1. (15%) Let $h(x)=\sqrt{x-x^2}+\sin^{-1}\left(\sqrt{x}\right)$, $0\le x\le 1$.

   (a) Find the length of the curve $y=h(x)$.

   (b) Find the area of the surface generated by rotating the curve $y=h(x)$ about the x-axis.

Solution:

$$h(x)=\sqrt{x-x^2}+\sin^{-1}(\sqrt{x})$$

$$\Rightarrow h'(x)=\frac{1-2x}{2\sqrt{x-x^2}}+\frac{\frac{1}{2\sqrt{x}}}{2\sqrt{1-x}}=\frac{2(1-x)}{\sqrt{x(1-x)}}=\sqrt{\frac{1-x}{x}},\ 0<x<1 \text{.................} (3pts)$$

$$ds=\sqrt{1+[h'(x)]^2} \text{.................} (3pts)$$

$$=\sqrt{\frac{1}{x}}=\frac{1}{\sqrt{x}} \text{.................} (2pts)$$

$$\text{Length}=\int_0^1\sqrt{1+[h'(x)]^2}dx=\int_0^1\frac{1}{\sqrt{x}}dx=2\sqrt{x}\ |_0^1=2 \text{.................} (2pts)$$

$$\text{Area of the surface}=\int_0^1 2\pi h(x)\sqrt{1+[h'(x)]^2}dx \text{.................} (3pts)$$

$$=2\pi[\int_0^1(\sqrt{x-x^2}+\sin^{-1}(\sqrt{x}))\frac{1}{\sqrt{x}}dx]$$

$$=2\pi[\int_0^1\sqrt{1-x}dx+\int_0^1\frac{\sin^{-1}(\sqrt{x})}{\sqrt{x}}]dx$$

$$=2\pi[\ \frac{-2}{3}(1-x)^{\frac{3}{2}}\ |_0^1+\int_0^1\frac{\sin^{-1}(\sqrt{x})}{\sqrt{x}}dx\ ]$$

$$=2\pi[\ \frac{2}{3}+2\int_0^1\sin^{-1}(\sqrt{x})d\sqrt{x}\ ]$$

$$=2\pi[\ \frac{2}{3}+2(\sqrt{x}\sin^{-1}(\sqrt{x})\ |_0^1-\int_0^1\sqrt{x}\frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}}dx)\ ]$$

$$=2\pi[\ \frac{2}{3}+2\sin^{-1}(1)-\int_0^1\frac{1}{\sqrt{1-x}}dx\ ]$$

$$=2\pi[\ \frac{2}{3}+\pi+2\sqrt{1-x}\ |_0^1\ ]$$

$$=2\pi[\ \frac{2}{3}+\pi-2\ ]=2\pi(\pi-\frac{4}{3}) \text{.................} (2pts)$$

2. (10%) Solve $xy' - 3y = 5x^3$

(a) with the initial condition $y(1) = 2$.

(b) with the initial condition $y(-1) = 2$.

Solution:

$$xy' - 3y = 5x^3$$
$$\Rightarrow y' - \frac{3}{x}y = 5x^2 (*)$$

And since

$$\int -\frac{3}{x}dx = -3\ln|x| + K$$

Let

$$I = e^{-3\ln x} = x^{-3} (3\%)$$

(*) Multiply by I on both sides, then we have

$$(x^{-3}y)' = 5x^{-1} \Rightarrow x^{-3}y = 5\ln|x| + C (3\%)$$

2(a)

$\because y(1) = 2 \Rightarrow 2 = 0 + C \Rightarrow C = 2$

$$\therefore y = 5x^3 \ln|x| + 2x^3 (2\%)$$

2(b)

$\because y(-1) = 2 \Rightarrow -2 = 0 + C \Rightarrow C = -2$

$$\therefore y = 5x^3 \ln|x| - 2x^3 (2\%)$$

3. (15%) Let $y = h(x)$ be decreasing on $[0, \frac{\pi}{2})$ and is continuously differentiable on $(0, \frac{\pi}{2})$ with $h(0) = 0$. Let $s(x)$ denote the arc length of $y = h(x)$ from $(0, 0)$ to $(x, h(x))$.

