1. (10%) Solve the initial-value problem:

$$\begin{cases} (\sec x)y' + y = (\tan x)e^{\cos x - \sin x}, & 0 \le x < \frac{\pi}{2}, \\ y(0) = 0. \end{cases}$$

Solution:

$(\sec x)y' + y = (\tan x)e^{\cos x - \sin x}$

$\Rightarrow y' + (\cos x)y = (\sin x)e^{\cos x - \sin x}$ (2 pts)

since $\int \cos x dx = \sin x + c'$($c'$ is a constant), we mutiply $e^{\sin x}$

$\Rightarrow e^{\sin x}y' + e^{\sin x}(\cos x)y = (\sin x)e^{\cos x}$ (2 pts)

$\Rightarrow (e^{\sin x}y)' = (\sin x)e^{\cos x}$

$\Rightarrow e^{\sin x}y = \int (\sin x)e^{\cos x}dx = -e^{\cos x} + c$ (c is a constant)(4 pts)

(-1 pt if you have not write down what c is)

$y(0) = 0 \Rightarrow 0 = -e + c$

$\Rightarrow c = e$

$\therefore y = (-e^{\cos x} + e)e^{-\sin x} = e^{1-\sin x} - e^{\cos x - \sin x}$(2 pts)

2. (10%) Solve the initial-value problem:

$$\begin{cases} (xy^2 + y^2 + x + 1)dx + (y-1)dy = 0, \\ y(2) = 0. \end{cases}$$

Solution:

$$(x+1)(y^2+1)dx = (1-y)dy$$

$$(x+1)dx = \frac{1-y}{y^2+1}dy \quad \text{..................................(3 pts)}$$

$$\int (x+1)dx = \int \frac{1-y}{y^2+1}dy$$

$$\frac{1}{2}x^2 + x = \int \frac{1}{y^2+1}dy - \int \frac{y}{y^2+1}dy \quad \text{...........................(1 pt)}$$

$$\frac{1}{2}x^2 + x = \tan^{-1} y - \frac{1}{2}\ln(y^2+1) + C \quad \text{....................................(4 pts)}$$

$$y(0) = 2 \;\Rightarrow\; \frac{1}{2}2^2 + 2 = \tan^{-1}(0) - \ln(1) + C \;\Rightarrow C = 4 \quad \text{..................(2 pts)}$$

The solution is

$$\frac{1}{2}x^2 + x = \tan^{-1} y - \frac{1}{2}\ln(y^2+1) + 4.$$

3. (10%) Consider the region bounded by the curves:

$$y = \sin(\pi x/2) \text{ and } y = 6/(x^2+3x+2) \text{ for } 0 \le x \le 1.$$

Note that the two curves meet at $x = 1$. Find the volume of revolving the region about (a)the y-axis, (b)the x-axis.

Solution:

Let $f(x) = \sin\frac{x\pi}{2}$, $\quad g(x) = \frac{6}{x^2+3x+2} = 6\left(\frac{1}{x+1} - \frac{1}{x+2}\right)$

Note that $f(x) \le g(x) \quad \forall x \in [0, 1])$

(a) $V = 2\pi \int_0^1 x(g(x) - f(x))dx = 2\pi \int_0^1 \frac{6x}{x^2+3x+2} - x\sin\frac{x\pi}{2}dx$ (2%)

$$\int_0^1 x\sin\frac{x\pi}{2} = -\frac{2}{\pi}x\cos\frac{x\pi}{2}\Big|_0^1 + \int_0^1 \frac{2}{\pi}\cos\frac{x\pi}{2}dx = 0 + \frac{4}{\pi^2}\sin\frac{x\pi}{2}\Big|_0^1 = \frac{4}{\pi^2} \quad (1\%)$$

$$\int_0^1 \frac{6x}{x^2+3x+2}dx = 6\int_0^1 \frac{-1}{x+1} + \frac{2}{x+2}dx = 6\ln\left|\frac{(x+2)^2}{x+1}\right|_0^1 = 6\ln\frac{9}{8} \quad (2\%)$$

