1032微甲01-04班期末考解答和評分標準

1. (5%) Determine the statement is true (◯) or false (×).

   (a) If $f(x,y)$ is continuous on the rectangle $R = \{(x,y) |\ a \le x \le b,\ c \le y \le d\}$ except for finitely many points, then $f(x,y)$ is integrable on R and
   $$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx.$$

   (b) If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on an open connected region D, then $\mathbf{F}$ is conservative on D.

   (c) If curl $\mathbf{F}$=curl $\mathbf{G}$ on $\mathbb{R}^3$, then $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{G}\cdot d\mathbf{r}$ for all closed path C.

   (d) If $\mathbf{F}$ and $\mathbf{G}$ are vector fields and curl $\mathbf{F}$=curl $\mathbf{G}$, div $\mathbf{F}$=div $\mathbf{G}$, then $\mathbf{F} - \mathbf{G}$ is a constant vector field.

   (e) Let B be a rigid body rotating about the z-axis with constant angular speed ω. If $\mathbf{v}(x,y,z)$ is the velocity at point $(x,y,z) \in B$, then curl $\mathbf{v}$ is parallel to $\mathbf{k}$.

   Answer. (每小題各 1 分)

(a)	(b)	(c)	(d)	(e)
×	×	×	×	◯

2. (10%) Write the integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)dzdydx$ in 5 other orders.

   Answer. (每小題錯一格扣 1 分，錯兩格以上全錯)

   (a) $\int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{y^2}} \int_{\boxed{0}}^{\boxed{1-y}} f(x,y,z)dzdxdy$

   (b) $\int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{1-z}} \int_{\boxed{0}}^{\boxed{y^2}} f(x,y,z)dxdydz$

   (c) $\int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{1-y}} \int_{\boxed{0}}^{\boxed{y^2}} f(x,y,z)dxdzdy$

   (d) $\int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{1-\sqrt{x}}} \int_{\boxed{\sqrt{x}}}^{\boxed{1-z}} f(x,y,z)dydzdx$

   (e) $\int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{(1-z)^2}} \int_{\boxed{\sqrt{x}}}^{\boxed{1-z}} f(x,y,z)dydxdz$

3. (15%) Evaluate the integrals.

(a) $\int_0^1 \int_{\tan^{-1} y}^{\frac{\pi}{4}} \cos x \cdot \tan(\cos x) dx dy$.

(b) $\int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x+y}{x^2+y^2} dx dy + \int_0^1 \int_{1-y}^1 \frac{x+y}{x^2+y^2} dx dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \frac{x+y}{x^2+y^2} dy dx$.

Solution:

(a)

$$\begin{aligned}
\int_0^1 \int_{tan^{-1}y}^{\frac{\pi}{4}} \cos x \tan(\cos x) dx dy &= \int_0^{\frac{\pi}{4}} \int_0^{\tan x} \cos x \tan(\cos x) dy dx \quad \text{(3pt)} \\
&= \int_0^{\frac{\pi}{4}} \sin x \tan(\cos x) dx \ \text{(Let } u = \cos x,\ du = -\sin x dx\text{)} \\
&= -\int_1^{\frac{1}{\sqrt{2}}} \tan u du \\
&= \ln(\cos u)\Big|_1^{\frac{1}{\sqrt{2}}} \quad \text{(2pt)} \\
&= \ln(\cos \frac{1}{\sqrt{2}}) - \ln(\cos 1) \quad \text{(1pt)}
\end{aligned}$$

(b)

$$\begin{aligned}
& \int_1^{\sqrt{2}} \int^{\sqrt{2-y^2}} \frac{x+y}{x^2+y^2} dx dy + \int_0^1 \int_{1-y}^1 \frac{x+y}{x^2+y^2} dx dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \frac{x+y}{x^2+y^2} dy dx \\
= & \iint_D \frac{x+y}{x^2+y^2} dA, \quad \text{where } D \text{ is bounded by } x^2+y^2 = 2 \text{ and } x+y = 1.
\end{aligned}$$

