1. (14%) Evaluate the limit, if it exists.

   (a) $\lim\limits_{x\to\infty} x\left(\sqrt{x^6-3x^5+1}-x^3\right)\tan\frac{1}{x^3}$,

   (b) $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^{x^2}\cdot e^x$.

Solution:

(a)
$$\lim_{x\to\infty} x\left(\sqrt{x^6-3x^5+1}-x^3\right)\tan\frac{1}{x^3} = \lim_{x\to\infty}\frac{\sqrt{x^6-3x^5+1}-x^3}{x^2}\frac{\tan\frac{1}{x^3}}{\frac{1}{x^3}}$$
$$= \lim_{x\to\infty}\frac{-3x^5+1}{x^2(\sqrt{x^6-3x^5+1}+x^3)}\frac{\tan\frac{1}{x^3}}{\frac{1}{x^3}} \quad (2\ \%)$$

$\because \lim\limits_{x\to\infty}\frac{-3x^5+1}{x^2(\sqrt{x^6-3x^5+1}+x^3)} = \frac{-3}{2}(2\ \%)$ and $\lim\limits_{x\to\infty}\frac{\tan\frac{1}{x^3}}{\frac{1}{x^3}} = 1(2\ \%)$

$\therefore \lim\limits_{x\to\infty} x\left(\sqrt{x^6-3x^5+1}-x^3\right)\tan\frac{1}{x^3} = \frac{-3}{2}\cdot 1 = \frac{-3}{2}(1\ \%)$

(b) $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^{x^2}\cdot e^x = \lim\limits_{x\to\infty} e^{x^2\ln(1-\frac{1}{x})+x} = e^{\lim\limits_{x\to\infty} x^2\ln(1-\frac{1}{x})+x}$

$\Rightarrow \lim\limits_{x\to\infty} x^2\ln(1-\frac{1}{x})+x(2\ \%) = \lim\limits_{x\to 0}\frac{\ln(1-x)+x}{x^2}(\because\frac{0}{0})$

By L'Hôpital's Rule (2 %) $= \lim\limits_{x\to 0}\frac{\frac{-1}{(1-x)}+1}{2x}(2\ \%)$

$= \lim\limits_{x\to 0}\frac{-x}{2x(1-x)} = \frac{-1}{2}$

$\Rightarrow \lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^{x^2}\cdot e^x = e^{\frac{-1}{2}} = \frac{1}{\sqrt{e}}(1\ \%)$

2. (10%) Find all horizontal, vertical and slant asymptotes, if any, of $f(x) = \frac{[\![x^3 - \sqrt{x^6}]\!]}{x^2}$ where $[\![x]\!]$ denotes the greatest integer function.

Solution:

$\because \sqrt{x^6} = |x^3| \quad \therefore f(x) = \begin{cases} 0 & \text{if } x > 0 \\ \frac{[\![2x^3]\!]}{x^2} & \text{if } x < 0 \end{cases}$ (2 %)

(a) Because $f(x) = 0$ as $x > 0$, so there is a horizontal asymptote is $y = 0$ (1 %)

(b) Consider $\lim_{x\to 0^-} f(x)$ as $x \to 0^-$ then $-1 < 2x^3 < 0 \Rightarrow \lim_{x\to 0^-} \frac{[\![2x^3]\!]}{x^2} = \lim_{x\to 0^-} \frac{-1}{x^2} = -\infty$, (2 %)
so there is a vertical asymptote is $x = 0$ (1 %)

