1. (15%) Evaluate the flux integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where

$$\mathbf{F}(x,y,z)=(x+z)\,\mathbf{i}-(z+y)\,\mathbf{j}+(y+z^3)\,\mathbf{k},$$

and S is the sphere $(x-2)^2+y^2+z^2=4$ with outward normal.

Solution:

Method 1: Divergence Theorem (平移球坐標去參數化球體)

V: the solid $(x-2)^2+y^2+z^2\le 4$ enclosed by S.

Parametrize V by

$$\begin{cases} x=2+\rho\sin\phi\cos\theta & ,\ 0\le\rho\le 2\\ y=\rho\sin\phi\sin\theta & ,\ 0\le\phi\le\pi \\ z=\rho\cos\phi & ,\ 0\le\theta\le 2\pi.\end{cases} \quad (3\text{ pt})$$

$$|J|=\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right|=\rho^2\sin\phi \quad (\text{不算分, 因為平移不影響}Jacobian)$$

$$\nabla\cdot F=(\partial_x,\partial_y,\partial_z)\cdot\left(x+z,-z-y,y+z^3\right)=1-1+3z^2=3z^2 \quad (3\text{ pt})$$

$$\begin{aligned}\iint_S F\cdot n\,dS &= \iiint_V \nabla\cdot F\,dV=\iiint_V 3z^2dV \quad \text{by divergence theorem} \quad (3\text{ pt})\\ &=3\int_0^{2\pi}\int_0^{\pi}\int_0^2 \rho^2\cos^2\phi\cdot\rho^2\sin\phi\,d\rho d\phi d\theta \quad (3\text{ pt})\\ &=3\underbrace{\left(\int_0^2\rho^4d\rho\right)}_{\frac{2^5}{5}}\underbrace{\left(\int_0^{2\pi}d\theta\right)}_{2\pi}\underbrace{\left(\int_0^{\pi}\cos^2\phi\sin\phi\,d\phi\right)}_{\frac{2}{3}}\\ &=\frac{128}{5}\pi \quad (3\text{ pt})\end{aligned}$$

Method 2: Divergence Theorem (平移直角坐標再用球坐標)

By translation $x'=x-2$ (3 pt), F in the new coordinate is

$$F(x',y,z)=\left(x'+z+2,-z-y,y+z^3\right).$$

S: the sphere $x'^2+y^2+z^2=4$ with outward normal.

V: the solid $x'^2+y^2+z^2\le 4$ enclosed by S.

$$\nabla_{(x',y,z)}\cdot F=(\partial_{x'},\partial_y,\partial_z)\cdot\left(x'+z+2,-z-y,y+z^3\right)=1-1+3z^2=3z^2 \quad (3\text{ pt})$$

$$\begin{aligned}\iint_S F\cdot n\,dS &= \iiint_V \nabla_{(x',y,z)}\cdot F\,dV=\iiint_V 3z^2dV \quad \text{by divergence theorem} \quad (3\text{ pt})\\ &=3\int_0^{2\pi}\int_0^{\pi}\int_0^2 \rho^2\cos^2\phi\cdot\rho^2\sin\phi\,d\rho d\phi d\theta \quad \text{by spherical coordinate} \quad (3\text{ pt})\\ &=\frac{128}{5}\pi \quad (3\text{ pt})\end{aligned}$$

(可直接說明 $\nabla\cdot F$ 只跟 z 有關, 所以球在 $x-y$ 平面上平移不影響積分結果)

Method 3: Divergence Theorem (直接用球坐標, 要注意積分上下界)

V: the solid $(x-2)^2+y^2+z^2\le 4$ enclosed by S.

