1041微乙01-05班期中考解答和評分標準

1. (13%) 計算下列函數之導函數

(a) (5%) $\ln\left(\sqrt{1-x^2}-x\right)$

(b) (8%) $(\tan x)^{\sin x}$

Solution:

(a) Let $f(x)=\ln(\sqrt{1-x^2}-x)$

$f'(x)=\dfrac{1}{\sqrt{1-x^2}-x}(\sqrt{1-x^2}-x)'$ (2 pts)

$=\dfrac{1}{\sqrt{1-x^2}-x}(\dfrac{-x}{\sqrt{1-x^2}}-1)$ (3 pts)

(b) Let $f(x)=(\tan x)^{\sin x}\Rightarrow \ln f(x)=\sin x\ln(\tan x)$

$(\ln f(x))'=\dfrac{f'(x)}{f(x)}=(\sin x\ln(\tan x))'$ (4 pts)

$=\cos x\ln\tan x+\sin x\dfrac{\sec^2 x}{\tan x}$

$\therefore f'(x)=(\tan x)^{\sin x}(\cos x\ln\tan x+\sin x\dfrac{\sec^2 x}{\tan x})$ (4 pts)

2. (10%) 計算下列之極限值

(a) (5%) $\lim\limits_{x\to 0} x\left(\cos x+\cos\dfrac{1}{x}\right)$

(b) (5%) $\lim\limits_{x\to 1}\dfrac{\frac{x}{\sqrt{x^2+1}}-\frac{\sqrt{2}}{2}}{x-1}$

Solution:

(a) $\forall x$, $-2\le\cos x+\cos\dfrac{1}{x}\le 2\Rightarrow -2|x|\le x(\cos x+\cos\dfrac{1}{x})\le 2|x|$

By pinching theorem, $\lim\limits_{x\to 0}x(\cos x+\cos\dfrac{1}{x})=0$

(b) Let $f(x)=\dfrac{x}{\sqrt{x^2+1}}$

Then $\lim\limits_{x\to 1}\dfrac{\frac{x}{\sqrt{x^2+1}}-\frac{\sqrt{2}}{2}}{x-1}=\lim\limits_{x\to 1}\dfrac{f(x)-f(1)}{x-1}=f'(1)$ (3 pts)

$f'(x)=\dfrac{1}{x^2+1}(\sqrt{x^2+1}-x^2(x^2+1)^{\frac{-1}{2}})$

$\therefore f'(1)=\dfrac{\sqrt{2}}{4}$ (2 pts)

3. (8%) 令 $f(x)=\dfrac{(x+1)(x^3+x^2+1)(x^2-x+1)}{e^{\frac{x+1}{x+2}}\sqrt{x^2+1}}$ 求 $f'(-1)$.

Solution:

Solution 1: (By definition of derivative)

$$\begin{aligned}
f'(-1)&=\lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)}\\
&=\lim_{x\to -1}\frac{(x+1)(x^3+x^2+1)(x^2-x+1)}{(x+1)e^{\frac{x+1}{x+2}}\sqrt{x^2+1}}\\
&=\lim_{x\to -1}\frac{(x^3+x^2+1)(x^2-x+1)}{e^{\frac{x+1}{x+2}}\sqrt{x^2+1}}\\
&=\frac{3\sqrt{2}}{2}
\end{aligned}$$

Solution 2: (Take logarithm of both sides)

$$
\begin{aligned}
f(x) &= \frac{(x+1)(x^3+x^2+1)(x^2-x+1)}{e^{\frac{x+1}{x+2}}\sqrt{x^2+1}} \\
\ln f(x) &= \ln(x+1)+\ln(x^3+x^2+1)+\ln(x^2-x+1)-\frac{x+1}{x+2}-\frac{1}{2}\ln(x^2+1) \\
\frac{1}{f(x)}\cdot f'(x) &= \frac{1}{x+1}+\frac{3x^2+3x}{x^3+x^2+1}+\frac{2x-1}{x^2-x+1}-\frac{1}{(x+2)^2}-\frac{x}{x^2+1} \\
\therefore f'(x) &= f(x)\cdot\Big(\frac{1}{x+1}+\frac{3x^2+3x}{x^3+x^2+1}+\frac{2x-1}{x^2-x+1}-\frac{1}{(x+2)^2}-\frac{x}{x^2+1}\Big) \\
&= \frac{(x^3+x^2+1)(x^2-x+1)}{e^{\frac{x+1}{x+2}}\sqrt{x^2+1}}+f(x)\Big(\frac{3x^2+3x}{x^3+x^2+1}+\frac{2x-1}{x^2-x+1}-\frac{1}{(x+2)^2}-\frac{x}{x^2+1}\Big) \\
\therefore f'(-1) &= \frac{3\sqrt{2}}{2}+0\cdot(0-1-1+\frac{1}{2})=\frac{3\sqrt{2}}{2}
\end{aligned}
$$

