1021微甲07-11班期中考解答

1. (9%) Find $\dfrac{d}{dx}(\sec x)^x$, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

Solution:

$(\sec x)^x = e^{x \ln \sec x}$ (2%)

$\dfrac{d}{dx}(\sec x)^x = \dfrac{d}{dx} e^{x \ln \sec x} = e^{x \ln \sec x} \dfrac{d}{dx}(x \ln \sec x)$ (3%)

$\dfrac{d}{dx}(\sec x)^x = (\sec x)^x \left(\ln \sec x + x \dfrac{\sec x \tan x}{\sec x} \right) = (\sec x)^x (\ln x + x \tan x)$ (4%)

2. (9%) Evaluate $\displaystyle\int_0^1 \frac{(2+x)^2}{1+x^2} dx$.

Solution:

$(2+x)^2 = x^2 + 4x + 4 \quad \Rightarrow \quad \displaystyle\int_0^1 \frac{(2+x)^2}{1+x^2} dx = \int_0^1 1 + \frac{4x+3}{1+x^2} dx$

$\displaystyle\int_0^1 \frac{(2+x)^2}{1+x^2} dx = \int_0^1 1 dx + 4\int_0^1 \frac{x}{1+x^2} dx + 3\int_0^1 \frac{1}{1+x^2} dx$ (3%)

$\displaystyle\int_0^1 1 dx = x\Big|_0^1 = 1$ (2%)

$\displaystyle 4\int_0^1 \frac{x}{1+x^2} dx = 2\ln(1+x^2)\Big|_0^1 = 2\ln 2$ (2%)

$\displaystyle 3\int_0^1 \frac{1}{1+x^2} dx = 4\arctan x\Big|_0^1 = 3\arctan 1 = 3 \cdot \frac{\pi}{4} = \frac{3}{4}\pi$ (2%)

$\displaystyle\int_0^1 \frac{(2+x)^2}{1+x^2} dx = 1 + 2\ln 2 + \frac{3}{4}\pi$

3. (9%) (a) Find $\displaystyle\lim_{t \to 0^+} \frac{t - \ln(1+t)}{t^2}$

(b) Use (a) to find $\displaystyle\lim_{t \to 0^+} \frac{\sqrt{t - \ln(1+t)}}{t}$

Solution:

(a)5 points

this is indeterminate form of type $\dfrac{0}{0}$ (1 point)

$$\lim_{t \to 0^+} \frac{t - \ln(1+t)}{t^2} = \lim_{t \to 0^+} \frac{1 - \frac{1}{1+t}}{2t} = \lim_{t \to 0^+} \frac{1}{2(1+t)} = \frac{1}{2}$$

(process 2 points,answer 2 points)

(b)4 points

because $f(x) = \sqrt{x}$ is continous at $x = 1/2$ and

$$\lim_{t \to 0^+} \frac{t - \ln(1+t)}{t^2} = \frac{1}{2}$$

so

$$\lim_{t \to 0^+} \frac{\sqrt{t - \ln(1+t)}}{t} = \lim_{t \to 0^+} \sqrt{\frac{t - \ln(1+t)}{t^2}} = \sqrt{\lim_{t \to 0^+} \frac{t - \ln(1+t)}{t^2}} = \sqrt{\frac{1}{2}}$$

(process 2 points,answer 2 points)

4. (9%) Evaluate $\lim_{x \to \infty} (2^x + 3^x + 5^x)^{\frac{1}{x}}$.

Solution:

解法一

$2^x + 3^x + 5^x = 5^x \left(1 + \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right)$ (2分)

$(2^x + 3^x + 5^x)^{\frac{1}{x}} = 5\left(1 + \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right)^{\frac{1}{x}}$ (3分)

$$\begin{cases} \ln\left(\lim_{x\to\infty}\left(1 + \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right)^{\frac{1}{x}}\right) = \lim_{x\to\infty} \frac{\ln\left(1 + \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right)}{x} = 0 \\ \quad \Rightarrow \lim_{x\to\infty}\left(1 + \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right)^{\frac{1}{x}} = e^0 = 1 \\ \text{或直接說} \\ \lim_{x\to\infty}\left(1 + \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right)^{\frac{1}{x}} = 1^0 = 1 \end{cases} \quad (3分)$$

$$\lim_{x\to\infty} (2^x + 3^x + 5^x)^{\frac{1}{x}} = 5 \times 1 = \boxed{5} \qquad (1\ 分)$$