By spherical coordinates,

$$(x-2)^2+y^2+z^2 \le 4 \implies x^2+y^2+z^2 \le 4x \implies \rho \le 4\sin\phi\cos\theta \quad (3\text{ pt})$$

$$\nabla \cdot F = 1-1+3z^2 = 3z^2 \quad (3\text{ pt})$$

$$\begin{aligned}
\iint_S F\cdot n\,dS &= \iiint_V \nabla\cdot F\,dV = \iiint_V 3z^2 dV \quad \text{by divergence theorem} \quad (3\text{ pt})\\
&= 3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{4\sin\phi\cos\theta} \rho^2\cos^2\phi\cdot\rho^2\sin\phi\,d\rho d\phi d\theta \quad (3\text{ pt})\\
&= 3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\frac{1}{5}\left(4\sin\phi\cos\theta\right)^5\cos^2\phi\sin\phi\,d\phi d\theta\\
&= \frac{3\cdot 2^{10}}{5}\underbrace{\left(\int_0^{\pi}\cos^2\phi\sin^6\phi\,d\phi\right)}_{\frac{5\pi}{128}}\underbrace{\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5\theta\,d\theta\right)}_{\frac{16}{15}}\\
&= \frac{128}{5}\pi \quad (3\text{ pt})
\end{aligned}$$

Method 4: **參數化曲面直接做面積分**

Parametrize the surface S by

$$\begin{cases} x = 2+2\sin\phi\cos\theta \\ y = 2\sin\phi\sin\theta & ,\ 0\le\phi\le\pi \qquad (3\text{ pt}) \\ z = 2\cos\phi & ,\ 0\le\theta\le 2\pi. \end{cases}$$

$$F = 2\left(1+\sin\phi\cos\theta+\cos\phi, -\cos\phi-\sin\phi\sin\theta, \sin\phi\sin\theta+4\cos^3\phi\right)$$

$$\begin{aligned}
r_\phi &= 2\left(\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\right)\\
r_\theta &= 2\left(-\sin\phi\sin\theta, \sin\phi\cos\theta, 0\right)\\
r_\phi\times r_\theta &= 4\left(\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\right) \quad (3\text{ pt})
\end{aligned}$$

$$\begin{aligned}
\frac{1}{8}F\cdot(r_\phi\times r_\theta) =& \sin^2\phi\cos\theta+\sin^3\phi\cos^2\theta\\
&+\sin^2\phi\cos\phi\cos\theta-\sin^3\phi\sin^2\theta+4\sin\phi\cos^4\phi \quad (3\text{ pt})\\
=& \sin^2\phi\cos\theta\left(1+\cos\phi\right)+\sin^3\phi\cos 2\theta+4\sin\phi\cos^4\phi
\end{aligned}$$

$\cos\theta$ **以及** $\cos 2\theta$ **積分一個週期皆為** 0**, 所以上式剩下第三項.**

$$\begin{aligned}
\iint_S F\cdot n\,dS &= \iint_S F\cdot(r_\phi\times r_\theta)\,dA\\
&= 8\int_0^{2\pi}\int_0^{\pi} 4\sin\phi\cos^4\phi d\phi d\theta\\
&= 8\cdot 4\cdot 2\pi\underbrace{\int_0^{\pi}\sin\phi\cos^4\phi d\phi}_{\frac{2}{5}}\\
&= \frac{128}{5}\pi \quad (6\text{ pt})
\end{aligned}$$

(要注意 r_ϕ **跟** r_θ **外積的順序,** $r_\theta\times r_\phi$ **是指向球心的, 這樣算會多一個負號)**

2. (15%) Evaluate $\displaystyle\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (y + \sin x)\,\mathbf{i} + (z^2 + \cos y)\,\mathbf{j} + x^3\,\mathbf{k}$ and where S is the surface $z = 2xy$ inside the cylinder $x^2 + y^2 = 1$ and with the normal pointing in the positive z-direction.

