1. (15%) Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin and also the shortest distance between the surface and the origin.

**Solution:**

*Solution.* Consider the Lagrange function $F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(xy^2z^3 - 2)$. The critical points of $F(x, y, z, \lambda)$ satisfy

$$\begin{aligned} F_x &= 2x - \lambda y^2 z^3 = 0 && (1)\\ F_y &= 2y - \lambda 2xyz^3 = 0 && (2)\\ F_z &= 2z - \lambda 3xy^2z^2 = 0 && (3)\\ F_\lambda &= -(xy^2z^3 - 2) = 0. && (4) \end{aligned}$$

(8pts)
To solve these equations, since $x \neq 0, y \neq 0, z \neq 0$ on the surface $xy^2z^3 = 2$, from (1), (2), and (3), we get

$$\lambda = \frac{2x}{y^2z^3} = \frac{2y}{2xyz^3} = \frac{2z}{3xy^2z^2}.$$

Second equality gives $2x^2 = y^2 \Rightarrow x = \pm\frac{\sqrt{2}}{2}y$. Third equality gives $2z^2 = 3y^2 \Rightarrow z = \pm\frac{\sqrt{6}}{2}y$. We put $x = \pm\frac{\sqrt{2}}{2}y$ and $z = \pm\frac{\sqrt{6}}{2}y$ into (4) and get

$$\left(\pm\frac{\sqrt{2}}{2}y\right)y^2\left(\pm\frac{\sqrt{6}}{2}y\right)^3 = 2 \Rightarrow y^6 = \left(\frac{2}{\sqrt{3}}\right)^3 \Rightarrow y^2 = \frac{2}{\sqrt{3}} \Rightarrow y = \pm\frac{\sqrt{2}}{\sqrt[4]{3}}.$$

(4pts)
So we get four critical points

$$\left(\frac{1}{\sqrt[4]{3}}, \frac{\sqrt{2}}{\sqrt[4]{3}}, \sqrt[4]{3}\right),\ \left(-\frac{1}{\sqrt[4]{3}}, \frac{\sqrt{2}}{\sqrt[4]{3}}, -\sqrt[4]{3}\right),\ \left(\frac{1}{\sqrt[4]{3}}, -\frac{\sqrt{2}}{\sqrt[4]{3}}, \sqrt[4]{3}\right),\ \left(-\frac{1}{\sqrt[4]{3}}, -\frac{\sqrt{2}}{\sqrt[4]{3}}, -\sqrt[4]{3}\right).$$

(2pts)
These four critical points have the same distance to the origin:

$$d = \sqrt{\left(\frac{1}{\sqrt[4]{3}}\right)^2 + \left(\frac{\sqrt{2}}{\sqrt[4]{3}}\right)^2 + \left(\sqrt[4]{3}\right)^2} = \sqrt{\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \sqrt{3}} = \sqrt{2\sqrt{3}} = \sqrt{2}\sqrt[4]{3}.$$

$(1pts)$

*Solution 2.* Another method to solve the system of equations (1) – (4) is comparing $6x\times(1)$, $3y\times(2)$, and $2z\times(3)$, then we get $12x^2 = 6y^2 = 4z^2$. So we also find relations $2x^2 = y^2$ and $2z^2 = 3y^2$.

*Solution 3.* Instead of finding the maximum or minimum values of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, we consider its square function $f^2(x, y, z) = x^2 + y^2 + z^2$ because they both attain maximum or minimum at the same places.

Since $xy^2z^3 = 1$, we get $y^2 = \frac{1}{xz^3}$, so the question reduce to find the absolute minimum of the following function of two variables:

$$f(x, z) = x^2 + \frac{2}{xz^3} + z^2.$$

The critical points of $f(x, z)$ satisfy

$$\begin{aligned} f_x &= 2x - \frac{2}{x^2z^3} = \frac{2x^3z^3 - 2}{x^2z^3} = \frac{2(x^3z^3 - 1)}{x^2z^3} = 0\\ f_z &= -\frac{6}{xz^4} + 2z = \frac{-6 + 2xz^5}{xz^4} = \frac{2(-3 + xz^5)}{xz^4} = 0. \end{aligned}$$

From $f_x = 0$, we get $(xz)^3 = 1 \Rightarrow xz = 1$. From $f_z = 0$, we get $xz^5 = 3 \Rightarrow z^4 = 3 \Rightarrow z = \pm\sqrt[4]{3}$.

(a) If $z = \sqrt[4]{3}$, then $x = \frac{1}{\sqrt[4]{3}}$, and $y^2 = \frac{1}{xz^3} = \frac{1}{\sqrt{3}} \Rightarrow y = \pm\frac{1}{\sqrt[4]{3}}$.

