1. (15%) (a) Let $\{b_n\}$ be a sequence of nonzero numbers such that $\lim_{n\to\infty} b_n = \infty$. Determine whether the series $\sum_{k=1}^{\infty}(b_{k+1}-b_k)$ and $\sum_{k=1}^{\infty}\left(\frac{1}{b_k}-\frac{1}{b_{k+1}}\right)$ are convergent or divergent. Explain your answer.

(b) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\left(n\sin\frac{1}{n}-1\right)$ is absolutely convergent, conditionally convergent or divergent.

(c) Find all values of p such that the series $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n^p}$ converges conditionally.

Solution:

(a) Observe that

$$a-b=a-c+c-b$$

then apply it to the finite sums(1+1pts) and take limits(3pts).

(b) Observe that

$$n(\sin(\frac{1}{n})-1)\approxeq\frac{-1}{6n^2}$$

by taylor expansion of $\sin x$.(2pts)

Then using limit comparison.(3pts)

(c) For converge on p¿o, by alternating series test.(2pts)

For absolute converge on p¿1, by a consideration

$$p=1+2\varepsilon$$

for any $\epsilon>0$.

then

$$\frac{\ln(n)}{n^p}=\frac{\ln(n)}{n^{\varepsilon}}\frac{1}{n^{1+\varepsilon}}$$

And observe that

$$\frac{\ln(n)}{n^{\varepsilon}}<1$$

for n large enough.

Now apply limit comparison with

$$\frac{1}{n^{1+\varepsilon}}$$

(3pts)

2. (10%) (a) Expand the function $f(x)=(8+x)^{\frac{1}{3}}$ as a power series centered at $x=0$. (You must write out the general terms.) Find the radius of convergence.

(b) Find the sum of the series $\sum_{n=2}^{\infty}\frac{n^2+1}{n!}$.

Solution:

(a)

$$f(x)=2(1+\frac{x}{8})^{\frac{1}{3}}=2\sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}(\frac{x}{8})^n$$

$$=2+\frac{x}{12}+2\sum_{n=2}^{\infty}\frac{(-1)^{n-1}[2\cdot5\cdot8\cdots(3n-4)]}{3^n n!}\cdot(\frac{x}{8})^n$$

The radius of convergence: 8

配分:

- $2\sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}(\frac{x}{8})^n$: 2分
- $2\sum_{n=2}^{\infty}\frac{(-1)^{n-1}[2\cdot 5\cdot 8\cdots(3n-4)]}{3^n n!}\cdot(\frac{x}{8})^n$: 2分
- $2+\frac{x}{12}$ and The radius of convergence is 8: 1分

(b)

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{n^2+1}{n!} &= \sum_{n=2}^{\infty}\frac{n(n-1)+n+1}{n!} = \sum_{n=2}^{\infty}\frac{1}{n-2!}+\frac{1}{n-1!}+\frac{1}{n!}\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}+\sum_{n=1}^{\infty}\frac{1}{n!}+\sum_{n=2}^{\infty}\frac{1}{n!} = (e)+(e-1)+(e-2)\\ &= 3e-3\end{aligned}$$

配分:

- (e): 1分
- (e-1): 2分
- (e-2): 2分

3. (15%) Let $\{f_n\}$ be the Fibonacci sequence defined by $f_1 = f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Define $a_n = \frac{f_{n+1}}{f_n}$, $n \geq 1$.
   (a) Show that $\{a_{2n}\}$ is decreasing while $\{a_{2n+1}\}$ is increasing and both $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ exist. Find the limits. (Hint. $\{a_n\}$ satisfies the recursive relation $a_{n+1} = 1 + \frac{1}{a_n}$, $n \geq 1$. Express a_{n+2} in terms of a_n.)
   (b) Find the radius of convergence of the power series $f(x) = \sum_{n=1}^{\infty} f_n x^n$.