Solution:

curl$\mathbf{F} =< -2z, -3x^2, -1 >$ (2%)

$r(x, y) =< x, y, 2xy >$

$r_x \times r_y =< -2y, -2x, 1 >$ (2%)

$$\iint_S \nabla \times \mathbf{F} \,\boldsymbol{.}\, d\mathbf{S}$$

$$= \iint_D 8xy^2 + 6x^3 - 1dA \text{ (2\%)}$$

$$= \int_0^{2\pi} \int_0^1 (8r^3 \cos\theta \sin^2\theta + 6r^3 \cos^3\theta - 1) r dr d\theta \text{ (2\%)}$$

$$= \int_0^{2\pi} (\frac{8}{5} r^5 \cos\theta \sin^2\theta + \frac{6}{5} r^5 \cos^3\theta - \frac{1}{2} r^2)|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{8}{5} \cos\theta \sin^2\theta + \frac{6}{5} \cos^3\theta - \frac{1}{2} d\theta$$

$$= \frac{8}{15} \sin^3\theta|_0^{2\pi} + \frac{6}{5} \int_0^{2\pi} \frac{\cos 3\theta + 3\cos\theta}{4} d\theta - \pi$$

$$= 0 + \frac{6}{5}(\frac{1}{12} \sin 3\theta + \frac{3}{4} \sin\theta)|_0^{2\pi} - \pi \text{ (6\% for all computation)}$$

$$= -\pi \text{ (1\%)}$$

3. (15%) Evaluate the line integral $\int_C \left(-x - y + \frac{y^2}{2}\right) dx + (x+2xy+3)dy$, where C consists of the arc C_1 of the quarter circle $x^2 + y^2 = 1, x \geq 0, y \leq 0$, from $(0, -1)$ to $(1, 0)$ followed by the arc C_2 of the quarter ellipse $4x^2 + y^2 = 4$, $x \geq 0$, $y \geq 0$, from $(1, 0)$ to $(0, 2)$. (Hint: You may use Green's Theorem, but note that C is not closed.)

Solution:

Method 1 use Green's Theorem

Let C_3 be the curve (part of y-axis) from $(0, 2)$ to $(0, -1)$. Then by Green's Theorem we have:

$$L = \int_C (-x - y + \frac{y^2}{2})dx + (x + 2xy + 3)dy$$
$$= \iint_D (1 + 2y) - (-1 + y)ds - \int_{C_3} (-x - y + \frac{y^2}{2})dx + (x + 2xy + 3)dy = A + B$$

where D is the region bounded by C and C_3. (5pts) for the equation above, however you can still get 3pts if you state Green's Theorem correctly.

$$A = \iint_D (1 + 2y) - (-1 + y)ds$$
$$= \int_0^1 \int_{-\sqrt{1-y^2}}^0 2 + y \; dydx + \int_0^1 \int_0^{\sqrt{4-4x^2}} 2 + y \; dydx$$
$$= (\frac{\pi}{2} - \frac{1}{3}) + (\pi + \frac{4}{3}) = \frac{3}{2}\pi + 1 \text{ (8pts)}$$

$$B = \int_{C_3} (-x - y + \frac{y^2}{2})dx + (x + 2xy + 3)dy = 0 + \int_2^{-1} 3 \; dt = -9 \text{ (2pts)}$$

$$\implies L = A + B = \frac{3}{2}\pi + 10$$

Method 2 Calculate directly

Let C_1 be curve $(\cos t, \sin t)$ t from $-\frac{\pi}{2}$ to 0.

Let C_2 be curve $(\cos t, 2\sin t)$ t from 0 to $\frac{\pi}{2}$.

$$L = \int_C (-x - y + \frac{y^2}{2})dx + (x + 2xy + 3)dy = \int_{C_1} + \int_{C_2}$$
$$= \int_{-\frac{\pi}{2}}^0 [-\cos t - \sin t + \frac{\sin^2 t}{2}](-\sin t) \; dt + [\cos t + 2\cos t \sin t + 3](\cos t) \; dt$$
$$+ \int_0^{\frac{\pi}{2}} [-\cos t - 2\sin t + 2\sin^2 t](-\sin t) \; dt + [\cos t + 4\cos t \sin t + 3](2\cos t) \; dt$$