- ⋆ At $P_1 = (\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \sqrt[4]{3})$, $d(P_1, O) = \sqrt{2\sqrt{3}} = \sqrt{2}\sqrt[4]{3}$.
- ⋆ At $P_2 = (\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \sqrt[4]{3})$, $d(P_2, O) = \sqrt{2\sqrt{3}} = \sqrt{2}\sqrt[4]{3}$.

(b) If $z = -\sqrt[4]{3}$, then $x = -\frac{1}{\sqrt[4]{3}}$, and $y^2 = \frac{1}{xz^3} = \frac{1}{\sqrt{3}} \Rightarrow y = \pm\frac{1}{\sqrt[4]{3}}$.

- ⋆ At $P_3 = (-\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\sqrt[4]{3})$, $d(P_3, O) = \sqrt{2\sqrt{3}} = \sqrt{2}\sqrt[4]{3}$.
- ⋆ At $P_4 = (-\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\sqrt[4]{3})$, $d(P_4, O) = \sqrt{2\sqrt{3}} = \sqrt{2}\sqrt[4]{3}$.

These four critical points have the shortest distance between the surface and the origin.

2. (12%) Find all the critical points of $f(x, y) = 4 + x^3 + y^3 - 3xy$. Then determine which gives a local maximum or a local minimum or a saddle point.

**Solution:**

- $f(x, y) = 4 + x^3 + y^3 - 3\,x\,y \implies (f_x, f_y) = (3\,x^2 - 3\,y, 3\,y^2 - 3\,x)$ ($f_x$: 1%, $f_y$: 1%)
- Solve
$$\begin{cases} f_x = 3x^2 - 3y = 0 \\ f_y = 3y^2 - 3x = 0. \end{cases}$$
to obtain $(x, y) = (0, 0), (1, 1)$. Therefore the critical points are $(x, y) = (0, 0), (1, 1)$.
($f_x = 0$ : 1%, $f_y = 0$ : 1%, solving: 1%)
- $(f_x, f_y) = (3\,x^2 - 3\,y, 3\,y^2 - 3\,x) \implies f_{xx} = 6\,x$, $f_{xy} = f_{yx} = -3$, $f_{yy} = 6\,y$, and
$$D(x, y) = \begin{vmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{vmatrix} = \begin{vmatrix} 6\,x & -3 \\ -3 & 6\,y \end{vmatrix} = 36\,x\,y - 9.$$
($f_{xx}$ : 1%, $f_{xy}$ : 1%, $f_{yx}$ : 1%, $f_{yy}$ : 1%, $D(x, y)$ : 1%)
- $D(0, 0) = -9 < 0 \implies (0, 0)$ is a saddle point. (1%)
- $f_{xx}(1, 1) = 6 > 0$ and $D(1, 1) = 27 > 0 \implies f(1, 1)$ is a local minimum. (1%)

3. (12%) Let the unit vectors **u** and **n** be respectively the tangent direction and the normal direction (with positive $x$-components) of the circle $x^2 + y^2 - 2x = 0$ at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Let $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$. Find $\nabla f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $D_{\mathbf{u}}f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $D_{\mathbf{n}}f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

**Solution:**

- $f_x(x, y) = \frac{1}{1+\frac{y^2}{x^2}} \frac{\partial}{\partial x}(\frac{y}{x}) = \frac{-y}{x^2+y^2}$. (2 points)
- $f_y(x, y) = \frac{x}{x^2+y^2}$. (2 points)
- $\nabla f(x, y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} \implies \nabla f(\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{-\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}$. (1 point)
- The equation $x^2 + y^2 - 2\,x = 0$ can be rewritten as $(x - 1)^2 + y^2 = 1$, which represents a circle centered at $(1, 0)$. Let $F(x, y) = x^2 + y^2 - 2\,x$.
  - The normal direction of the circle at $(x, y)$ is $\nabla F = (2x - 2, 2y) = 2(x - 1, y)$. (2 points)
  - The tangent direction of the circle at $(x, y)$ is $(y, -x + 1)$. (1 point)
  - At $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, the normal direction with the positive $x$-component is $\vec{n} = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. (1 point)

- At $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, the tangent direction with the positive $x$-component is $\vec{u} = (\frac{\sqrt{3}}{2}, \frac{1}{2})$. (1 point)

- $D_{\vec{n}} f(\frac{1}{2}, \frac{\sqrt{3}}{2}) = (-\frac{\sqrt{3}}{2}, \frac{1}{2}) \cdot (\frac{1}{2}, -\frac{\sqrt{3}}{2}) = -\frac{\sqrt{3}}{2}$. (1 point)
- $D_{\vec{u}} f(\frac{1}{2}, \frac{\sqrt{3}}{2}) = (-\frac{\sqrt{3}}{2}, \frac{1}{2}) \cdot (\frac{\sqrt{3}}{2}, \frac{1}{2}) = -\frac{1}{2}$. (1 point)

4. (a) (10%) Find the 4-th degree MacLaurin polynomials of $\sec x$ and of $\left(1 - x^2\right)^{-\frac{1}{2}}$ (5% each).