(c) Assume there is a slant asymptote $y = mx + b$ then $m = \lim_{x\to -\infty} \frac{f(x)}{x} = \lim_{x\to -\infty} \frac{[\![2x^3]\!]}{x^3}$,
Because $2x^3 - 1 \le [\![2x^3]\!] \le 2x^3 \Rightarrow -1 \le [\![2x^3]\!] - 2x^3 \le 0$, then by Pinching Theorem we have
$\lim_{x\to -\infty} \frac{2x^3-1}{x^3} \le \lim_{x\to -\infty} \frac{[\![2x^3]\!]}{x^3} \le \lim_{x\to -\infty} \frac{2x^3}{x^3} \Rightarrow m = \lim_{x\to -\infty} \frac{[\![2x^3]\!]}{x^3} = 2$ (2%)
and $|b| = \lim_{x\to -\infty} \left| \frac{[\![2x^3]\!]}{x^2} - 2x \right| = \lim_{x\to -\infty} \left| \frac{[\![2x^3]\!] - 2x^3}{x^3} \right| \le \lim_{x\to -\infty} \frac{1}{|x^3|} = 0$, (2 %)
so there is a slant asymptote is $y = 2x$

3. (10%) Let $f(x) = \begin{cases} ae^x + bx & x < 0 \\ m & x = 0 \\ e^{-\frac{1}{x^2}} + \sqrt[3]{x+5} & x > 0. \end{cases}$ Find the constants m, a, b, such that $f(x)$ is differentiable everywhere.

Solution:

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{-\frac{1}{x^2}} + \sqrt[3]{x+5} = \sqrt[3]{5}$ $(1pt)$

$\lim_{x\to 0^-} ae^x + bx = a$ $(1pt)$

$\therefore a = m = \sqrt[3]{5}$ $(2pt)$

$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}} + \sqrt[3]{x+5} - \sqrt[3]{5}}{x-0} \left(\frac{0}{0}\right) \overset{L'Hospital}{=\!=\!=} \lim_{x\to 0^+} \left(\frac{2e^{-\frac{1}{x^2}}}{x^3} + \frac{1}{3\sqrt[3]{(x+5)^2}}\right) = \frac{5^{-\frac{2}{3}}}{3}$ $(2pt)$

$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{5^{\frac{1}{3}} e^x + bx - 5^{\frac{1}{3}}}{x-0} \left(\frac{0}{0}\right) \overset{L'Hospital}{=\!=\!=} \lim_{x\to 0^-} 5^{\frac{1}{3}} e^x + b = 5^{\frac{1}{3}} + b$ $(2pt)$

$\Rightarrow b = 5^{-\frac{2}{3}} - 5^{\frac{1}{3}} = 5^{-\frac{2}{3}}\left(\frac{1}{3} - 5\right) = 5^{-\frac{2}{3}} \cdot \frac{-14}{3}$ $(2pt)$

4. (28%)

(a) (7%) Let $f(x) = \frac{\tan 2x \cdot \cos^{-1} x + \ln(1+x)}{3\sec^3 x + x^3 \sin^{-1} x}$. Find $f'(0)$.

(b) (7%) Let $f(x) = \sin^{-1}(\tanh x) + \tan^{-1}(\sinh x)$. Find $f'(x)$. [Make your answer as simple as possible.]

(c) (7%) Let $f(x) = a^{a^{\sqrt{x}}} + a^{x^{\sqrt{x}}}$, where $a > 0$ is a constant. Find $f'(x)$.

(d) (7%) Let $f(x) = \log_{2^x}(\log_{x^2} e)$. Find $f'(x)$.

Solution:

(a) By definition of derivative and $f(0) = 0$

$\lim_{x\to 0} \frac{f(x)-0}{x-0}$ (2pt) $= \lim_{x\to 0} \frac{1}{3\sec^2 x + x^3 \sin^{-1} x} \cdot \left(2\frac{\tan 2x}{2x}\cos^{-1} x + \frac{\ln(1+x)}{x}\right)$ (2pt)

$= \frac{1}{3}(1 \cdot 2 \cdot \frac{\pi}{2} + 1) = \frac{\pi+1}{3}$ (3pt)

(b) $f'(x) = \frac{\text{sech}^2 x}{\sqrt{1-\tanh^2 x}} + \frac{\cosh x}{1+\sinh^2 x}$ (4pt) $= \text{sech}\ x + \text{sech}\ x = 2\text{sech}\ x$ (3pt)