Solution 3: (By quotient rule)

$$
\begin{aligned}
f'(x) &= \frac{e^{\frac{x+1}{x+2}}\sqrt{x^2+1}\big((x+1)(x^3+x^2+1)(x^2-x+1)\big)'-\big(e^{\frac{x+1}{x+2}}\sqrt{x^2+1}\big)'(x+1)(x^3+x^2+1)(x^2-x+1)}{\big(e^{\frac{x+1}{x+2}}\sqrt{x^2+1}\big)^2} \\
&= \frac{(6x^5+5x^4+6x^2+2x)-(x+1)(x^3+x^2+1)(x^2-x+1)\big(\frac{1}{(x+2)^2}+\frac{x}{x^2+1}\big)}{e^{\frac{x+1}{x+2}}\sqrt{x^2+1}}
\end{aligned}
$$

$$\therefore f'(-1)=\frac{3\sqrt{2}}{2}$$

Marking Criteria:

1. quotient rule [2points]
2. $\big((x+1)(x^3+x^2+1)(x^2-x+1)\big)' = 6x^5+5x^4+6x^2+2x$ [1point]
3. $\big(e^{\frac{x+1}{x+2}}\sqrt{x^2+1}\big)' = e^{\frac{x+1}{x+2}}\sqrt{x^2+1}\big(\frac{1}{(x+2)^2}+\frac{x}{x^2+1}\big)$ [3points]

4. (12%) 令 $\frac{\sqrt{3}}{2}+xy=\sin y$.
   (a) (6%) 求過點 $(0,\frac{\pi}{3})$ 之切線方程式。
   (b) (6%) 求 $\frac{d^2y}{dx^2}$ 之公式及其在點 $(0,\frac{\pi}{3})$ 之值。

Solution:

(a) 對 $\frac{\sqrt{3}}{2}+xy=\sin y$ 做隱函數微分，得到:

$$y+xy'=\cos y\cdot y' \quad (3分)$$

將上式整理後，我們有

$$y'=\frac{y}{\cos y-x}$$

帶值 $(0,\frac{\pi}{3})$:

$$y'=\frac{\frac{\pi}{3}}{\cos\frac{\pi}{3}}=\frac{2}{3}\pi \quad (1分)$$

故切線方程式為: $y-\frac{\pi}{3}=\frac{2}{3}\pi x$ (2分)。

(b) 直接對(a)的式子做隱函數微分，得到:

$$12x^2-(2y^2+4xyy')-(4xyy'+2x^2y'y'+2x^2yy'')+(12y^2y'y'+4y^3y'')=0 \quad (3分)$$

或者

$$y'' = \frac{y'(\cos y - x) - y(-\sin y \cdot y' - 1)}{(\cos y - x)^2}$$

$\frac{d^2y}{dx^2}$ 的公式為:

$$\begin{aligned} y'' &= \frac{y'(\cos y - x) - y(-\sin y \cdot y' - 1)}{(\cos y - x)^2} \\ &= \frac{\frac{y}{\cos y - x}(\cos y - x) - y(-\sin y \cdot \frac{y}{\cos y - x} - 1)}{(\cos y - x)^2} \\ &= \frac{2y + \frac{\sin y \cdot y^2}{\cos y - x}}{(\cos y - x)^2} \\ &= \frac{2y\cos y - 2xy + y^2 \sin y}{(\cos y - x)^3} \quad (2分) \end{aligned}$$

帶値 $(0, \frac{\pi}{3})$:

$$y'' = \frac{8}{3}\pi + \frac{4\sqrt{3}}{9}\pi^2 \quad (1分)$$

5. (12%) 令 $f(x) = x^3 + x + \cos x$, $x \in \mathbb{R}$.
   (a) (6%) 說明 $f(x)$ 是一對一函數。
   (b) (6%) 令 $g(x)$ 是 $f(x)$ 之反函數，求 $g'(1)$。