By polar coordinate, we have

$$\begin{aligned}
\iint_D \frac{x+y}{x^2+y^2} dA &= \int_0^{\frac{\pi}{2}} \int_{\frac{1}{\cos\theta+\sin\theta}}^{\sqrt{2}} \frac{r\cos\theta + r\sin\theta}{r^2} \cdot r dr d\theta \ \text{(4pt)} \\
&= \int_0^{\frac{\pi}{2}} \int_{\frac{1}{\cos\theta+\sin\theta}}^{\sqrt{2}} (\cos\theta + \sin\theta) dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \sqrt{2}(\cos\theta + \sin\theta) - 1 d\theta \\
&= \sqrt{2}(\sin\theta - \cos\theta)\Big|_0^{\frac{\pi}{2}} - \frac{\pi}{2} \quad \text{(3pt)} \\
&= 2\sqrt{2} - \frac{\pi}{2} \quad \text{(2pt)}
\end{aligned}$$

4. (10%) Let S be the surface $x^2+y^2+z^2=a^2$, $x \geq 0$, $y \geq 0$, $z \geq 0$ $(a>0)$, and let C be the boundary of S. Find the centroid of C.

Solution:

For a quarter circle of radius a (named C') on a plane, its centroid can be found to be at $(\frac{2a}{\pi}, \frac{2a}{\pi})$ by either way:
(1) Parametrize the curve:
Parametrize C' by $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle, t \in [0, \frac{\pi}{2}]$.
$\Rightarrow |\mathbf{r}'(t)| = |\langle -a\sin t, a\cos t\rangle| = a$.
Arc length $s = \frac{1}{4}\cdot 2\pi a = \frac{1}{2}\pi a$.

$$\overline{x}\cdot s = \int_{C'} x \cdot ds = \int_0^{\frac{\pi}{2}} x(t)|\mathbf{r}'(t)|dt = \int_0^{\frac{\pi}{2}} a\cos(t)a dt = a^2$$
$$\therefore \overline{x} = \frac{a^2}{\frac{1}{2}\pi a} = \frac{2a}{\pi}$$

By the symmetry of the arc, $\overline{y} = \overline{x} = \frac{2a}{\pi}$.

(2) Pappus's Theorem:
Knowing that the surface area of a hemisphere of radius a is $2\pi a^2$ and the arc length of a quarter circle of radius a is $\frac{1}{2}\pi a$, if the quarter circle is in the first quadrant and is rotated about the x-axis, Pappus's Theorem gives

$$A = 2\pi\overline{y}\cdot s$$
$$\Rightarrow 2\pi a^2 = 2\pi\overline{y}\cdot\frac{\pi a}{2}$$
$$\Rightarrow \overline{y} = \frac{2a}{\pi}$$

By the symmetry of the arc, $\overline{x} = \overline{y} = \frac{2a}{\pi}$.

(7 points up to this point.)

The curve C is composed of quarter circles C_1, C_2, and C_3 on the xy-, yz-, and xz-planes, respectively. By the above discussion, their centroids are $(\frac{2a}{\pi}, \frac{2a}{\pi}, 0)$, $(0, \frac{2a}{\pi}, \frac{2a}{\pi})$, and $(\frac{2a}{\pi}, 0, \frac{2a}{\pi})$, respectively. Since they have equal masses, the centroid of C is the average of them, namely $(\frac{4a}{3\pi}, \frac{4a}{3\pi}, \frac{4a}{3\pi})$. (3 points)

(Note: if you misunderstood the problem but correctly calculated the centroid of the surface S to be at $(\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$, you still get 4 points. But no points will be given if you calculated the centroid of the part of the volume inside the sphere in the first octant.)