(c) Let $u(x) = \sqrt{x}$, $g(x) = a^{u(x)}$, and $h(x) = x^{u(x)}$. Then

$$f(x) = a^{g(x)} + a^{h(x)}.$$

So we have

$$f'(x) = (a^{g(x)})' + (a^{h(x)})'.$$

But

$$\begin{aligned}(a^{g(x)})' &= (a^{g(x)} \ln a)\ g'(x)\\ &= (a^{g(x)} \ln a)(a^{u(x)} \ln a)\ u'(x)\\ &= (a^{g(x)} \ln a)(a^{u(x)} \ln a)\frac{1}{2\sqrt{x}}\\ &= a^{a^{\sqrt{x}}+\sqrt{x}}(\ln a)^2 \frac{1}{2\sqrt{x}}\end{aligned}$$

and

$$\begin{aligned}(a^{h(x)})' &= (a^{h(x)} \ln a)\ h'(x)\\ &= (a^{h(x)} \ln a)(e^{u(x)\ln x})'\\ &= (a^{h(x)} \ln a)(e^{u(x)\ln x})(u(x)\ln x)'\\ &= (a^{h(x)} \ln a)(e^{u(x)\ln x})(u'(x)\ln x + u(x)\cdot \frac{1}{x})\\ &= a^{x^{\sqrt{x}}}(\ln a)\left[x^{\sqrt{x}}\left(\frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}}\right)\right]\end{aligned}$$

Hence,

$$f'(x) = a^{a^{\sqrt{x}}+\sqrt{x}}(\ln a)^2 \frac{1}{2\sqrt{x}} + a^{x^{\sqrt{x}}}(\ln a)\left[x^{\sqrt{x}}\left(\frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}}\right)\right].$$

(d) First we simplify $f(x)$.

$$\begin{aligned}f(x) &= \log_{2^x}(\log_{x^2} e)\\ &= \frac{\ln\left(\frac{1}{2\ln x}\right)}{x\ln 2}\\ &= \frac{-\ln(2\ln x)}{x\ln 2}\end{aligned}$$

Hence by the quotient rule of derivative,

$$\begin{aligned}f'(x) &= \frac{(x\ln 2)\left(-\frac{\frac{2}{x}}{2\ln x}\right) - (-\ln(2\ln x))(\ln 2)}{(x^2\ln 2)^2}\\ &= \frac{-\frac{1}{\ln x} + \ln(2\ln x)}{x^2\ln 2}\end{aligned}$$

5. (10%) If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, with $g(0) = 1$, $f(1) = 2$, $g'(0) = 2$, $f'(1) = -1$, $g''(0) = 1$ and $f''(1) = 3$. Find $\left.\frac{d^2y}{dx^2}\right|_{x=0}$.

Solution:

$$\begin{aligned} y = f(g(x)) &\Rightarrow y' = f'(g(x))g'(x) \\ &\Rightarrow y'' = f''(g(x))\left(g'(x)\right)^2 + f'(g(x))g''(x) \end{aligned}$$

Hence,

$$\begin{aligned} y''\,|_{x=0} &= f''(g(0))\left(g'(0)\right)^2 + f'(g(0))g''(0) \\ &= f''(1)\cdot 2^2 + f'(1)\cdot 1 \\ &= 3\cdot 4 + (-1) \\ &= 11. \end{aligned}$$

6. (10%) Suppose that the function $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), and $0 < a < b$. If $f(a) = ka$, $f(b) = kb$ for some k, show that there exists $c \in (a,b)$ such that the tangent line of $y = f(x)$ at c passes through the origin. [Apply Rolle's Theorem.]

Solution:

Overall Policies:
1. Any complete and valid proof is granted a full 10 points, whether or not the proof follows a similar structure of the suggested proof given below.
2. In the case your proof follows a similar structure of the suggested proof, jumping to conclusions while skipping some "necessary" details (as stated in the boxes below) can cost you the partial credits of those details.
3. Any statement that relies on intuition (for example, a graphical one) without sound reasoning is regarded as invalid.