(3pts) for the equation above.

$$= (\frac{\pi}{4} - \frac{1}{6}) + (\frac{\pi}{4} + \frac{7}{3}) \text{ (6pts)}$$
$$+ (\frac{\pi}{2} + \frac{5}{6}) + (\frac{\pi}{2} + \frac{26}{3}) \text{ (6pts)}$$
$$= \frac{3}{2}\pi + 10$$

4. (10%) Evaluate the surface integral $\iint_S (x^2+y^2)dS$, where S is the surface $z=\sqrt{x^2+y^2}$ with $0\le z\le 1$.

Solution:

Solution 1. We parameterize the surface S by

$$\mathbf{r}(u,v)=u\cos v\,\mathbf{i}+u\sin v\,\mathbf{j}+u\,\mathbf{k},\quad 0\le u\le 1,\ 0\le v<2\pi.\quad \text{(寫出参數式得 2 分)}$$

Then we have

$$\begin{aligned}
\mathbf{r}_u(u,v)&=\cos v\,\mathbf{i}+\sin v\,\mathbf{j}+1\,\mathbf{k}\\
\mathbf{r}_v(u,v)&=-u\sin v\,\mathbf{i}+u\cos v\,\mathbf{j}+0\,\mathbf{k}\\
\mathbf{r}_u\times\mathbf{r}_v&=-u\cos v\,\mathbf{i}-u\sin v\,\mathbf{j}+u\,\mathbf{k}\Rightarrow|\mathbf{r}_u\times\mathbf{r}_v|=\sqrt{2}u.\quad \text{(一個算式 1 分，共 4 分)}
\end{aligned}$$

So the surface integral is

$$\int_{v=0}^{v=2\pi}\int_{u=0}^{u=1}\sqrt{2}u\,\mathrm{d}u\,\mathrm{d}v=\frac{\sqrt{2}}{4}\cdot 2\pi=\frac{\sqrt{2}}{2}\pi.$$

(列式正確得 2 分，最後計算完成得 2 分。)

- 前面三個部分獨立算分；最後一個計算的給分必須前三個部分都完正確下才會考量。

Solution 2. For the function $f(x,y)=\sqrt{x^2+y^2}$, we compute

$$\begin{aligned}
\frac{\partial f}{\partial x}&=\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}}\\
\frac{\partial f}{\partial y}&=\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}}.\quad \text{(算出偏微分，共 2 分)}
\end{aligned}$$

So the surface integral is

$$\begin{aligned}
\iint_S (x^2+y^2)\,\mathrm{d}S&=\iint_{x^2+y^2\le 1}(x^2+y^2)\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,\mathrm{d}A\quad \text{(3分)}\\
&=\sqrt{2}\iint_{x^2+y^2\le 1}(x^2+y^2)\,\mathrm{d}A\ \text{(1分)}=\sqrt{2}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}r^2\cdot r\,\mathrm{d}r\,\mathrm{d}\theta\ \text{(2分)}\\
&=\sqrt{2}\int_{\theta=0}^{\theta=2\pi}\left[\frac{1}{4}r^4\right]\Bigg|_{r=0}^{r=1}\mathrm{d}\theta=\frac{\sqrt{2}}{4}\cdot 2\pi=\frac{\sqrt{2}}{2}\pi.\text{(2分)}
\end{aligned}$$

- 前面四個部分獨立算分；最後一個計算的給分必須前四個部分都完正確下才會考量。

5. (12%) (a) (8%) Find the potential function of the vector field

$$\mathbf{F}(x,y,z) = \frac{2x(1-\mathrm{e}^y)}{(1+x^2)^2}\,\mathbf{i} + \left(\frac{\mathrm{e}^y}{1+x^2} + (y+1)\mathrm{e}^y\right)\mathbf{j} + \mathbf{k}.$$

(b) (4%) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is any curve from $(0,0,0)$ to $(1,1,1)$.