5. (10%) Let C be the curve of intersection of $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 = 2x$, $z \geq 0$, oriented C to be counterclockwise when viewed from above. Evaluate $\int_C y^2 dx + z^2 dy + x^2 dz$.

Solution:

- Solution 1: Using line integral to solve this problem directly.
  $r(\theta) =< 1 + \cos\theta, \sin\theta, 2\sin\frac{\theta}{2} >,\ \theta \in [0, 2\pi].$ (3 points)
  The original equation $= \int_0^{2\pi} [-\sin^3\theta + \sin^2\frac{\theta}{2}\cos\theta + (1+\cos\theta)^2\cos\frac{\theta}{2}]\ \mathrm{d}\theta$ (3 points)
  By symmetry, the first and the third term will be zero in the end.
  Therefore, the above equation will change as follows:
  $4\int_0^{2\pi} \sin^2\frac{\theta}{2}\ \mathrm{d}\theta$
  $= 4\int_0^{2\pi} \frac{1-\cos\theta}{2}\cos\theta\ \mathrm{d}\theta$
  $= -2\pi$ (4 points)

- Solution 2: Using Stokes' Theorem to solve this problem.
  $F =< y^2, z^2, x^2 >$
  $\nabla \times F =< -2z, -2x, -2y >$ (2 points)

  $r(x,y) =< x, y, \sqrt{4 - x^2 - y^2} >$
  $r_x =< 1, 0, \frac{-x}{\sqrt{4-x^2-y^2}} >$
  $r_y =< 0, 1, \frac{-y}{\sqrt{4-x^2-y^2}} >$
  $r_x \times r_y =< \frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1 >$ (2 points)

  By Stokes' Theorem,
  $\oint_c F\cdot\ \mathrm{d}r = \iint_S (\nabla \times F)\cdot\ \mathrm{d}S$
  $= \iint_D < -2\sqrt{4 - x^2 - y^2}, -2x, -2y > \cdot < \frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1 >\ \mathrm{d}A$
  $= \iint_D (-2x - \frac{2xy}{\sqrt{4-x^2-y^2}} - 2y)\ \mathrm{d}A$ (3 points)

  By symmetry, the second and the third term will be zero in the end.
  Therefore, the above equation will change as follows:
  $-2\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\cos\theta r \mathrm{d}r\mathrm{d}\theta$
  $-2\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos\theta\ (\frac{1}{3}r^3)\ |_0^{2\cos\theta}\ r\mathrm{d}r\mathrm{d}\theta$
  $= \frac{-16}{3}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\ \mathrm{d}\theta$
  $= \frac{-32}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta\ \mathrm{d}\theta$
  $= \frac{-32}{3}\int_0^{\frac{\pi}{2}} (\frac{\cos 2\theta - 1}{2})^2\ \mathrm{d}\theta$
  $= \frac{-8}{3}\int_0^{\frac{\pi}{2}} (\cos^2 2\theta - 2\cos 2\theta + 1)\ \mathrm{d}\theta$

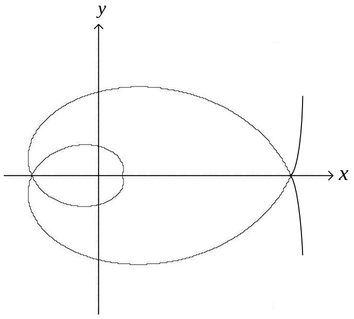
  By symmetry, the second term will be zero in the end.
  Therefore, the above equation will change as follows:
  $= \frac{-8}{3} \cdot \frac{3}{2} \cdot \frac{\pi}{2}$
  $= -2\pi.$ (3 points)

6. (20%) Let $\mathbf{F} = \frac{(x-y)^2 y}{(x^2+y^2)^2}\mathbf{i} + \frac{-(x-y)^2 x}{(x^2+y^2)^2}\mathbf{j}$.

(a) Verify that $\mathbf{F}$ is conservative on the right half plane $x > 0$. Find a potential function of $\mathbf{F}$ on the right half plane.