Solution:

(a) $f'(x) = 3x^2 + 1 - \sin x$ (1 pt)
Since $\forall x \in \mathbb{R}, -1 \le \sin x \le 1 \Rightarrow 3x^2 + 1 - \sin x \ge 0$ (3 pts)
and there is no solution such that $f'(x) = 0$.
So, $f'(x) > 0, \forall x \in \mathbb{R} \Rightarrow f$ is strictly increasing.
Hence, f is a one-to-one function. (2 pts)

Note that you can get 2 points if you wrote down the correct definition of one-to-one function. And you can get all 6 points even though you didn't prove that f is "strictly" increasing.

(b) $g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = 1$. (4 pts)
Note that $f(0) = 1$, so $g(1) = 0$. (2 pts)

Note that you can't get any point if you wrote the wrong formula without deriving it.

6. (10%) 說明 $y = 1 - x$ 與 $y = \cos x$ 的圖形只交於一點。

Solution:

Since the range of $y = \cos x$ is $[-1, 1]$, and $-1 \le 1 - x \le 1 \Rightarrow 0 \le x \le 2$.
so we only need to prove that:
$y = 1 - x$ and $y = \cos x$ intersect at only one point when $x \in [0, 2]$. (2 pts)
First, we let $h(x) = 1 - x - \cos x$ and it is clear that 0 is a root of h,
i.e. $h(0) = 1 - 0 - \cos 0 = 0$. (3 pts)
$h'(x) = -1 + \sin x$ (2 pts)
Consider $x \in [0, 2]$, $h'(x) \le 0$ and $h'(x) = 0$ only when $x = \frac{\pi}{2}$ (2 pts)
We have $h(0) = 0$, $h(x)$ is decreasing on $[0, 2]$, and $h'(0) = -1 < 0$. (1 pt)
Hence, $h(x)$ has only one root 0, i.e. $y = 1 - x$ and $y = \cos x$ intersect at only one point $(0, 1)$.

Note that you can get 2 points if you only sketched the graph.

7. (25%) 若 $y = f(x) = \dfrac{x(x-1)+2}{x+1}$

(a) $y = f(x)$ 在 ________________ (區間)遞增

$y = f(x)$ 在 ________________ (區間)遞減 (6%)

(b) $y = f(x)$ 在 ________________ (區間)凹向上

$y = f(x)$ 在 ________________ (區間)凹向下 (6%)

(c) $y = f(x)$ 之極大值(若存在的話): ________________ (座標)

$y = f(x)$ 之極小值(若存在的話): ________________ (座標) (6%)

(d) $y = f(x)$ 之所有漸近線為 ________________ (4%)

(e) 畫出 $y = f(x)$ 之圖形 (3%)

Solution:

(a)

$$y = \frac{x(x-1)+2}{x+1}$$
$$y' = \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2}$$

f is increasing if $y' > 0$, decreasing if $y' < 0$
so in $(-\infty,-3) \cup (1,\infty)$, f is increasing, while in $(-3,-1) \cup (-1,1)$, f is decreasing.

(b)

$$y'' = \frac{8}{(x+1)^3}$$

f is concave upward if $y'' > 0$, concave downward if $y'' < 0$.
so in $(-1,\infty)$, f is concave upward, while in $(-\infty,-1)$, f is concave downward.

(c)