   (b) (4%) Find $\lim_{x \to 0} \frac{\sec x - \left(1 - x^2\right)^{-\frac{1}{2}}}{x^4}$.

**Solution:**

(a) The 4-th degree MacLaurin polynomial of $\sec x$ can be derived from the cosine function: since

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots,$$

the MacLaurin polynomial of $\sec x$ can be obtained using long division or by comparing the coefficients in

$$1 = (\cos x) \cdot (\sec x) = (1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots) \cdot (a_0 + a_2 x^2 + a_4 x^4 + \cdots)$$

Then we have

$$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots.$$

(Since only a finite number of terms are required, you may also use the definition of MacLaurin polynomial: $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$ and perform the required differentiation to get the answer.)

On the other hand, by the binomial expansion

$$\begin{aligned}(1 - x^2)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n \\ &= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \cdots\end{aligned}$$

Grading policy: 5 points for each polynomial. Three points are credited if only two terms are correct.

(b) From the results in part (a)

$$\begin{aligned}\lim_{x \to 0} \frac{\sec x - (1 - x^2)^{-\frac{1}{2}}}{x^4} &= \lim_{x \to 0} \frac{(1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots) - (1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \cdots)}{x^4} \\ &= \lim_{x \to 0} \frac{-\frac{1}{6}x^4 + \cdots}{x^4} \\ &= -\frac{1}{6} \text{ (4 points)}\end{aligned}$$

5. (a) (12%) Find the radius of convergence and the interval of convergence of the power series $f(x) = \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n}$.

   (b) (3%) Evaluate $f^{(3)}(0)$.

**Solution:**

(a) By Ratio Test, f(x) converges absolutely if $\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| < 1$ (4%)

$\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = \lim_{n \to \infty} |\frac{(-1)^{n+1} \frac{x^{n+1}}{4^{n+1} \ln n+1}}{(-1)^n \frac{x^n}{4^n \ln n}}| = \lim_{n \to \infty} |\frac{x \ln n}{4 \ln n + 1}| = |\frac{x}{4}| < 1$ (2%)

Since $\lim_{y \to \infty} |\frac{\ln y}{\ln y + 1}| = \lim_{y \to \infty} |\frac{y+1}{y}| = 1$

Hence, the radius of convergence is 4.

for $x = 4$, $f(x) = \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ converges.

Since (i) $\lim_{n\to\infty} \frac{1}{\ln n} = 0$, (ii) $\frac{1}{\ln n} > \frac{1}{\ln n + 1} \Rightarrow$ converges by Alternating Series Test. (3%)

for $x = -4$, $f(x) = \sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$ diverges.

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ is p-series of $p = 1 \Rightarrow$ diverges (3%)

(b) $f'(x) = \sum_{n=2}^{\infty} (-1)^n \frac{x^{n-1}}{4^n \ln n} \times n$

$f''(x) = \sum_{n=2}^{\infty} (-1)^n \frac{x^{n-2}}{4^n \ln n} \times n(n-1)(1\%)$

$f^{(3)}(x) = \sum_{n=3}^{\infty} (-1)^n \frac{x^{n-3}}{4^n \ln n} \times n(n-1)(n-2)(1\%)$

$f^{(3)}(0) = (-1)^3 \frac{1}{4^3 \ln 3} \times 3 \times 2 \times 1 = -\frac{6}{4^3 \ln 3}(1\%)$

< **Solution2** > $\frac{f^{(3)}(0)}{3!}(2\%) =$ coefficient of $x^3 = \frac{(-1)^3}{4^3 \ln 3}(1\%) \Rightarrow f^{(3)}(0) = -\frac{6}{4^3 \ln 3}$

6. (10%) Suppose that $z = f(x, y)$ is a smooth function and let $x = uv$, and $y = v - u$. Express $\frac{\partial^2 z}{\partial u \partial v}$ in terms of $x, y, f_x, f_y, f_{xx}, f_{xy}$, and $f_{yy}$.