$$V = 12\pi\ln\frac{9}{8} - \frac{8}{\pi}$$

(b) $V = \pi\int_0^1 g(x)^2 - f(x)^2 dx = \pi\int_0^1 36\left(\frac{1}{x+1} - \frac{1}{x+2}\right)^2 - \sin^2\frac{x\pi}{2}dx$ (2%)

$$\int_0^1 -\sin^2\frac{x\pi}{2}dx = \int_0^1 \frac{1}{2} - \frac{1}{2}\cos\pi x\,dx = \frac{1}{2} \quad (1\%)$$

$$\int_0^1 36\left(\frac{1}{x+1} - \frac{1}{x+2}\right)^2 dx = 36\int_0^1 \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{-2}{x+1} + \frac{2}{x+2}dx$$

$$= 36\left[-(x+1)^{-1} - (x+2)^{-1} + 2\ln\left|\frac{x+2}{x+1}\right|\right]_0^1$$

$$= 36\left(\frac{2}{3} + 2\ln\frac{3}{4}\right) \quad (2\%)$$

$$V = \pi\left(\frac{47}{2} + 72\ln\frac{3}{4}\right)$$

4. (10%) Find the area both inside $r^2 = 2\cos 2\theta$ and inside $r = 1$.

Solution:

$1. r^2 = 2cos2\theta$ and $r = 1$

so $\theta = \frac{\pi}{6}; \frac{5\pi}{6}; \frac{7\pi}{6}; \frac{11\pi}{6} (4points)$

$2. r = 0; \theta = \frac{\pi}{4}(3points)$

3. $4[\int_0^{\frac{\pi}{6}} \frac{1}{2}1d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2}2cos2\theta d\theta](2points)$

$= \frac{\pi}{3} + 2 - \sqrt{3}(1point)$

5. (10%) Find the area of the surface generated by revolving the curve $r = \sin\theta$, $0 \le \theta \le \pi/2$ about the x-axis.

Solution:

$1. x = sin\theta cos\theta; y = sin^2\theta$

$x' = cos^2\theta - sin^2\theta; y' = 2sin\theta cos\theta(6points)$

$2. A = \int_0^{\frac{\pi}{2}} 2\pi y\sqrt{(x')^2 + (y')^2}d\theta$

$= 2\pi\int_0^{\frac{\pi}{2}} sin^2\theta\sqrt{(cos2\theta)^2 + (sin2\theta)^2}d\theta(3points)$

$= \frac{\pi^2}{2}(1point)$

6. (12%) The cycloid is the curve parametrized by $x(\theta) = R(\theta - \sin\theta)$, $y(\theta) = R(1 - \cos\theta)$.

(a) Find the arclength of the cycloid for $0 \le \theta \le 2\pi$.

(b) Find the area under the cycloid and above the x-axis for $0 \le \theta \le 2\pi$.

(c) Find the volume of the solid generated by revolving the region in (b) about the x-axis.

Solution:

(a)

Some basic setting.(*2%*)

$$\begin{cases} x'(\theta) &= R(1-\cos\theta) \\ y'(\theta) &= R\sin\theta \end{cases} \implies \sqrt{(x')^2+(y')^2} = 2R\sqrt{\frac{1-\cos\theta}{2}} = 2R\sin\frac{\theta}{2}$$

Calculate the arclength.(*2%*)

$$\begin{aligned} L &= \int_0^{2\pi} 2R\sin\frac{\theta}{2}\,d\theta = 4R(-\cos\frac{\theta}{2})|_0^{2\pi} \\ &= 8R \end{aligned}$$