ⓐ Define $g(x) = \frac{f(x)}{x}$ on $[a,b]$. **(3 points)**

Policies for part ⓐ:
1. Any other $g(x)$ that also leads to a proof of the conclusion, such as "$g(x) = \frac{f(x)}{x} - k$ on $[a,b]$", is granted 3 points.
2. Any other $g(x)$ that doesn't lead to a proof of the conclusion, such as "$g(x) = f(x) - kx$ on $[a,b]$", is granted 0 point.

ⓑ $g(x)$ is continuous on $[a,b]$ and differentiable on (a,b). **(1 point)**

(Reason: $f(x)$ and the function $h(x) := x$ are both continuous on $[a,b]$ and differentiable on (a,b), and $h(x) \neq 0$ on $[a,b]$ since $0 < a < b$.)

Policies for part ⓑ:
1. Writing down this fact about $g(x)$ without reasoning is still granted 1 point.
2. Writing down this fact for any $g(x)$ you defined previously is also granted 1 point, as long as Rolle's Theorem or MVT (Mean Value Theorem) is applied to it later.

ⓒ Also, $g(a) = g(b)$. Thus $g(x)$ satisfies the assumptions of Rolle's Theorem.

By Rolle's Theorem, there exists a $c \in (a,b)$ such that $g'(c) = 0$. **(3 points)**

Policies for part ⓒ:
1. Applying Rolle's Theorem correctly to any $g(x)$ you defined previously is also granted 3 points.
2. Applying MVT correctly to any $g(x)$ you defined previously (maybe without the condition $g(a) = g(b)$, which is unnecessary for MVT) is also granted 3 points.
3. Writing down the complete statement of Rolle's Theorem or MVT without application to any specific $g(x)$ is also granted 3 points.

ⓓ The tangent line of $y = f(x)$ at c is L: $y - f(c) = f'(c)(x - c)$. **(1 point)**

Policies for part ⓓ:
1. Any equivalent form is granted 1 point.
2. 0 point is granted upon any substitution of c, $f(c)$ or $f'(c)$ by another constant without further explaining the relation between this constant and c, $f(c)$ or $f'(c)$.

ⓔ An equivalent condition that L passes through $(0,0)$ is: $f(c) = cf'(c)$. **(1 point)**

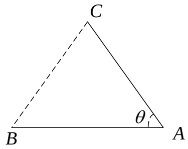
Policies for part ⓔ:
1. Any equivalent condition is granted 1 point.
2. If an equivalent condition is correctly derived without stating and/or using the equation of the tangent line in part ⓓ, a total of 2 points is granted for part ⓓ and part ⓔ combined.

ⓕ This condition is equivalent to $g'(c) = \frac{cf'(c)-f(c)}{c^2} = 0$ **(1 point)**, completing the proof.

Policies for part ⓕ:

1. 0 point is granted if the correct formula of the derivative $g'(x) = \frac{xf'(x)-f(x)}{x^2}$ or $g'(c) = \frac{cf'(c)-f(c)}{c^2}$ is not given.
2. For parts ⓓ, ⓔ and ⓕ, any valid argument that "$g'(c) = 0$" is equivalent to the condition that "the tangent line of $y = f(x)$ at c passes through the origin" will be granted a total of 3 points combined.

7. (10%) The lengths of line segments $\overline{AB}$ and $\overline{AC}$ are fixed but the angle θ between them decreases with time t so that the area of the triangle $\triangle ABC$ decays exponentially. Suppose that T_0 is the time required for half of the area to decay and at time $t = 0$, the angle θ is $\frac{\pi}{3}$. How fast is θ decreasing when $t = 2T_0$?