(b) Evaluate $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r}$ where C_1 is the ellipse $\frac{x^2}{4} + (y-2)^2 = 1$.

(c) Evaluate $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ where C_2 is the curve with polar equation $r = e^{|\theta|}$, $-\frac{9\pi}{4} \le \theta \le \frac{9\pi}{4}$.



Solution:

(a) (6%) $\partial_y f = \frac{-(x-y)^2 x}{(x^2+y^2)^2} \implies f = \frac{-x^2}{(x^2+y^2)^2} - \tan^{-1}(\frac{y}{x}) + g(x)$

$\implies \partial_x f = \frac{-2xy^2}{(x^2+y^2)^2} + \frac{y}{(x^2+y^2)^2} + g'(x) = \frac{(x-y)^2 y}{(x^2+y^2)^2} + g'(x) = \frac{(x-y)^2 y}{(x^2+y^2)^2}$

$\implies g'(x) = 0 \implies g$ is constant

$\implies f = \frac{-x^2}{(x^2+y^2)^2} - \tan^{-1}(\frac{y}{x})$ or $1 + \frac{-x^2}{(x^2+y^2)^2} - \tan^{-1}(\frac{y}{x}) = \frac{y^2}{(x^2+y^2)^2} - \tan^{-1}(\frac{y}{x})$ (6%)

Other point: $P_y = Q_x = \frac{x^4-4x^3y+4xy^3-y^4}{(x^2+y^2)^3}$ (2%); "$P_y = Q_x$" implies f is conservative (1%)

(b) (4%) Since $\{y > 0\}$ is simple connected, $\mathbf{F}$ is conservative on $\{y > 0\}$.
On the other hand, C_1 is closed curve on $y > 0$ (1%); therefore, $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$ (3%)

(c) (10%) "method 1"
the integral on C_2 is equal to the integral on unit circle times two and integral on the tail.
$\int_D \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} -(\cos\theta - \sin\theta)^2 d\theta = -2\pi$ (4%), where D is unit circle.
The integral on tail is independent of path, which equals to
$f(\cos\frac{9\pi}{4}e^{\frac{9\pi}{4}}, \sin\frac{9\pi}{4}e^{\frac{9\pi}{4}}) - f(\cos\frac{-9\pi}{4}e^{\frac{-9\pi}{4}}, \sin\frac{-9\pi}{4}e^{\frac{-9\pi}{4}}) = -\frac{\pi}{2}$(2%), where f is potential function of $\mathbf{F}$
Therefore, $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = -4\pi - \frac{\pi}{2} = -\frac{9\pi}{2}$(2%)
"method2"
$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\gamma_1} \mathbf{F}\cdot d\mathbf{r} + \int_{\gamma_2} \mathbf{F}\cdot d\mathbf{r}$, where γ_1 is the "$\theta \ge 0$" part of C_2, γ_2 is "$\theta < 0$" part of C_2,
in which $x(\theta)$ and $y(\theta)$ is differentiable.
$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^{\frac{9\pi}{4}} -(\cos\theta - \sin\theta)^2 d\theta + \int_{-\frac{9\pi}{4}}^0 -(\cos\theta - \sin\theta)^2 d\theta$ (4%)
$= \int_{-\frac{9\pi}{4}}^{\frac{9\pi}{4}} -(\cos\theta - \sin\theta)^2 d\theta = -\frac{9\pi}{2}$ (6%) (the answer worth 2 point)

7. (10%) Evaluate $\int_C (y+\sin^3 x)dx+(z^2+\cos^4 y)dy+(x^3+\tan^5 z)dz$ where C is the curve $\mathbf{r}(t)=\sin t\ \mathbf{i}+\cos t\ \mathbf{j}+\sin 2t\ \mathbf{k}$, $0\le t\le 2\pi$. [*Hint*: C lies on the surface $z=2xy$.]