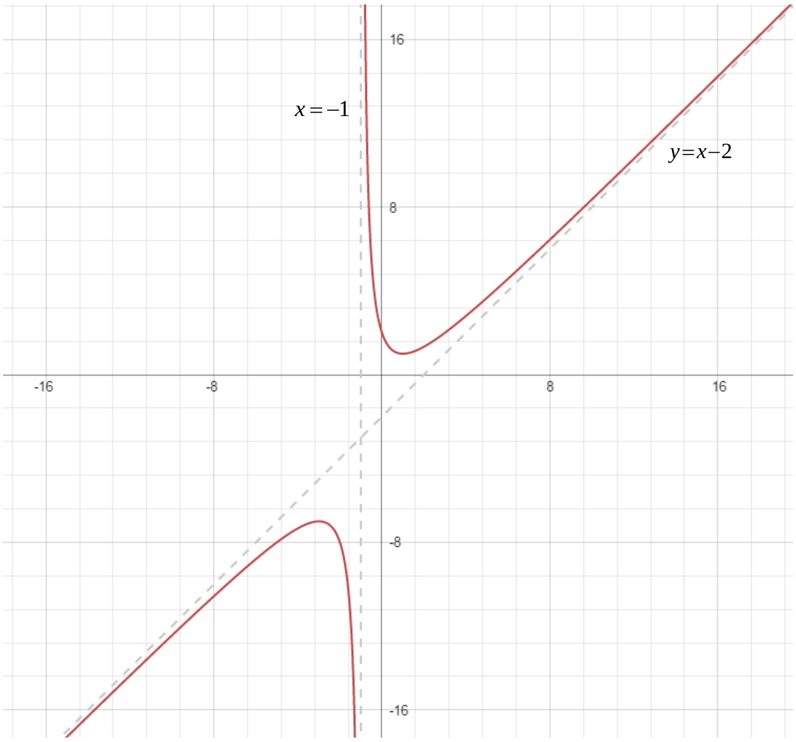
To be local maximum or minimum, $y' = 0$. So the possible points are in $x = 1$ or $x = -3$.
For $x = 1$, $y''(1) > 0$, so it is a local minimum. Since $y(1) = 1$, its coordinate is $(1,1)$.
For $x = -3$, $y''(-3) < 0$, so it is a local maximum. Since $y(-3) = -7$, its coordinate is $(-3,-7)$.

(d)

Since $\lim\limits_{x\to -1} y(x) = \pm\infty$, $x = -1$ is a vertical asymptote.

Also, let the oblique asymptote be $y = mx + k$. Since $\lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x} = 1$ and $k = \lim\limits_{x\to\pm\infty} (f(x) - x) = -2$, the oblique asymptote is $y = x - 2$.

(e) The graph is drawn by wolfram alpha.



評分標準:
(a) 一次微分算對+2 ,範圍正確 +4
(b) 二次微分算對+2 ,範圍正確 +4
(c) 極大極小判斷正確才給分,x 點正確但函數值代錯各給部分分1分
(d) 每條漸近線各2
(e) 漸近線,各區間的遞增遞減性質,凹凸性錯一個扣一分,到扣完為止

8. (10%) 有一船在 5 : 00 PM 駛離碼頭，以時速 20 公里向北航行；同時，另一船則以時速 15 公里向西航行並於 6 : 00 PM 抵達同一個碼頭。求兩船距離最近的時刻。

Solution:

設定區(佔5分)
令向北行的船名為A，向西行的船名為B。設碼頭座標為(0,0)，已知B向西行駛一小時後抵達碼頭，故可令B在5:00PM的座標為(15,0)。
現在，可以設A與B從5:00PM到6:00PM的座標分別為(0,20t),(15-15t,0), $0 \le t \le 1$ (4分)
距離函數:$d(t) = \sqrt{(20t)^2 + (15 - 15t)^2}$ (1分)
題目所求兩船距離最近的時刻即為求d(t)在 $0 \le t \le 1$ 區間內最小值發生處。

求d(t)最小值發生處(佔5分)
方法一
注意到 d(t)≥ 0 ，所以 D(t)=(d(t))^2 與 d(t) 最小值發生處相同。
$D'(t) = 1250t - 450 = 0$
t=$\frac{9}{25}$ is a critical point.(3分)
$D'(t) < 0$ as $0 \le t < \frac{9}{25}$. $D'(t) > 0$ as $\frac{9}{25} < t \le 1$.
$\Rightarrow$ t=$\frac{9}{25}$ is the absolute minimum point.(2 分)
(若能說明此二次多項式領導係數為正，故為凹向上拋物線，切線斜率為0處必為最小值，也能拿分。)

方法二

$$
\begin{aligned}
d(t) &= \sqrt{(20t)^2 + (15 - 15t)^2} \\
&= \sqrt{625t^2 - 450t + 225} \\
&= \sqrt{625(t - \frac{9}{25})^2 + 144}
\end{aligned}
$$

$\Rightarrow$ t=$\frac{9}{25}$ is the absolute minimum point.(5 分)

兩船最接近的時刻: 5:21:36 PM