解法二

$\ln(2^x + 3^x + 5^x)^{\frac{1}{x}} = \frac{\ln(2^x + 3^x + 5^x)}{x}$ (2分)

$\lim_{x\to\infty} \frac{\ln(2^x + 3^x + 5^x)}{x} \overset{l'Hospital}{=\!=} \lim_{x\to\infty} \frac{\frac{(\ln 2)2^x + (\ln 3)3^x + (\ln 5)5^x}{2^x + 3^x + 5^x}}{1}$ (3分)

$= \lim_{x\to\infty} \frac{(\ln 2)\left(\frac{2}{5}\right)^x + (\ln 3)\left(\frac{3}{5}\right)^x + (\ln 5)}{\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x + 1}$ (2分)

$= \ln 5$ (1分)

$\lim_{x\to\infty} (2^x + 3^x + 5^x)^{\frac{1}{x}} = e^{\ln 5} = \boxed{5}$ (1 分)

解法三

When $x \geq 1$

$2^x + 3^x + 5^x < 5^x + 5^x + 5^x = 3 \times 5^x$ (2分)

$5^x < 2^x + 3^x + 5^x$ (2分)

Because
$\lim_{x\to\infty}(5^x)^{\frac{1}{x}} = 5$ (1分)
$\lim_{x\to\infty}(3\times 5^x)^{\frac{1}{x}} = (\lim_{x\to\infty} 3^{\frac{1}{x}})\times 5 = 1\times 5 = 5$ (3分)
By Squeeze Theorem
$\lim_{x\to\infty}(2^x+3^x+5^x)^{\frac{1}{x}} = \boxed{5}$ (1 分)

5. (9%) Find the linear approximation of the function

$$g(x) = \sin^{-1}\left(\frac{x-1}{x+1}\right) - \tan^{-1}\left(\sqrt{x}\right) \text{ at the point } x = 3$$

Solution:

Linear approximation at $x = 3$ is

$$g(x) \approx g(3) + g'(3)\cdot(x-3) \qquad (1\%)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1}(\frac{x-1}{x+1})) = \frac{1}{\sqrt{1-(\frac{x-1}{x+1})^2}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\frac{x-1}{x+1})$$

$$= \frac{1}{\sqrt{x}(x+1)} \qquad \text{(1\% for the chain rule term, 3\% in total)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1}(\sqrt{x})) = \frac{1}{1+(\sqrt{x})^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{x}(x+1)} \qquad \text{(1\% for the chain rule term, 3\% in total)}$$

$$\Rightarrow g'(x) = \frac{1}{2\sqrt{x}(x+1)} \text{ and } g'(3) = \frac{1}{8\sqrt{3}} \qquad (1\%)$$

$$g(3) = \sin^{-1}(\frac{1}{2}) - \tan^{-1}(\sqrt{3})$$

$$= \frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6} \qquad (1\%)$$

$$\Rightarrow g(x) \approx -\frac{\pi}{6} + \frac{1}{8\sqrt{3}}(x-3)$$

6. (10%) Let $y = f(x)$ satisfy $x^3 + 2xy + y^3 = 13$. Find y' and y'' at the point $x = 1$, $y = 2$.

Solution:

$$x^3 + 2xy + y^3 = 13$$

Implicit differentiation gives

$$3x^2 + 2y + 2xy' + 3y^2y' = 0 \quad (*) \qquad 4\text{ 分}$$

$$y' = \frac{-3x^2 - 2y}{2x + 3y^2}$$

At $(x, y) = (1, 2)$, $y' = \dfrac{-3-4}{2+12} = -\dfrac{1}{2}$ 2 分

Differentiate (*) once more we have

$$6x + 2y' + 2y' + 2xy'' + 6y(y')^2 + 3y^2y'' = 0 \qquad 2 \text{ 分}$$

At $(x, y) = (1, 2)$,

$$6 + 4\left(-\frac{1}{2}\right) + 2y'' + 6 \cdot 2 \cdot \left(-\frac{1}{2}\right)^2 + 3 \cdot 4y'' = 0$$

$$4 + 3 + 14y'' = 0 \qquad y'' = -\frac{1}{2} \qquad 2 \text{ 分}$$

7. (10%) Evaluate $\displaystyle\lim_{x\to 0^+} \frac{\int_0^{x^2} e^t\sqrt{t}\sin\sqrt{t}dt + x\cos x - x}{x^3}$.