Solution:

First observe that $r(t)=\sin t\ \boldsymbol{i}+\cos t\ \boldsymbol{j}+\sin 2t\ \boldsymbol{k}$ is negative oriented.
Thus by Stoke's theorem:

$$\int_C (y+\sin^3 x)\ dx+(z^2+\cos^4 y)\ dy+(x^3+\tan^5 z)\ dz=-\iint_S \nabla F\cdot dS \qquad (2\%)$$

where S is the surface $z=2xy$ bounded by $D=\{x^2+y^2\le 1\}$

$$\nabla F=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+\sin^3 x & z^2+\cos^4 y & x^3+\tan^5 z \end{vmatrix}=-2z\ \boldsymbol{i}-3x^2\ \boldsymbol{j}-\boldsymbol{k} \qquad (2\%)$$

$$\vec{n}=(-\frac{\partial z}{\partial x},-\frac{\partial z}{\partial y},1)/\|(-\frac{\partial z}{\partial x},-\frac{\partial z}{\partial y},1)\|=(-2y,-2x,1)/\|(-2y,-2x,1)\| \qquad (2\%)$$

$$\begin{aligned}\iint_S \nabla F\cdot dS &= \iint_D (-2z,-3x^2,-1)\cdot(-2y,-2x,1)\,dA \\ &= \iint_D 4yz+6x^3-1\,dA \\ &= \iint_D 8x^2y+6x^3-1\,dA \\ &= \int_0^{2\pi}\int_0^1 (8r^3\cos\theta\sin^2\theta+6r^3\cos^3\theta-1)r\,dr\,d\theta \\ &\ \ \vdots \\ &= -\pi \qquad (4\%)\end{aligned}$$

$$\int_C (y+\sin^3 x)\ dx+(z^2+\cos^4 y)\ dy+(x^3+\tan^5 z)\ dz=\pi$$

8. (10%) Evaluate $\iint_S x dS$ where S is the part of the cone $z = \sqrt{2(x^2+y^2)}$ that lies below the plane $z = 1 + x$.

Solution:

Step1.

Find the projection onto the xy-plane of the curve of intersection of the cone $z = \sqrt{2(x^2+y^2)}$ and the plane $z = 1 + x$.

$$\begin{cases} z = \sqrt{2(x^2+y^2)} \\ z = 1 + x \end{cases}$$

$$\Rightarrow \quad 2(x^2+y^2) = (x+1)^2$$

$$\Rightarrow \quad \left(\frac{x-1}{\sqrt{2}}\right)^2 + y^2 = 1 \quad \textbf{(1pt)}$$

Step2.

If we regard x and y as parameters, then we can write the parametric equations of S as

$$x = x \quad y = y \quad z = \sqrt{2(x^2+y^2)} \quad \textbf{(1pt)}$$

where

$$1 - \sqrt{2(1-y^2)} \le x \le 1 + \sqrt{2(1-y^2)} \quad , \quad -1 \le y \le 1 \quad \textbf{(1pt)}$$

and the vector equation is

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + \sqrt{2(x^2+y^2)}\mathbf{k}$$

Step3.

Find $|\mathbf{r}_x \times \mathbf{r}_y|$.

$$\begin{cases} \mathbf{r}_x = 1\mathbf{i} + 0\mathbf{j} + \frac{\sqrt{2}x}{\sqrt{x^2+y^2}}\mathbf{k} \\ \mathbf{r}_y = 0\mathbf{i} + 1\mathbf{j} + \frac{\sqrt{2}y}{\sqrt{x^2+y^2}}\mathbf{k} \end{cases}$$

$$\Rightarrow \quad \mathbf{r}_x \times \mathbf{r}_y = \frac{-\sqrt{2}x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{-\sqrt{2}y}{\sqrt{x^2+y^2}}\mathbf{j} + 1\mathbf{k} \quad \textbf{(2pts)}$$

$$\Rightarrow \quad |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{3} \quad \textbf{(1pt)}$$

Step4.