Solution:

(a) We prove it by induction, given $f_1 = f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5$
when $n = 1$,
$\Rightarrow a_1 = 1, a_2 = 2, a_3 = \frac{3}{2}, a_4 = \frac{5}{3}$.
$a_1 < a_3,\ a_2 > a_4$.
when $n = k$, suppose $a_{2k} > a_{2k+2}$,

$$1+\frac{1}{a_{2k}} < 1+\frac{1}{a_{2k+2}} \Rightarrow a_{2k+1} < a_{2k+3} \Rightarrow 1+\frac{1}{a_{2k+1}} > 1+\frac{1}{a_{2k+3}} \Rightarrow a_{2k+2} > a_{2k+4}$$

accroding to upper proof,

$n = k+1$ is also hold for $[a_{2n}]$, so $[a_{2n}]$ is decreasing.

similar proof to $[a_{2n+1}]$, so $[a_{2n+1}]$ is increasing. (5 points)

On the other hand, clearly f_n is increasing and all terms are positive,

$$a_n = \frac{f_{n+1}}{f_n} \geq 1 \Rightarrow 0 \leq a_{n+1} = 1+\frac{1}{a_n} \leq 2, \forall n.$$

so $[a_{2n+1}]$ and $[a_{2n}]$ is bounded. (3 points)

Apply monotonic theorem, both limit exist, and

$$\lim_{n\to\infty} a_{2n} = L;\ \lim_{n\to\infty} a_{2n+1} = M$$

$$a_{2n+2} = 1 + \frac{1}{1+\frac{1}{a_{2n}}} \Rightarrow L = 1 + \frac{1}{1+\frac{1}{L}} \Rightarrow L = \frac{1+\sqrt{5}}{2}$$

similarly to $K = \dfrac{1+\sqrt{5}}{2}$.

finally we conclude that

$$\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2} \quad \text{(3 points)}$$

(b) use ratio test,

$$\lim_{n\to\infty} \left|\frac{f_{n+1}}{f_n}\right| |x| = \lim_{n\to\infty} |a_n|\,|x| = \frac{1+\sqrt{5}}{2}\,|x|$$

since series is convergence,

$$\frac{1+\sqrt{5}}{2}\,|x| < 1 \Rightarrow |x| < \frac{\sqrt{5}-1}{2}$$

hence **R**= $\dfrac{\sqrt{5}-1}{2}$ (4 points)

4. (20%) A curve C is defined by $\mathbf{r}(t) = \left\langle \cos t + t\sin t, \sin t - t\cos t, t^2 \right\rangle$, $t \geq 0$.
   (a) Find the arc length function $s(t)$ with the starting point $(1,0,0)$.
   (b) Find the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$ and the unit binormal vector $\mathbf{B}$.
   (c) Find the curvature of C.

Solution:

(a) $r'(t) = (t\cos t, t\sin t, 2t)(1pt)$

$|r'(t)| = \sqrt{5}t(1pt)$

Hence $s(t) = \displaystyle\int_0^t |r'(s)|ds(1pt) = \int_0^t \sqrt{5}sds = \frac{\sqrt{5}t^2}{2}(2pt)$

(b) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}(1pt) = \dfrac{1}{\sqrt{5}}(\cos t, \sin t, 2)(2pt)$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}(1pt) = (-\sin t, \cos t, 0)(2pt)$

$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)(2pt) = \dfrac{-1}{\sqrt{5}}(2\cos t, 2\sin t, -1)(2pt)$

(c) $\kappa(t) = |\dfrac{d\mathbf{T}}{ds}|(1pt) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}(1pt) = \dfrac{1}{5t}(3pt)$

5. (10%) Find the limit, if it exists, or show that it does not exist.
   (a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy - x - y + 1}{x^2 + y^2 - 2x - 2y + 2}$.
   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}$.