Solution:
We want to find $|\frac{d\theta}{dt}|$ at $t = 2T_0$. We now give three methods to establish the relation between θ and t. (Method II and III are from students.)
Method I
Since the area A of $\triangle ABC$ decays exponentially, we have

$$A(t) = A_0 e^{-ct} \text{ (2 points)}$$

where A_0 denotes the initial area and c is some constant. Moreover the area is halved at $t = T_0$, so

$$A(T_0) = \frac{A_0}{2} = A_0 e^{-cT_0}.$$

Hence $c = \frac{\ln 2}{T_0}$ (1 point) and $A(t) = A_0 e^{-\frac{\ln 2}{T_0}t}$.
On the other way, the area A is related to the angle θ by the equation

$$A(\theta) = \frac{2}{\sqrt{3}} A_0 \sin\theta \text{ (2 points).}$$

Note that θ depends on t and $\theta(0) = \frac{\pi}{3}$. Furthermore $A(2T_0) = \frac{A_0}{4} = \frac{2}{\sqrt{3}} A_0 \sin\theta(2T_0)$, thus $\cos\theta(2T_0) = \frac{\sqrt{61}}{8}$ (1 point). Now by the chain rule

$$\frac{dA(t)}{dt} = \frac{dA(\theta)}{d\theta}\frac{d\theta}{dt} \text{ (2 points)}$$
$$-\frac{\ln 2}{T_0} A_0 e^{-\frac{\ln 2}{T_0}t} = \frac{2}{\sqrt{3}} A_0 \cos\theta \frac{d\theta}{dt},$$

hence when $t = 2T_0$

$$-\frac{\ln 2}{T_0} A_0 e^{-\frac{\ln 2}{T_0}\cdot 2T_0} = \frac{2}{\sqrt{3}} A_0 \cos\theta(2T_0) \left.\frac{d\theta}{dt}\right|_{t=2T_0}$$
$$\left.\frac{d\theta}{dt}\right|_{t=2T_0} = -\sqrt{\frac{3}{61}}\frac{\ln 2}{T_0}.$$

We conclude that the decreasing rate is $\sqrt{\frac{3}{61}}\frac{\ln 2}{T_0}$ when $t = 2T_0$ (2 points). (Notice the positive sign here.)
Method II
Another way to express the decay process is

$$A(t) = A_0 \left(\frac{1}{2}\right)^{t/T_0} \text{ (2 points)}$$

where A_0 denotes the initial area. As in Method I, the area A is a function of θ by

$$A(\theta) = \frac{2}{\sqrt{3}} A_0 \sin\theta \text{ (2 points)}$$

whose θ depends on t with the relation $\theta(0) = \frac{\pi}{3}$. Moreover $A(2T_0) = \frac{A_0}{4} = \frac{2}{\sqrt{3}} A_0 \sin\theta(2T_0)$, thus $\cos\theta(2T_0) = \frac{\sqrt{61}}{8}$ (2 points). Now by the chain rule

$$\frac{dA(t)}{dt} = \frac{dA(\theta)}{d\theta}\frac{d\theta}{dt} \text{ (2 points)}$$
$$\frac{A_0}{T_0}\ln\frac{1}{2}\cdot\left(\frac{1}{2}\right)^{t/T_0} = \frac{2}{\sqrt{3}} A_0 \cos\theta \frac{d\theta}{dt},$$

hence when $t = 2T_0$

$$-\frac{A_0 \ln 2}{T_0} \cdot \frac{1}{4} = \frac{2}{\sqrt{3}} A_0 \cos\theta(2T_0) \left.\frac{d\theta}{dt}\right|_{t=2T_0}$$

$$\left.\frac{d\theta}{dt}\right|_{t=2T_0} = -\sqrt{\frac{3}{61}} \frac{\ln 2}{T_0}.$$

So the decreasing rate is $\sqrt{\frac{3}{61}} \frac{\ln 2}{T_0}$ when $t = 2T_0$ (2 points). (Notice again the positive sign.)