Evaluate $\iint_S x dS$.

$$\begin{aligned} \iint_S x dS &= \iint_D x \cdot |\mathbf{r}_x \times \mathbf{r}_y| dx dy \quad \textbf{(2pts)} \\ &= \sqrt{3} \int_{-1}^{1} \int_{1-\sqrt{2(1-y^2)}}^{1+\sqrt{2(1-y^2)}} x dx dy \\ &= \sqrt{3} \cdot 2\sqrt{2} \int_{-1}^{1} \sqrt{1-y^2} dy \\ &= 2\sqrt{6} \cdot \frac{\pi}{2} \\ &= \sqrt{6}\pi \quad \textbf{(2pts)} \end{aligned}$$

9. (10%) Let S be the surface of the solid bounded by $x^2+y^2+z^2=1$ and $z \geq \frac{1}{2}$. Find the total flux of $\mathbf{F}(x,y,z) = x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$ across S.

Solution:

(Method I)

Let $V = \left\{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 \leq 1, z \geq \frac{1}{2}\right\}$, then by Divergence Theorem,

$$\text{Flux of } \mathbf{F} = \iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V \text{div}\mathbf{F}\ d\mathbf{S} = \iiint_V 2(x+y+z)\ d\mathbf{S} \quad (3\%)$$

From the symmetry of V, we have $\iiint_V x\ d\mathbf{S} = \iiint_V y\ d\mathbf{S} = 0$.

Therefore, $\iiint_V \text{div}\mathbf{F}\ d\mathbf{S} = \iiint_V 2z\ d\mathbf{S} = 2\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\frac{1}{2}\sec\phi}^1 \rho\cos\phi\rho^2\sin\phi d\rho d\phi d\theta$

$$\begin{aligned}\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\frac{1}{2}\sec\phi}^1 \rho\cos\phi\rho^2\sin\phi d\rho d\phi d\theta &= \frac{1}{4}\int_0^{2\pi}\int_0^{\frac{\pi}{3}} \rho^4\cos\phi\sin\phi\Big|_{\frac{1}{2}\sec\phi}^1 d\phi d\theta \\ = \frac{2\pi}{4}\int_0^{\frac{\pi}{3}} \cos\phi\sin\phi - \frac{1}{16}\tan\phi\sec^2\phi\ d\phi &= \frac{2\pi}{4}\left(\frac{1}{2}\sin^2\phi - \frac{1}{32}\tan^2\phi\right)_0^{\frac{\pi}{3}} = \frac{9\pi}{64}\end{aligned}$$

$\Rightarrow \iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V 2z\ d\mathbf{S} = 2\cdot\frac{9\pi}{64} = \frac{9}{32}\pi \quad (7\ \%)$

(Method II)

Let $S_1 = \left\{(x,y,z)\in\mathbb{R}^3 | x^2+y^2+z^2 = 1, z \geq \frac{1}{2}\right\}$ and $S_2 = \left\{(x,y,z)\in\mathbb{R}^3 | x^2+y^2 \leq \frac{3}{4}, z = \frac{1}{2}\right\}$

$$\begin{aligned}\text{Flux of } \mathbf{F} &= \iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}(x^2,y^2,z^2)\cdot d\mathbf{S} + \iint_{S_2}(x^2,y^2,z^2)\cdot d\mathbf{S} \\ &= \iint_{S_1}(x^2,y^2,z^2)\cdot(x,y,z)\ dS + \iint_{S_2}(x^2,y^2,z^2)\cdot(0,0,-1)\ dS. \\ &= \iint_{S_1} x^3+y^3+z^3\ dS - \iint_{S_2} z^2\ dS = \iint_{S_1} x^3+y^3+z^3\ dS - \frac{1}{4}\text{Area}(S_2). \quad (3\%)\end{aligned}$$

From the symmetry of S_1, we have $\iint_{S_1} x^3 dS = \iint_{S_1} y^3 dS = 0$.

