points otherwise.

Remark.

You don't have to claim that $x = \dfrac{\pi}{4}$ is the unique solution to $\dfrac{\tan x}{x} = \dfrac{4}{\pi}$ on $(0, \frac{\pi}{2})$.

8. 令函數$f(x) = e^{-x^2}$.

(a) (10%) 求 $f(x)$ 對 $x = 0$ 的泰勒展開式, 並寫出一般項.

(b) (10%) 以(a)中非零的前三項估計積分 $\int_0^{\frac{1}{2}} f(x)\, dx$, 誤差忽略不計.

Solution:

a.

We know

$$e^x = \sum_0^{\infty} \frac{x^n}{n!}$$

Therefore,

$$e^{-x^2} = \sum_0^{\infty} \frac{(-x^2)^n}{n!}$$

b.

$$\int_0^{\frac{1}{2}} e^{-x^2} dx \approx \int_0^{\frac{1}{2}} (1 - x^2 + \frac{x^4}{2!}) dx$$
$$= \frac{443}{960}$$