Method III

In the last method, we only present the difference from the above. That is, one can solve $\theta(t)$ directly from $A(t)$ and $A(\theta)$ as below

$$A(\theta) = A(t)$$

$$\frac{2}{\sqrt{3}} A_0 \sin\theta = A_0 \left(\frac{1}{2}\right)^{t/T_0}.$$

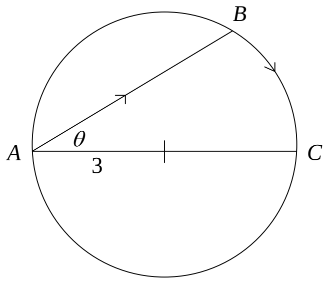
Hence

$$\sin\theta = \frac{\sqrt{3}}{2} \left(\frac{1}{2}\right)^{t/T_0}$$

$$\theta(t) = \arcsin\left(\frac{\sqrt{3}}{2} \left(\frac{1}{2}\right)^{t/T_0}\right).$$

We then differentiate $\theta(t)$. After some calculation, we get $\left.\frac{d\theta}{dt}\right|_{t=2T_0} = -\sqrt{\frac{3}{61}} \frac{\ln 2}{T_0}$ and thus the decreasing rate is $\sqrt{\frac{3}{61}} \frac{\ln 2}{T_0}$.

8. (10%) A woman at a point A on the shore of a circular lake with radius 3 km wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at the rate of 6 km/h and row a boat at v km/h. How should she proceed? [Discuss the cases according to v.]



Solution:

$$\overline{AB} = 2(3\cos\theta) = 6\cos\theta \text{ (km)} \tag{1}$$

$$\widehat{BC} = 3(2\theta) = 6\theta \text{ (km)} \tag{2}$$

Case 1: $v = 0$

It's trivial that the only way is $\widehat{AC}= 3\pi$ (km).

It spends $\frac{3\pi}{6} = \frac{\pi}{2}$ hours.

Case 2: $0 < v < \infty$

$$T(\theta) = \frac{\overline{AB}}{v} + \frac{\widehat{BC}}{6} \text{ (h)} = \frac{6}{v}\cos\theta + \theta \text{ (h)} \tag{3}$$

$$\text{domain: } 0 \le \theta \le \frac{\pi}{2} \tag{4}$$

Get 3 points with equation (3).

Get 1 point with equation (4).

We need to find the absolute minimum value of $T(\theta)$ on the domain $[0, \frac{\pi}{2}]$.

By the Closed Interval Method, the absolute minimum value is the minimum value of critical numbers and the endpoints.

Let's consider two endpoints.

$$T(0) = \frac{6}{v} + 0 = \frac{6}{v} \tag{5}$$

$$T(\frac{\pi}{2}) = 0 + \frac{\pi}{2} = \frac{\pi}{2} \tag{6}$$

Let's consider the critical numbers. If $(c, T(c))$ is a critical number of $T(\theta)$ for some $c \in (0, \frac{\pi}{2})$, then $T'(c) = 0$ because $T(\theta)$ is differentiable in $(0, \frac{\pi}{2})$.

$$0 = T'(c) = -\frac{6}{v}\sin c + 1 \tag{7}$$

$$\iff \sin c = \frac{v}{6} \tag{8}$$

Get 1 point with equation (8).

If $\frac{v}{6} > 1$, then there is no critical number of $T(\theta)$.

If $\frac{v}{6} \le 1$, then we claim **that $T(c)$ is impossible to be the minimum value**.

The following are three possible ways to prove our claim.