(a) Write down the formula for $s(x)$.

(b) Suppose that $s(x)$ is also given by $s(x) = \int_0^x e^{-h(t)} dt$. Find the function $h(x)$ explicitly.

(c) Find the function $s(x)$ explicitly.

Solution:

(a)

$$s(x) = \int_0^x \sqrt{1 + (h'(t))^2}\, dt \quad (3\%)$$

(b)

$$\int_0^x \sqrt{1 + (h'(t))^2}\, dt = \int_0^x e^{-h(t)}\, dt$$

$$\Rightarrow \quad 1 + (h'(x))^2 = e^{-2h(x)} \quad (3\%)$$

$$y' = -\sqrt{e^{-2y} - 1} = \frac{dy}{dx} \quad (2\%)$$

$$-\int dx = \int \frac{1}{\sqrt{e^{-2y} - 1}}\, dy \quad (1\%)$$

Let $e^{-y} = \sec\theta \Rightarrow -e^{-y} = \sec\theta\tan\theta d\theta$

$$\begin{aligned} -x &= -\int \frac{\tan\theta}{\tan\theta}\, d\theta \\ &= -\theta + C \end{aligned}$$

$$\Rightarrow x = \theta + C = \sec^{-1}(e^{-y}) + C$$

take into $(x, y) = (0, 0)$, then

$$0 = \sec^{-1} 1 + C = C \quad (1\%)$$

Therefore

$$\begin{aligned} & e^{-y} = \sec x \\ \Rightarrow \quad & -y = \ln\sec(-x) \\ \Rightarrow \quad & h(x) = y = \ln\cos x \quad (2\%) \end{aligned}$$

(c)

$$\begin{aligned} s(x) &= \int_0^x \sec t\, dt \\ &= \ln|\sec x + \tan x| \quad (3\%) \end{aligned}$$

4. (15%) Let Ω be the region bounded by $y = \cos x$, $y = 0$, $x = 0$ and $x = \frac{\pi}{2}$.

(a) Find the volume of the solid obtained by revolving Ω about x-axis.

(b) Find the volume of the solid obtained by revolving Ω about y-axis.

(c) Find the centroid of Ω.

Solution:

(a) $V = \int_0^{\pi/2} \pi y^2 dx = \pi \int_0^{\pi/2} \cos^2 x dx$ (2%)

$= \pi \int_0^{\pi/2} \frac{\cos 2x + 1}{2} dx$ (1%)

$= \pi(\frac{\sin 2x}{4} + \frac{x}{2})|_0^{\pi/2}$ (1%)

$= \frac{\pi^2}{4}$ (1%)

(b) $V = \int_0^1 \pi x^2 dy = -\pi \int_0^{\pi/2} x^2(-\sin x) dx$ (1%)

$= -\pi(x^2 \cos x|_0^{\pi/2} - 2\int_0^{\pi/2} x \cos x dx)$ (1%) (or $\mathrm{V} = 2\pi \int_0^{\pi/2} xy dx$)

$= 2\pi(x \sin x|_0^{\pi/2} - \int_0^{\pi/2} \sin x dx)$ (1%)

$= 2\pi(\frac{\pi}{2} + \cos x|_0^{\pi/2})$ (1%)

$= \pi^2 - 2\pi$ (1%)

(c) $A = \int_0^{\pi/2} \cos x dx = \sin x|_0^{\pi/2} = 1$(1%)

$\bar{x} = \frac{1}{A}\int_0^{\pi/2} x \cos x dx$(1%) $= \frac{\pi}{2} - 1$(1%)