Solution:

$\displaystyle\lim_{x\to 0^+} \frac{\int_0^{x^2} e^t\sqrt{t}\sin\sqrt{t}dt + x\cos x - x}{x^3} \left(\frac{0}{0}\right)$

$\displaystyle= \lim_{x\to 0^+} \frac{(e^{x^2}x\sin x)\cdot 2x + (\cos x - x\sin x - 1)}{3x^2}$

$\displaystyle= \lim_{x\to 0^+} \frac{(2x^2e^{x^2}\sin x) - x\sin x + (\cos x - 1)}{3x^2}$

$\displaystyle= \lim_{x\to 0^+} \left(\frac{2}{3}e^{x^2}\sin x - \frac{x\sin x}{3x^2} + \frac{\cos x - 1}{3x^2}\right)$

$\displaystyle= \lim_{x\to 0^+} \frac{2}{3}e^{x^2}\sin x - \lim_{x\to 0^+} \frac{1}{3}\frac{\sin x}{x} + \lim_{x\to 0^+} \frac{1}{3}\frac{\cos x - 1}{x^2}$

$\displaystyle= 0 - \frac{1}{3} + \frac{1}{3}\cdot\left(\frac{-1}{2}\right)$

$\displaystyle= -\frac{1}{2}.$

Grading Policy:

Identify $\left(\frac{0}{0}\right)$ type: 3 points

$\displaystyle\frac{d}{dx}\int_0^{x^2} e^t\sqrt{t}\sin\sqrt{t}dt$: 2 points

$(x\cos x - x)'$: 2 points

Three limits: each get 1 point

8. (15%) The minute hand (分針) on a clock is 8cm long and the hour hand (時針) is 4cm long. How fast is the distance between the tips of the hands changing at two o'clock? Give your answer in the unit cm/hour.

Solution:

Let θ be the angle of minute hand and hour hand.
Let X be the distance of minute hand and hour hand.
By cosine theorem:[3 pts]

$$X^2 = 8^2 + 4^2 - 2 \times 8 \times 4 \times \cos\theta$$

At 2 o'clock. $\theta = \frac{\pi}{3}$ and $X = 4\sqrt{3}$. [2 pts]

The angle vector of minute hand is 2π and the angle vector of hour hand is $\frac{\pi}{6}$ [4 pts]

Moreover, the angle vector of θ is $\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$. [2 pts]

By differntating: [2 pts]

$$2X\frac{dX}{dt} = 64\sin\theta\frac{d\theta}{dt}$$

To end up, $\frac{dX}{dt} = -\frac{22\pi}{3}$ [2 pts].

9. (20%) Let $y = f(x) = x - \frac{x^2}{6} - \frac{2\ln x}{3}$, $x > 0$.
Answer the following questions and give your reasons (including computations). Put "**None**" in the blank if the item asked does **not** exist.

(a) (5%) Find the interval(s) on which f is increasing.

Answer: ______

(b) (4%) Find the local maximal point(s) and minimal point(s) of f, if any.

Answer: local maximal point(s) $(x, y) =$ ______

local minimal point(s) $(x, y) =$ ______

(c) (5%) Find the interval(s) on which f is concave up.

Answer: ______

(d) (2%) Find the inflection point(s) if any.

Answer: ______

(e) (4%) Sketch the graph of f. Indicate all information in (a)-(d).

Solution:

a. $f'(x) > 0$
$\Rightarrow 1 - \frac{x}{3} - \frac{2}{3x} > 0$
$\Rightarrow x - \frac{x^2}{3} - \frac{2}{3} > 0, (x > 0)$
$\Rightarrow 1 < x < 2$

(b) by (a)local maximal point $(x, y) = (2, \frac{4 - 2ln2}{3})$
local minimal point $(x, y) = (1, \frac{5}{6})$

(c) $f''(x) > 0$
$\Rightarrow \dfrac{-1}{3} + \dfrac{2}{3x^2} > 0, (x > 0)$
$\Rightarrow 0 < x < \sqrt{2}$

(d) by (c) the inflection point $(x, y) = (\sqrt{2}, \sqrt{2} - \dfrac{1 - ln2}{3})$

(e) f is concave up on$(0, \sqrt{2})$, f is concave down on $(\sqrt{2}, \infty)$,and $\lim_{x \to 0} f(x) = \infty$.
If you are totally right of (a),(b),(c),(d),I think you will get the full scores of (e).

Sketching the graph of f by yourself.