One possible way is the second derivative test.

$$T''(c) = -\frac{6}{v}\cos c < 0 \tag{9}$$

Another way is the first derivative test.

$$T'(\theta) = -\frac{6}{v}\sin\theta + 1 > 0 \text{ for } \theta \in [0, c). \tag{10}$$

$$T'(\theta) = -\frac{6}{v}\sin\theta + 1 < 0 \text{ for } \theta \in (c, \frac{\pi}{2}]. \tag{11}$$

The other way is to prove $T(c) \geq T(0) = \frac{6}{v}$ directly.

$$\iff \frac{6}{v}\cos c + c = T(c) \geq \frac{6}{v} \tag{12}$$

$$\iff c \geq \frac{6}{v}(1 - \cos c) \tag{13}$$

$$\iff c(1 + \cos c) \geq \frac{6}{v}(1 - \cos c)(1 + \cos c) = \frac{6}{v}\sin^2 c \tag{14}$$

Becasue $\sin c = \frac{v}{6}$, we have $\frac{6}{v}\sin^2 c = \frac{v}{6} = \sin c$.

$$\iff c(1 + \cos c) \geq \sin c \tag{15}$$

$$\iff c(1 + \cos c) = c + c\cos c \geq c \geq \sin c \tag{16}$$

Get 2 points with proof of our claim by one of three possible ways.

Therefore, we claim that the absolute minimum value is the minimum value of two endpoints.

Here is the final answer.

If $0 < v \leq \frac{12}{\pi}$, then $T(\frac{\pi}{2}) = \frac{\pi}{2} \leq \frac{6}{v} = T(0)$ and the absolute minimum value is $T(\frac{\pi}{2}) = \frac{\pi}{2}$.

If $\frac{12}{\pi} < v < \infty$, then $T(\frac{\pi}{2}) = \frac{\pi}{2} > \frac{6}{v} = T(0)$ and the absolute minimum value is $T(0) = \frac{6}{v}$.

Get 3 points with the final answer.

9. (18%) Let $y = 3\cos^2 x + \sin x,\ x \in [-\pi, \pi]$.

(a) Find the intervals of increase or decrease.

(b) Find the intervals of concavity.

(c) Find the local maximum and minimum values.

(d) Find the global maximum and minimum values.

(e) Find the inflection points.

(f) Sketch the graph of $y = f(x)$.

Solution:

$$\begin{aligned} y(x) &= 3\cos^2 x + \sin x, x \in [-\pi, \pi] \\ y'(x) &= \cos x(1 - 6\sin x) \\ y''(x) &= 12\sin^2 x - \sin x - 6 = (3\sin x + 2)(4\sin x - 3) \end{aligned}$$

$$\begin{aligned} y'(x) = 0 &\Rightarrow x = -\frac{\pi}{2}, \sin^{-1}(\frac{1}{6}), \frac{\pi}{2}, \pi - \sin^{-1}(\frac{1}{6}) \\ y''(x) = 0 &\Rightarrow x = -\pi + \sin^{-1}(\frac{2}{3}), -\sin^{-1}(\frac{2}{3}), \sin^{-1}(\frac{3}{4}), \pi - \sin^{-1}(\frac{3}{4}) \end{aligned}$$

(8% in total up to this point. Grading policy: for $y'(x)$ and $y''(x)$, 2% each for differentiation, 1% each for solving $\sin x$ (or $\cos x$), 1% each for solving x (including correct use of inverse functions.)

Discussing the signs of $y'(x)$ and $y''(x)$ in the intervals between the above points and finding $y(x)$ at these points, we have the following chart:

x	$-\pi$		$-\pi+\beta$		$-\pi/2$		$-\beta$		α		γ		$\pi/2$		$\pi-\gamma$		$\pi-\alpha$		π
$f'(x)$		−		−	0	+		+	0	−		−	0	+		+	0	−	
$f''(x)$		−	0	+		+	0	−		−	0	+		+	0	−		−	
$f(x)$	3	╮	1	╰	−1	╯	1	╭	$\frac{37}{12}$	╮	$\frac{33}{16}$	╰	1	╯	$\frac{33}{16}$	╭	$\frac{37}{12}$	╮	3

where $\alpha = \sin^{-1}(\frac{1}{6}), \beta = \sin^{-1}(\frac{2}{3}), \gamma = \sin^{-1}(\frac{3}{4})$,

Thus we conclude that

(a) (2%)

$y(x)$ is increasing on $[-\frac{\pi}{2}, \sin^{-1}(\frac{1}{6})]$ and $[\frac{\pi}{2}, \pi - \sin^{-1}(\frac{1}{6})]$.