(b)

$$\begin{aligned} A &= \int_0^{2\pi R} y\,dx = \int_0^{2\pi} y(\theta)x'(\theta)d\theta \\ &= \int_0^{2\pi} R(1-\cos\theta)R(1-\cos\theta)d\theta \quad \text{(Formula of area } \mathit{2\%}\text{)} \\ &= R^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta \\ &= R^2\int_0^{2\pi}(1-2\cos\theta+\frac{1+\cos 2\theta}{2})\,d\theta \\ &= R^2(2\pi+0+\pi) \\ &= 3\pi R^2 \quad (\mathit{2\%} \text{ for exactly right answer}) \end{aligned}$$

(c)

$$\begin{aligned} V &= \int_0^{2\pi R} y^2\pi dx = \int_0^{2\pi} y^2(\theta)\pi x'(\theta)\,d\theta \\ &= \pi\int_0^{2\pi} R^2(1-\cos\theta)R(1-\cos\theta)\,d\theta \quad \text{(formula of volume } \mathit{2\%}\text{)} \\ &= \pi R^3\int_0^{2\pi}(1-3\cos\theta+3\cos^2\theta-\cos^3\theta)\,d\theta \\ &= \pi R^3(2\pi+0+3\pi+0) \\ &= 5\pi^2R^3 \quad (\mathit{2\%} \text{ for exactly right answer}) \end{aligned}$$

7. (10%) Let $A = \int_0^\infty e^{-x^2}dx$. Compute the limit

$$\lim_{x\to\infty} xe^{x^2}\left(A - \int_0^x e^{-t^2}\,dt\right).$$

Solution:

Let $A = \int_0^\infty e^{-x^2}dx$. Compute the limit $\lim\limits_{x\to\infty} xe^{x^2}(A - \int_0^x e^{-t^2}dt)$.

Sol: $\lim\limits_{x\to\infty} xe^{x^2}(A - \int_0^x e^{-t^2}dt) = \lim\limits_{x\to\infty}\frac{A-\int_0^x e^{-t^2}dt}{x^{-1}e^{-x^2}}$. The limit is $\frac{0}{0}$ type.

Then $\lim\limits_{x\to\infty}\frac{A-\int_0^x e^{-t^2}dt}{x^{-1}e^{-x^2}}$

$= \lim\limits_{x\to\infty}\frac{-e^{-x^2}}{-x^{-2}e^{-x^2}-2e^{-x^2}}$ (*4points*)

$= \lim_{x\to\infty} \frac{1}{2+x^{-2}}$ $(4points)$

$= \frac{1}{2}$ $(2points)$

8. (10%) Evaluate (a) $\int_0^1 x\tan^{-1}(x^2)\,dx$ and (b) $\int_0^1 x(\tan^{-1}x)^2\,dx$.

Solution:

(a)

$$\begin{aligned}\int_0^1 x\tan^{-1}(x^2)\,\mathrm{d}x &= \frac{x^2}{2}\tan^{-1}(x^2)\Big|_0^1 - \int_0^1 \frac{x^2}{2}\frac{2x}{(1+(x^2)^2)}\mathrm{d}x \qquad (2\%)\\ &= \frac{\pi}{8} - \frac{1}{4}\int_0^1 \frac{1}{1+x^4}\mathrm{d}(1+x^4)\\ &= \frac{\pi}{8} - \frac{1}{4}\ln(1+x^4)\Big|_0^1 \qquad (1\%)\\ &= \frac{\pi}{8} - \frac{1}{4}\ln 2 \qquad (2\%)\end{aligned}$$

(b)