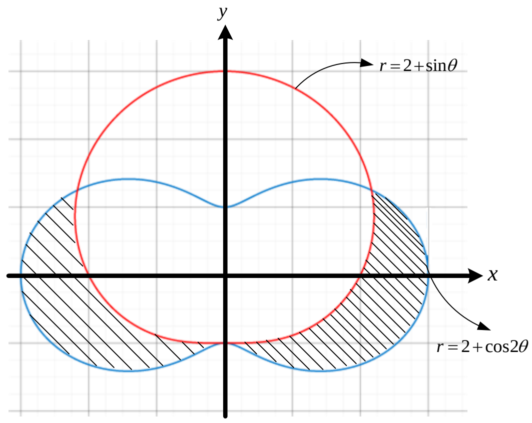
$\bar{y} = \frac{1}{A}\int_0^{\pi/2} \frac{1}{2}\cos^2 x dx$(1%) $= \frac{\pi}{8}$(1%)

or use Pappus Theorem.

$2\pi A\bar{x} = V$(revolve about y-axis)

$2\pi A\bar{y} = V$(revolve about x-axis)

5. (10%) Find the area of the region that lies inside the curve $r = 2 + \cos 2\theta$ but outside the curve $r = 2 + \sin\theta$.



Solution:

find intersection

$2 + \sin\theta = 2 + \cos 2\theta$

$\rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$

$\sin\theta = -1 or \frac{1}{2},\ \theta = -\frac{\pi}{2}, \frac{\pi}{6} or \frac{5\pi}{6}$

(2 pts)

$Area = \frac{1}{2}\int_{(\frac{5}{6}-2)\pi}^{\frac{\pi}{6}} (2+\cos\theta)^2 - (2+\sin\theta)^2$ (3 pts)

$(= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (2+\cos\theta)^2 - (2+\sin\theta)^2 d\theta)$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} 4\cos 2\theta + cos^2 2\theta - 4\sin\theta - sin^2\theta d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} 4\cos 2\theta + \frac{1+\cos 4\theta}{2} - 4\sin\theta - \frac{1-\cos 2\theta}{2} d\theta$ (3 pts)

$= \frac{9}{4}\sin 2\theta + \frac{1}{8}\sin 4\theta + 4\cos\theta \Big|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{6}}$

$= \frac{51}{16}\sqrt{3}$ (2 pts)

6. (10%) Find the arc length of the curve $x = t\sin 2t,\ \ y = t\cos 2t,\ \ 0 \le t \le 1$.

Solution:

Since (+3 points)

$$\begin{aligned} x' &= \sin 2t + 2t\cos 2t, \\ y' &= \cos 2t - 2t\sin 2t. \end{aligned}$$

We have

$$\begin{aligned} x'^2 + y'^2 &= (\sin 2t + 2t\cos 2t)^2 + (\cos 2t - 2t\sin 2t)^2 \\ &= \sin^2 2t + \cos^2 2t + 4t^2\left(\sin^2 2t + \cos^2 2t\right) \\ &= 1 + 4t^2. \end{aligned}$$

The arc length is (+2 points)

$$L := \int_0^1 \sqrt{x'^2 + y'^2}dt = \int_0^1 \sqrt{1+4t^2}dt.$$

Let $t = \frac{1}{2}\tan\theta$, then we have $dt = \frac{1}{2}\sec^2\theta d\theta$ and

$$\begin{aligned} L &= \frac{1}{2}\int_0^{\tan^{-1}2} |\sec\theta|\sec^2\theta dt \\ &= \frac{1}{2}\int_0^{\tan^{-1}2} |\sec\theta|^3 dt \text{ (+2 points)} \\ &= \frac{1}{4}\left(\tan\theta\sec\theta + \ln|\sec\theta + \tan\theta|\right)|_0^{\tan^{-1}2} \\ &= \frac{1}{4}\left(2\sqrt{5} + \ln\left|\sqrt{5}+2\right|\right).\text{(+3 points)} \end{aligned}$$

Note

$$\begin{aligned} \int \sec^3\theta d\theta &= \tan\theta\sec\theta - \int \tan^2\theta\sec\theta d\theta \\ &= \tan\theta\sec\theta - \int \left(\sec^2\theta - 1\right)\sec\theta d\theta \\ &= \tan\theta\sec\theta - \int \sec^3\theta + \int \sec\theta d\theta.\text{(+2 points)} \end{aligned}$$

Hence

$$\begin{aligned} \int \sec^3\theta d\theta &= \frac{1}{2}\left(\tan\theta\sec\theta + \int \sec\theta d\theta\right) \\ &= \frac{1}{2}\left(\tan\theta\sec\theta + \ln|\sec\theta + \tan\theta|\right) + C. \end{aligned}$$

Ps. If you didn't write down the calculation of $\int \sec^3$ but a wrong formula, you won't get any points.