$y(x)$ is decreasing on $[-\pi, -\frac{\pi}{2}], [\sin^{-1}(\frac{1}{6}), \frac{\pi}{2}]$, and $[\pi - \sin^{-1}(\frac{1}{6}), \pi]$.

(b) (2%)

$y(x)$ is concave up on $(-\pi + \sin^{-1}(\frac{2}{3})), -\sin^{-1}(\frac{2}{3}))$ and $(\sin^{-1}(\frac{3}{4}), \pi - \sin^{-1}(\frac{3}{4}))$.

$y(x)$ is concave down on $(-\pi, -\pi + \sin^{-1}(\frac{2}{3})), (-\sin^{-1}(\frac{2}{3}), \sin^{-1}(\frac{3}{4}))$, and $(\pi - \sin^{-1}(\frac{3}{4}), \pi)$.

(c)(d) (2%)

Local minimum: $y(-\frac{\pi}{2}) = -1$, $y(\frac{\pi}{2}) = 1$, and $y(\pi) = 3$.

Local maximum: $y(-\pi) = 3, y(\sin^{-1}(\frac{1}{6})) = \frac{37}{12}$, and $y(\pi - \sin^{-1}(\frac{1}{6})) = \frac{37}{12}$.

Global maximum: $y(\sin^{-1}(\frac{1}{6})) = y(\pi - \sin^{-1}(\frac{1}{6})) = \frac{37}{12}$.

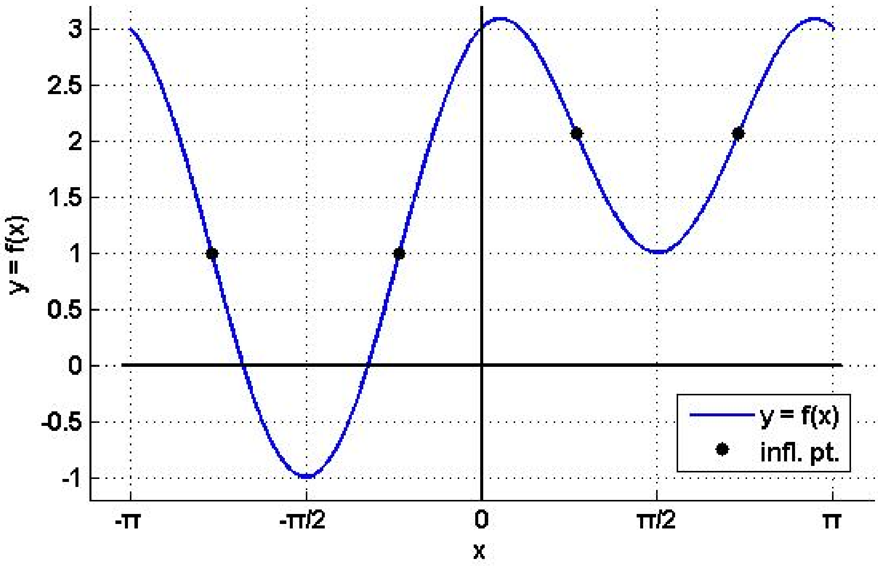
Global minimum: $y(-\frac{\pi}{2}) = -1$.

(e) (2%)

Inflection points: $(-\pi + \sin^{-1}(\frac{2}{3}), 1), (-\sin^{-1}(\frac{2}{3}), 1), (\sin^{-1}(\frac{3}{4}), \frac{33}{16})$, and $(\pi - \sin^{-1}(\frac{3}{4}), \frac{33}{16})$.

(f) (2%)

Sketch of $f(x)$:


3
2.5
2
1.5
1
0.5
0
-0.5
-1
y = f(x)
-π
-π/2
0
π/2
π
x
y = f(x)
infl. pt.