Solution:

(a)

$$\begin{aligned}\lim_{(x,y)\to(1,1)} \frac{xy - x - y + 1}{x^2 + y^2 - 2x - 2y + 2} &= \lim_{(x,y)\to(1,1)} \frac{(x-1)(y-1)}{(x-1)^2 + (y-1)^2} \\ &= \lim_{(u,v)\to(0,0)} \frac{uv}{u^2 + v^2}\end{aligned}$$

But

$$u = v, \quad \lim_{(u,u)\to(0,0)} \frac{u^2}{u^2+u^2} = \frac{1}{2}$$
$$v = 0, \quad \lim_{(u,0)\to(0,0)} \frac{0}{u^2} = 0$$

So the limit doesn't exist.
配分: 全對或全錯,計算錯誤扣1 2分.

(b)

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \left|\frac{x^3+y^3}{x^2+y^2}\right| &\le \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}\cdot|x| + \frac{y^2}{x^2+y^2}\cdot|y| \\ &\le \lim_{(x,y)\to(0,0)} |x|+|y| \\ &= 0\end{aligned}$$

So

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$$

配分: 全對或全錯,少絶對直扣1分.

6. (20%) Let $f(x,y) = \int_1^{2y-x^2} e^{t^2}\,dt$.

(a) Find the rate of change of f at the point $P(1,1)$ in the direction from P to $Q(6,13)$.
(b) In what direction does f have the maximum rate of change? What is this rate of change.
(c) Find the tangent plane and the normal line to the surface $S:\ z = f(x,y)$ at the point $(1,1,0)$.
(d) The sphere $x^2+y^2+z^2 = 2$ intersects S in a curve C. Find the equations for the tangent line to C at the point $(1,1,0)$.

Solution:

We calculate ∇f first. (Maximum 5 points for finding ∇f correctly.)
$\nabla f = (-2x\cdot e^{(2y-x^2)^2}, 2e^{(2y-x^2)^2})$
$\quad = 2e^{(2y-x^2)^2}(-x,1).$

(a) $\overrightarrow{PQ} = (6,13)-(1,1) = (5,12)$
$\vec{u} = \frac{1}{\sqrt{5^2+12^2}}(5,12) = (\frac{5}{13},\frac{12}{13})$ (1 point)
$D_{\vec{u}}\cdot f = \nabla f\cdot\vec{u} = (-2e,2e)\cdot(\frac{5}{13},\frac{12}{13}) = \frac{14}{13}e.$ (2 points)

(b) $\vec{u} = (\frac{x}{\sqrt{x^2+1}},\frac{1}{\sqrt{x^2+1}})$ (2 points)
$|\nabla f(x,y)| = 2e^{(2y-x^2)^2}\sqrt{x^2+1}.$ (2 points)

(c)Normal line: $x = 1-2et,\quad y = 1+2et,\quad z = t,\quad t\in R$ (1 point)
Tangent plame:
$z - 0 = f_x(1,1)(x-1)+f_y(1,1)(y-1)$ (1 point)
$= -2e(x-1)+2e(y-1)$
$= -2ex+2ey.$ (1 point)

(d) Let $g(x,y,z) = x^2+y^2+z^2-2$
$\nabla g(1,1,0) = (2,2,0)$ (1 point)
$\nabla S(1,1,0) = (-2e,2e,-1)$ (1 point)
$\nabla S\times\nabla g = (1,-1,-4e)$ (2 points)
$\Rightarrow x = 1+t,\quad y = 1-t,\quad z = -4et,\quad t\in R$ (1 point)

7. (10%) Let $f(x,y) = \sin x\cos(x+y)$ and $D = \left\{(x,y)|0\le x\le\frac{\pi}{2}, 0\le y\le\frac{\pi}{2}\right\}$. Classify all the critical points of f on D.