**Solution:**

The chain rule gives

$$\begin{aligned} \frac{\partial z}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} && \text{(3 points)} \\ &= \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}(1) && \text{(2 points)} \\ &= u\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \end{aligned}$$

Apply the product rule and again the chain rule, and also note that $\frac{\partial}{\partial u}(u) = 1$,

$$\begin{aligned} \frac{\partial^2 z}{\partial u \partial v} &= \frac{\partial}{\partial u}\left(u\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) \\ &= \frac{\partial f}{\partial x} + u\frac{\partial^2 f}{\partial u \partial x} + \frac{\partial^2 f}{\partial u \partial y} \\ &= \frac{\partial f}{\partial x} + u\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u}\right) + \left(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial u}\right) \\ &= \frac{\partial f}{\partial x} + u\left(\frac{\partial^2 f}{\partial x^2}v + \frac{\partial^2 f}{\partial y \partial x}(-1)\right) + \left(\frac{\partial^2 f}{\partial x \partial y}v + \frac{\partial^2 f}{\partial y^2}(-1)\right) \\ &= uv\frac{\partial^2 f}{\partial x^2} + v\frac{\partial^2 f}{\partial x \partial y} - u\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x} \end{aligned}$$

Since $f$ is a smooth function, $f_{xy} = f_{yx}$. Therefore

$$\begin{aligned} \frac{\partial^2 z}{\partial u \partial v} &= uvf_{xx} + (v - u)f_{xy} - f_{yy} + f_x && \text{(4 points)} \\ &= xf_{xx} + yf_{xy} - f_{yy} + f_x && \text{(1 point)} \end{aligned}$$

(Because $f$ satisfies the condition of Clairaut's theorem, you can first calculate $z_u$ and $z_{uv}$, and then claim that $z_{uv} = z_{vu}$. This approach yields the same solution as above.)

7. (10%) Find the equation of the tangent plane to the elliptic paraboloid $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ at the point $(a, b, 2c)$.

**Solution:**

Let $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c}$
then $f_x = \frac{2x}{a^2}$, $f_y = \frac{2y}{b^2}$, $f_z = \frac{-1}{c}$
at $(a,b,2c)$, we have $f_x = \frac{2}{a}$, $f_y = \frac{2}{b}$, $f_z = \frac{-1}{c}$
Hence the tangent plane is $\frac{2}{a}(x-a) + \frac{2}{b}(x-b) - \frac{1}{c}(x-2c) = 0$

$f$ 對 $x$ 和 $y$ 和 $z$ 的偏微分各 1 分
把點帶進去 4 分
切平面方程式 3 分

8. (a) (2%) Parametrize the curve of intersection of the parabolic cylinders $x = y^2$ and $z = x^2$ by setting $t = y$.
   (b) (10%) Find the unit tangent $\mathbf{T}$ and the curvature $\kappa$ at the point $(1,1,1)$.

**Solution:**

(a) Since $x = y^2$ ,$z = x^2$, and $y = t$ , we can see $x(t) = t^2$ and $z(t) = t^4$.

$$\begin{cases} x = t^2 \\ y = t, \qquad t \in \mathbb{R} \\ z = t^4 \end{cases}$$

or write as $\mathbf{r}(t) = (t^2, t, t^4)$, $t \in \mathbb{R}$.
Although I do not deduction any points, you should still wirte down range of $t$.

(b) By formula $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|r'(t)|}$, and easy to know $\mathbf{r}'(t) = (2t, 1, 4t^3)$, so $\mathbf{r}'(1) = (2,1,4)$, and $|r'(t)| = \sqrt{21}$. So we get $\mathbf{T}(1) = \frac{(2,1,4)}{\sqrt{21}}$.
If you do perfect, you get 5 points. If you compute some error, you will get from 1 to 4 points, depending your answer. If you use wrong formula, you will get 0 or 1 point.

By formula $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$, so we only need to compute $\kappa(1) = \dfrac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3}$.

First, $\mathbf{r}''(t) = (2, 0, 12t^2)$, so $\mathbf{r}'(1) \times \mathbf{r}''(1) = (2,1,4) \times (2,0,12) = (12, -16, -2)$. So $|\mathbf{r}'(1) \times \mathbf{r}''(1)| = \sqrt{404} = 2\sqrt{101}$, and $|\mathbf{r}'(1)|^3 = 21\sqrt{21}$. We get the answer is $\kappa 1 = \dfrac{2\sqrt{101}}{21\sqrt{21}}$.
If you do perfect, you get 5 points. If you compute some error, you will get from 1 to 4 points, depending your answer. If you use wrong formula, you will get 0 or 1 point.