$$\begin{aligned}&\int_0^1 x(\tan^{-1}x)^2\,\mathrm{d}x\\ =\ & \frac{x^2}{2}(\tan^{-1}x)^2\Big|_0^1 - \int_0^1 \frac{x^2}{2}2(\tan^{-1}x)\,(\frac{1}{1+x^2})\,\mathrm{d}x \qquad (2\%)\\ =\ & \frac{\pi^2}{32} - \int_0^1 (\tan^{-1}x)(1-\frac{1}{1+x^2})\,\mathrm{d}x\\ =\ & \frac{\pi^2}{32} - \int_0^1 \tan^{-1}x\,\mathrm{d}x + \int_0^1 (\tan^{-1}x)\frac{1}{1+x^2}\,\mathrm{d}x\\ =\ & \frac{\pi^2}{32} - \left(x\tan^{-1}x\right)_0^1 + \int_0^1 \frac{x}{1+x^2}\mathrm{d}x + \int_0^1 (\tan^{-1}x)\,\mathrm{d}(\tan^{-1}x) \qquad (1\%)\\ =\ & \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\mathrm{d}(1+x^2) + \frac{(\tan^{-1}x)^2}{2}\Big|_0^1 \qquad (1\%)\\ =\ & \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2}\ln(1+x^2)\Big|_0^1 + \frac{(\pi/4)^2}{2}\\ =\ & \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\ln 2 \qquad (1\%)\end{aligned}$$

9. (10%) Evaluate the improper integral $\int_0^2 \frac{1}{x}\sqrt{x(2-x)}\,dx$.

Solution:

Because $\lim_{x\to 0^+} \frac{1}{x}\sqrt{x(2-x)} = \infty$, $\int_0^2 \frac{1}{x}\sqrt{x(2-x)}dx$ is an improper integral.

Hence $\int_0^2 \frac{1}{x}\sqrt{x(2-x)}dx = \lim_{t\to 0^+}\int_t^2 \frac{1}{x}\sqrt{x(2-x)}dx$.

Write $\int_t^2 \frac{1}{x}\sqrt{x(2-x)}dx$ as $\int_t^2 \frac{1}{x}\sqrt{x(2-x)}dx$

let $x-1=\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, then $dx=\cos\theta d\theta$ and

$$\begin{aligned}\int_t^2 \frac{1}{x}\sqrt{x(2-x)}dx = \int_t^2 \frac{1}{x}\sqrt{1-(x-1)^2}dx &= \int_{\sin^{-1}(t-1)}^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\sqrt{1-\sin^2\theta}\cos\theta d\theta\\ &= \int_{\sin^{-1}(t-1)}^{\frac{\pi}{2}} \frac{1-\sin^2\theta}{1+\sin\theta}d\theta = \int_{\sin^{-1}(t-1)}^{\frac{\pi}{2}} 1-\sin\theta d\theta = \theta+\cos\theta\Big|_{\sin^{-1}(t-1)}^{\frac{\pi}{2}}\end{aligned}$$

$$= \frac{\pi}{2} - \sin^{-1}(t-1) - \sqrt{1-(t-1)^2}$$

$$\lim_{t\to 0^+} \int_t^2 \frac{1}{x}\sqrt{x(2-x)}dx = \lim_{t\to 0^+} \frac{\pi}{2} - \sin^{-1}(t-1) - \sqrt{1-(t-1)^2} = \frac{\pi}{2} - \sin^{-1}(-1) = \pi$$

Hence $\int_0^2 \frac{1}{x}\sqrt{x(2-x)}dx = \pi$

10. (8%) Find (a) $\lim_{n\to\infty} \left(\sin\frac{1}{n}\right)^{\frac{1}{n}}$ and (b) $\lim_{n\to\infty} \left(1+\frac{1}{n^2}\right)^n$.

Solution:

(a)

$$\lim_{n\to\infty} \sin\frac{1}{n}^{\frac{1}{n}} = e^{\lim_{n\to\infty}\frac{1}{n}\ln\sin\frac{1}{n}} = e^{\lim_{n\to\infty}\frac{\cos\frac{1}{n}}{\sin\frac{1}{n}}(-\frac{1}{n^2})} = e^0 = 1$$

(b)

$$\lim_{n\to\infty}(1+\frac{1}{n^2})^n = e^{\lim\limits_{n\to\infty} n\ln(1+\frac{1}{n^2})} = e^{\lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n^2}}(\frac{-2}{n^3})}{-\frac{1}{n^2}}} = e^{\lim_{n\to\infty}\frac{2}{n(1+\frac{1}{n^2})}} = e^0 = 1$$