7. (15%) Let $P(x) = x^3 + x^2 + x + 1$ and $Q(x) = x^3 - x^2 + x + 1$. Evaluate $\int \frac{Q(x)}{P(x)} dx$. Note that $P(-1) = 0$.

Solution:

Note that $P(x) = (x^2 + 1)(x + 1)$. (2pts)
$\frac{Q}{P} = 1 + \frac{A}{x+1} + \frac{Bx + C}{x^2 + 1}$ (5 pts)
$\Rightarrow A = B = -1$, $C = 1$. (3 pts)
Hence

$$\begin{aligned}\int \frac{Q}{P} dx &= \int 1 dx + \int \frac{-1}{x+1} dx + \int \frac{1-x}{x^2+1} dx \\ &= x - \ln|x+1| + \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c(\text{constant})(5 \text{ pts})\end{aligned}$$

8. (10%) Determine the values of $\alpha > 0$ such that $\int_1^\infty \frac{\ln x}{x^\alpha} dx$ is convergent.

Solution:

1° For $\alpha \neq 1$

$$\int \frac{\ln x}{x^\alpha} dx = \int \ln x d(\frac{x^{1-\alpha}}{1-\alpha}) = \frac{x^{1-\alpha}}{1-\alpha} \ln x - \frac{1}{1-\alpha} \int x^{-\alpha} dx = \frac{x^{1-\alpha}}{1-\alpha} \ln x - \frac{1}{(1-\alpha)^2} x^{1-\alpha} + C \qquad (5\%)$$

$$\therefore \int_1^\infty \frac{\ln x}{x^\alpha} dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^\alpha} dx = \lim_{t\to\infty} \left(\frac{t^{1-\alpha}}{1-\alpha} \ln t - \frac{1}{(1-\alpha)^2} t^{1-\alpha} + \frac{1}{(1-\alpha)^2} \right)$$

(case1) If $0 < \alpha < 1$

$$\lim_{t\to\infty} \frac{(1-\alpha)\ln t - 1}{(1-\alpha)^2 \ t^{\alpha-1}} = \infty \quad \therefore \int_1^\infty \frac{\ln x}{x^\alpha} dx \text{ is divergent.} \qquad (1\%)$$

(case2) If $\alpha > 1$

$$\lim_{t\to\infty} \frac{(1-\alpha)\ln t - 1}{(1-\alpha)^2 \ t^{\alpha-1}} \left[\frac{\infty}{\infty}\right] = \lim_{t\to\infty} \frac{\frac{1}{t}}{(1-\alpha)^2(\alpha-1) \ t^{\alpha-2}} = 0 \text{ by L'Hospital Rule.}$$

$$\therefore \int_1^\infty \frac{\ln x}{x^\alpha} dx \text{ is convergent.} \qquad (1\%)$$

2° For $\alpha = 1$

$$\int \frac{\ln x}{x} dx = \int \ln x d(\ln x) = \frac{1}{2}(\ln x)^2 + C \qquad (2\%)$$

$$\therefore \int_1^\infty \frac{\ln x}{x} dx = \lim_{t\to\infty} \frac{1}{2}(\ln t)^2 = \infty \quad \Rightarrow \int_1^\infty \frac{\ln x}{x} \text{ is divergent.} \qquad (1\%)$$

Thus, $\int_1^\infty \frac{\ln x}{x^\alpha} dx$ is convergent for $\alpha > 1$.