Solution:

$f_x(x,y) = \cos x \cos(x+y) - \sin x \sin(x+y) = \cos(2x+y)$ and $f_y(x,y) = -\sin x \sin(x+y) = \dfrac{\cos(2x+y) - \cos y}{2}$.
Then let $f_x(x,y) = f_y(x,y) = 0 \Rightarrow \cos(2x+y) = \cos y = 0 \Rightarrow y = \frac{\pi}{2}$ and $x = 0$ or $\frac{\pi}{2}$. So critical points of f on D are $(0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \frac{\pi}{2})$. Then $f_{xx}(x,y) = -2\sin(2x+y)$, $f_{xy}(x,y) = -\sin(2x+y)$ and $f_{yy}(x,y) = -\sin x \cos(x+y)$. $D(0, \frac{\pi}{2}) = -2 \cdot 0 - (-1)^2 = -1 < 0$ and $D(\frac{\pi}{2}, \frac{\pi}{2}) = 2 \cdot 1 - (1)^2 = 1 > 0$. Hence $(0, \frac{\pi}{2})$ is saddle point and $(\frac{\pi}{2}, \frac{\pi}{2})$ is local minimum.

評分標準：
f 對 x 和 y 的偏微分有算出來的有 2 分

在 D 上解方程式有解出來的有 4 分(包含求出 critical point)

算出二階偏微分的有 2 分

帶入判別式分出 critical point 是哪一種的有 2 分

8. (10%) Find the maximum and minimum values of $xy + z^2$ on the ball $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 \leq 1$.

Solution:

Let $f = xy + z^2$ and $g = x^2 + y^2 + (z - \frac{1}{2})^2 - 1$.
For the points inside the ball $x^2 + y^2 + (z - \frac{1}{2})^2 \leq 1$, consider critical points of f:

$$\nabla f = \langle y, x, 2z \rangle = 0 \Rightarrow (x,y,z) = (0,0,0).$$

We have $f(0,0,0) = 0$. (1 pt)
For the points on the boundary, that is, points satisfy $g = 0$, consider $\nabla f = \lambda \nabla g$:

$$\begin{cases} y = \lambda(2x) & (1) \\ x = \lambda(2y) & (2) \\ 2z = \lambda(2(z - \frac{1}{2})) & (3) \\ x^2 + y^2 + (z - \frac{1}{2})^2 = 1 & (4) \end{cases} . \qquad \text{(2 pts)}$$

From (1)(2), $x = \lambda(2\lambda(2x)) \Rightarrow x(4\lambda^2 - 1) = 0 \Rightarrow x = 0$ or $\lambda = \pm\frac{1}{2}$.
(i) $x = 0 \Rightarrow y = 0$, from (4), $0 + 0 + (z - \frac{1}{2})^2 = 1 \Rightarrow z = -\frac{1}{2}$ or $\frac{3}{2}$.
$f(0,0,-\frac{1}{2}) = \frac{1}{4}$, $f(0,0,\frac{3}{2}) = \frac{9}{4}$. (2pts)
(ii) $\lambda = \frac{1}{2}$, from (1)(2), $x = y$; from (3), $z = -\frac{1}{2}$.
$\Rightarrow$ From (4), $x^2 + x^2 + (-\frac{1}{2} - \frac{1}{2})^2 = 1 \Rightarrow x = y = 0$.
$f(0,0,-\frac{1}{2}) = \frac{1}{4}$. (2pts)
(iii) $\lambda = -\frac{1}{2}$, from (1)(2), $x = -y$; from (3), $z = \frac{1}{6}$.
$\Rightarrow$ From (4), $x^2 + x^2 + (\frac{1}{6} - \frac{1}{2})^2 = 1 \Rightarrow x = \pm\frac{2}{3}$ and $y = \mp\frac{2}{3}$.
$f(\frac{2}{3}, -\frac{2}{3}, \frac{1}{6}) = f(-\frac{2}{3}, \frac{2}{3}, \frac{1}{6}) = -\frac{15}{36}$. (2pts)
Therefore, the maximum is $\frac{9}{4}$ and the minimum is $-\frac{15}{36}$. (1pt)