Thus, we only need to calculate $\iint_{S_1} z^3\ dS$, then by Spherical coordinate

$$\iint_{S_1} z^3\ dS = \int_0^{2\pi}\int_0^{\frac{\pi}{3}} \cos\phi^3\sin\phi\ d\phi d\theta = 2\pi\cdot\left(\frac{-1}{4}\cos^4\phi\right)_0^{\frac{\pi}{3}} = \frac{-\pi}{2}\left[\left(\frac{1}{2}\right)^4 - 1\right] = \frac{15}{32}\pi.$$

$\Rightarrow \iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} z^3\ dS - \frac{1}{4}\text{Area}(S_2) = \frac{15\pi}{32} - \frac{1}{4}\cdot\left(\frac{\sqrt{3}}{2}\right)^2\pi = \frac{9}{32}\pi \quad (7\ \%)$

10. (10%) Solve the differential equation

$$y'' + y = x^2 e^x + \tan x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Solution:

Complementary equation: $y'' + y = 0$.
Auxiliary equation: $r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_c = c_1 \sin x + c_2 \cos x$. (2 points)

For the particular solution:
(1) For $y'' + y = x^2 e^x$ it's a better idea to use the method of undetermined coefficients:
Let $y_{p_1} = (Ax^2 + Bx + C)e^x$,
$y'_{p_1} = [Ax^2 + (2A + B)x + (B + C)]e^x$,
$y''_{p_1} = [Ax^2 + (4A + B)x + (2A + 2B + C)]e^x$.

$y''_{p_1} + y_{p_1} = [2Ax^2 + (4A + 2B)x + (2A + 2B + 2C)]e^x \equiv x^2 e^x$
$\Rightarrow A = \frac{1}{2}, B = -1, C = \frac{1}{2}$
$\therefore y_{p_1} = (\frac{1}{2}x^2 - x + \frac{1}{2})e^x$.
(2 points for the formulation, 2 points for solving the coefficients.)

(2) For $y'' + y = \tan x$ we use the method of variation of parameters:
Let $y_{p_2} = u_1 \sin x + u_2 \cos x$, $y'_{p_2} = (u'_1 \sin x + u'_2 \cos x) + u_1 \cos x - u_2 \sin x$.
Setting $u'_1 \sin x + u'_2 \cos x = 0$ (equation 1), we have $y''_{p_2} = u'_1 \cos x - u'_2 \sin x - u_1 \sin x - u_2 \cos x$.
$\Rightarrow y''_{p_2} + y_{p_2} = u'_1 \cos x - u'_2 \sin x \equiv \tan x$ (equation 2).

Solving the system of equations 1 and 2, we have

$$\begin{cases} u'_1 \sin x + u'_2 \cos x &= 0 \\ u'_1 \cos x - u'_2 \sin x &= \tan x \end{cases}$$
$$\Rightarrow \begin{cases} u'_1(x) &= \sin x, \\ u'_2(x) &= -\tan x \sin x = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x \end{cases}$$
$$\Rightarrow \begin{cases} u_1(x) &= -\cos x, \\ u_2(x) &= \sin x - \ln|\sec x + \tan x| = \sin x - \ln(\sec x + \tan x) \text{ for } x \in (-\frac{\pi}{2}, -\frac{\pi}{2}) \end{cases}$$

$$\Rightarrow y_{p_2}(x) = u_1 \sin x + u_2 \cos x = -(\cos x)\ln(\sec x + \tan x)$$

(2 points for the system of equations, 2 points for solving and integrating them.)

Combining the above results, we have the general solution

$$y(x) = c_1 \sin x + c_2 \cos x + (\frac{1}{2}x^2 - x + \frac{1}{2})e^x - (\cos x)\ln(\sec x + \tan x)$$