Solution:

(a) Suppose there is an potential function $\Phi(x,y,z)$such that $\nabla\Phi(x,y,z) = \mathbf{F}(x,y,z)$

Then $\Phi(x,y,z) = \int \frac{2x(1-e^y)}{(1+x^2)^2}\,dx + g(y,z) = -\frac{1-e^y}{1+x^2} + g(y,z)$

Since $\frac{\partial\Phi}{\partial y} = \frac{e^y}{1+x^2} + (1+y)e^y$,we get $\frac{\partial g}{\partial y} = (1+y)e^y$

This impled $g(y,z) = y\,e^y + h(z)$, and $\Phi(x,y,z) = -\frac{1-e^y}{1+x^2} + y\,e^y + h(z)$

Finally we use $\frac{\partial\Phi}{\partial z} = 1$, then it's easy to find $h(z) = z + C$, where C is arbitary costant.

Let $C = 0$, we get $\Phi(x,y,z) = -\frac{1-e^y}{1+x^2} + y\,e^y + z$

(b) By theorem of potential function

$\int_C \mathbf{F}\,d\mathbf{r} = \Phi(1,1,1) - \Phi(0,0,0) = (-\frac{1-e}{2} + e + 1) - 0 = \frac{3e+1}{2}$

6. (12%) Evaluate the double integral $\displaystyle\iint_R (x+y)^2\sin^2(x-y)\,dA$, where R is the square region with vertices $(\frac{\pi}{2},0)$, $(\pi,\frac{\pi}{2})$, $(\frac{\pi}{2},\pi)$, and $(0,\frac{\pi}{2})$.

Solution:

Solution 1. Let $u = x+y$ and $v = x-y$ (2分), then $\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right| = 2$ and $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{2}$. (2分) The region R is mapped to the rectangle region $S = \{(u,v) | \frac{\pi}{2} \le u \le \frac{3\pi}{2}, -\frac{\pi}{2} \le v \le \frac{\pi}{2}\}$. So we have

$$\begin{aligned}\iint_R (x+y)^2\sin^2(x-y)\,\mathrm{d}A &= \int_{u=\frac{\pi}{2}}^{u=\frac{3\pi}{2}}\int_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}} u^2\sin^2 v\cdot\frac{1}{2}\,\mathrm{d}u\,\mathrm{d}v \text{ (2+1+3 分)}\\ &= \frac{1}{2}\left[\frac{1}{3}u^3\right]\Big|_{u=\frac{\pi}{2}}^{u=\frac{3\pi}{2}}\cdot\int_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}}\left(\frac{1-\cos 2v}{2}\right)\mathrm{d}v = \frac{13}{48}\pi^4. \text{ (2分)}\end{aligned}$$

- (2+1+3 分) 代表：範圍正確得 2 分；函數寫對得 1 分；Jacobian 有加絕對值，並代入正確得 3 分。
- 前面五個部分獨立算分；最後一個計算的給分必須前五個部分都完正確下才會考量。

Solution 2. Let $u = (x+y)^2$ and $v = x-y$ (2分), then $\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right| = 4(x+y)$ and $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{4(x+y)} = \dfrac{1}{4\sqrt{u}}$. (2分) The region R is mapped to the rectangle region $S = \{(u,v) | \left(\frac{\pi}{2}\right)^2 \le u \le \left(\frac{3\pi}{2}\right)^2, -\frac{\pi}{2} \le v \le \frac{\pi}{2}\}$. So we have

$$\begin{aligned}\iint_R (x+y)^2\sin^2(x-y)\,\mathrm{d}A &= \int_{u=\left(\frac{\pi}{2}\right)^2}^{u=\left(\frac{3\pi}{2}\right)^2}\int_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}} u\sin^2 v\cdot\frac{1}{4\sqrt{u}}\,\mathrm{d}u\,\mathrm{d}v \text{ (2+1+3 分)}\\ &= \int_{u=\left(\frac{\pi}{2}\right)^2}^{u=\left(\frac{3\pi}{2}\right)^2}\frac{\sqrt{u}}{4}\,\mathrm{d}u\cdot\int_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}}\sin^2 v\,\mathrm{d}v\\ &= \left[\frac{1}{6}u^{\frac{3}{2}}\right]\Big|_{u=\left(\frac{\pi}{2}\right)^2}^{u=\left(\frac{3\pi}{2}\right)^2}\cdot\int_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}}\left(\frac{1-\cos 2v}{2}\right)\mathrm{d}v = \frac{13}{48}\pi^4.\end{aligned}$$

- (2+1+3 分) 代表：範圍正確得 2 分；函數寫對得 1 分；Jacobian 有加絕對值，並代入正確得 3 分。
- 前面五個部分獨立算分；最後一個計算的給分必須前五個部分都完正確下才會考量。

7. (12%) Evaluate $\iint_D y \sin\left(\frac{\pi x}{y}\right) dA$, where D is the region bounded by $y = x$ and $y = \sqrt{x}$.

Solution:

Since $y \sin\left(\frac{\pi x}{y}\right)$ is bounded on D, by Fubini's theorem, we have:

$$\iint_D y \sin\left(\frac{\pi x}{y}\right) dA = \int_0^1 \int_x^{\sqrt{x}} y \sin\left(\frac{\pi x}{y}\right) dydx = \int_0^1 \int_{y^2}^{y} y \sin\left(\frac{\pi x}{y}\right) dxdy \text{ (4 pts)}$$

(The upper and lower bounds each accounts for 2 pts. If the order is reversed, you'll only get 3 pts.)

$$\begin{aligned} &= -\int_0^1 \left[y\frac{y}{\pi} \cos\frac{\pi x}{y}\right]_{y^2}^{y} dy = \frac{1}{\pi}\int_0^1 y^2 \cos \pi y + y^2 dy \text{ (4 pts. 3 pts if signs incorrect)} \\ &= \frac{1}{\pi}\left(-\frac{2}{\pi^2} + \frac{1}{3}\right) = \frac{1}{3\pi} - \frac{2}{\pi^3} \text{ (4 pts.)} \end{aligned}$$

In last step, the computation of $\int_0^1 y^2 \cos \pi y$ worths 3 pts in total. If you correctly integrate by parts twice but fail to get correct coefficients, you'll get 2 pts. Here's the computation:

$$\begin{aligned} &\int_0^1 y^2 \cos \pi y dy \\ =\ &\frac{1}{\pi}\int_0^1 y^2 d \sin \pi y = \frac{1}{\pi}\left[y^2 \sin \pi y\right]_0^1 - \frac{2}{\pi}\int_0^1 y \sin \pi y dy \\ =\ &\frac{2}{\pi^2}\int_0^1 y d \cos \pi y = \frac{2}{\pi^2}\left[y \cos \pi y\right]_0^1 - \frac{2}{\pi^2}\int_0^1 \cos \pi y dy = -\frac{2}{\pi^2} \end{aligned}$$

8. (9%) Let C be the closed curve formed by $y = x^2$, where $0 \le x \le 1$, and $x = y^2$, where $0 \le y \le 1$. Given C the counterclockwise orientation, evaluate $\int_C xy\, ds$ and $\oint_C xy\, dx$.

Solution:
Let $C = C_1 \cup C_2$, where

$$C_1 : \mathbf{r}_1(t) = t\,\mathbf{i} + t^2\,\mathbf{j},\ 0 \le t \le 1, \quad \mathbf{r}_1'(t) = 1\,\mathbf{i} + 2t\,\mathbf{j} \quad |\mathbf{r}_1'(t)| = \sqrt{1+(2t)^2}$$
$$C_2 : \mathbf{r}_2(t) = t^2\,\mathbf{i} + t\,\mathbf{j},\ 0 \le t \le 1, \quad \mathbf{r}_2'(t) = 2t\,\mathbf{i} + 1\,\mathbf{j} \quad |\mathbf{r}_2'(t)| = \sqrt{1+(2t)^2}$$

(a)

We compute

$$\begin{aligned}\int_C xy\,\mathrm{d}s &= \int_{C_1} xy\,\mathrm{d}s + \int_{C_2} xy\,\mathrm{d}s \\ &= \int_0^1 t\cdot t^2\sqrt{1+(2t)^2}\,\mathrm{d}t + \int_0^1 t^2\cdot t\sqrt{1+(2t)^2}\,\mathrm{d}t \\ &= 2\int_0^1 t^3\sqrt{1+(2t)^2}\,\mathrm{d}t. \quad (3\%)\end{aligned}$$

Note that since we integrate respect to arc length, so the value must be positive.

Let $u = 1+4t^2$, then $\mathrm{d}u = 8t\,\mathrm{d}t$, the upper limit gives $u = 5$ and lower limit implies $u = 1$, so

$$\begin{aligned}\int_C xy\,\mathrm{d}s = 2\int_0^1 t^3\sqrt{1+(2t)^2}\,\mathrm{d}t &= \frac{1}{4}\int_1^5 \frac{u-1}{4}\cdot\sqrt{u}\,\mathrm{d}u \\ &= \frac{1}{16}\int_1^5 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,\mathrm{d}u = \frac{1}{16}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right)\Big|_1^5 \\ &= \frac{5}{12}\sqrt{5} + \frac{1}{60}. \quad (2\%)\end{aligned}$$

(b)

We compute

$$\begin{aligned}\oint_C xy\,\mathrm{d}x = \int_{C_1} xy\,\mathrm{d}x + \int_{C_2} xy\,\mathrm{d}x &= \int_0^1 t\cdot t^2\,\mathrm{d}t + \int_1^0 t^2\cdot t\cdot 2t\,\mathrm{d}t \quad (3\%) \\ &= \int_0^1 (t^3 - 2t^4)\,\mathrm{d}t = \left[\frac{1}{4}t^4 - \frac{2}{5}t^5\right]\Bigg|_{t=0}^{t=1} = \frac{1}{4} - \frac{2}{5} = -\frac{3}{20}. \quad (1\%)\end{aligned}$$

Also, you can apply Green's Theorem.
Let $P = xy$ and $Q = 0$, then $Q_x = 0$ and $P_y = x$. By Green's Theorem, we have

$$\begin{aligned}\oint_C xy\,\mathrm{d}x = \iint_D -x\,\mathrm{d}A &= \int_{y=0}^{y=1}\int_{x=y^2}^{x=\sqrt{y}} -x\,\mathrm{d}x\,\mathrm{d}y = \int_{y=0}^{y=1}\left[-\frac{1}{2}x^2\right]\Bigg|_{x=y^2}^{x=\sqrt{y}}\mathrm{d}y \\ &= -\frac{1}{2}\int_0^1 (y - y^4)\,\mathrm{d}y = -\frac{1}{2}\left[\frac{1}{2}y^2 - \frac{1}{5}y^5\right]\Bigg|_{y=0}^{y=1} = -\frac{1}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = -\frac{3}{20}.\end{aligned}$$

or

$$\begin{aligned}\oint_C xy\,\mathrm{d}x = \iint_D -x\,\mathrm{d}A &= \int_{x=0}^{x=1}\int_{y=x^2}^{y=\sqrt{x}} -x\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{x=1}\left[-xy\right]\Bigg|_{y=x^2}^{y=\sqrt{x}}\mathrm{d}x \\ &= \int_0^1 \left(-x^{\frac{3}{2}} + x^3\right)\mathrm{d}x = \left[-\frac{2}{5}x^{\frac{5}{2}} + \frac{1}{4}x^4\right]\Bigg|_{x=0}^{x=1} = -\frac{2}{5} + \frac{1}{4} = -\frac{3}{20}.\end{aligned}$$