1. (16%) Compute the following integrals.

(a) $\int \frac{\cos\theta - 2\sin\theta}{2\cos\theta + \sin\theta} d\theta$.

(b) $\int \frac{\sqrt{1+x^2}}{x^2} dx$.

Solution:

1. (Method I)Let $u = 2\cos\theta + \sin\theta \Rightarrow du = \cos\theta - 2\sin\theta d\theta$

$$\Rightarrow \int \frac{\cos\theta - 2\sin\theta}{2\cos\theta + \sin\theta} d\theta = \int \frac{1}{u} du = \ln|u| + c = \ln|2\cos\theta + \sin\theta| + c$$

(Method II) Let $t = \tan\frac{\theta}{2} \Rightarrow d\theta = \frac{2}{1+t^2}dt,\ \sin\theta = \frac{2t}{1+t^2},\ \cos\theta = \frac{1-t^2}{1+t^2}$

$$\Rightarrow \int \frac{\cos\theta - 2\sin\theta}{2\cos\theta + \sin\theta} d\theta = \int \frac{\left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{2t}{1+t^2}\right)}{2\left(\frac{1-t^2}{1+t^2}\right) + \left(\frac{2t}{1+t^2}\right)} \cdot \frac{2}{1+t^2} dt = \int \frac{2-2t^2-8t}{(2+2t-2t^2)(1+t^2)} dt$$

$$\frac{1-t^2-4t}{(1+t-t^2)(1+t^2)} = \frac{At+B}{1+t^2} + \frac{Ct+D}{1+t-t^2} = \frac{(At+B)(1+t-t^2) + (Ct+D)(1+t^2)}{(1+t-t^2)(1+t^2)}$$

$$\begin{cases} C - A = 0 \\ D - B + A = -1 \\ A + B + C = -4 \\ B + D = 1 \end{cases} \Rightarrow \begin{cases} A = -2 \\ B = 0 \\ C = -2 \\ D = 1 \end{cases}$$

$$\Rightarrow \int \frac{2-2t^2-8t}{(2+2t-2t^2)(1+t^2)} dt = \int -\frac{2t}{1+t^2} - \frac{2t-1}{1+t-t^2} dt$$

$$= -\ln\left|t^2+1\right| + \ln\left|-t^2+t+1\right| + c = \ln\left|\frac{1-t^2+t}{1+t^2}\right| + c = \ln\left|\frac{1-t^2}{1+t^2} + \frac{1}{2}\left(\frac{2t}{1+t^2}\right)\right| + c$$

$$= \ln\left|\cos\theta + \frac{1}{2}\sin\theta\right| + c = \ln|2\cos\theta + \sin\theta| - \ln 2 + c = \ln|2\cos\theta + \sin\theta| + c'$$

(Method III) $\int \frac{\cos\theta - 2\sin\theta}{2\cos\theta + \sin\theta} d\theta = \int \frac{\sin(\alpha-\theta)}{\cos(\alpha-\theta)} d\theta = -\int \tan(\alpha-\theta) d(\alpha-\theta)$, where $\tan\alpha = \frac{1}{2}$

$$\Rightarrow -\int \tan(\alpha-\theta) d(\alpha-\theta) = -\ln|\sec(\alpha-\theta)| + c = \ln|2\cos\theta + \sin\theta| + c'$$

2. Let $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$$\int \frac{\sqrt{1+x^2}}{x^2} dx = \int \frac{\sec^3\theta}{\tan^2\theta} d\theta = \int \frac{1}{\sin^2\theta\cos\theta} d\theta$$

(Method I) $\int \frac{1}{\sin^2\theta\cos\theta} d\theta = \int \frac{\cos\theta}{\sin^2\theta\cos^2\theta} d\theta$

Let $u = \sin\theta \Rightarrow du = \cos\theta d\theta \Rightarrow u = \frac{x}{\sqrt{1+x^2}}$

$$\Rightarrow \int \frac{\cos\theta}{\sin^2\theta\cos^2\theta} d\theta = \int \frac{1}{u^2(1-u^2)} du = \int \frac{1}{u^2} + \frac{1}{1-u^2} du$$

$$= \int \frac{1}{u^2} + \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right) du = -\frac{1}{u} + \ln\left|\frac{1+u}{1-u}\right| + c$$

$$= -\frac{\sqrt{1+x^2}}{x} + \frac{1}{2}\ln\left|\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right| + c$$

$$= -\frac{\sqrt{1+x^2}}{x} + \frac{1}{2}\ln\left|\frac{\left(\sqrt{1+x^2}+x\right)^2}{(1+x^2)-x^2}\right| + c = \ln\left|\sqrt{1+x^2}+x\right| - \frac{\sqrt{1+x^2}}{x} + c$$

(Method II) $\int \frac{1}{\sin^2\theta\cos\theta} d\theta = \int \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos\theta} d\theta = \int \frac{1}{\cos\theta} + \frac{\cos\theta}{\sin^2\theta} d\theta$

$$= \int \sec\theta + \cot\theta\csc\theta d\theta = \ln|\sec\theta + \tan\theta| - \csc\theta + c$$

$$= \ln\left|x + \sqrt{1+x^2}\right| - \frac{\sqrt{1+x^2}}{x} + c$$

(Method III) $\int \frac{1}{\sin^2\theta\cos\theta} d\theta = \int \csc^2\theta\sec\theta d\theta = -\int \sec\theta d\cot\theta = -\sec\theta\cot\theta + \int \sec\theta\tan\theta\cot\theta$

$$= -\csc\theta + \ln|\sec\theta + \tan\theta| + c = \ln\left|x + \sqrt{1+x^2}\right| - \frac{\sqrt{1+x^2}}{x} + c$$

(Method IV) Let $x = \sinh\theta \Rightarrow dx = \cosh\theta d\theta$

$$\int \frac{\sqrt{1+x^2}}{x^2}dx = \int \coth^2\theta d\theta = \int \operatorname{csch}^2\theta + 1 d\theta = -\coth\theta + \theta + c = \frac{\sqrt{1+x^2}}{x} + \sinh^{-1}x + c$$

$\left(\because \sinh^{-1}x = \ln\left|x + \sqrt{1+x^2}\right|\right)$

2. (16%) Compute the following integrals.

(a) $\int \sqrt{1-x^2}\sin^{-1} x dx$.

(b) $\int \ln(\sqrt{x}+\sqrt{1+x})dx$.

Solution:

note :

$$\int u\ dv = uv - \int v\ du$$

(a)

(i) Let $x = \sin\theta,\ dx = \cos\theta d\theta$

$$\int \sqrt{1-x^2}\sin^{-1} x\ dx$$

$$= \int (\cos^2\theta)\theta\ d\theta = \int (\frac{1+\cos 2\theta}{2})\theta\ d\theta$$

$$= \int \frac{1}{2}\theta\ d\theta + \int \frac{1}{2}\theta\cos(2\theta)\ d\theta$$

Let $u=\theta; v = \frac{1}{2}\sin 2\theta$. So we can use integration by part to get

$$\int \frac{1}{2}\theta\ d\theta + \int \frac{1}{2}\theta\cos(2\theta)\ d\theta = \frac{1}{4}\theta^2 + \frac{1}{2}[\frac{1}{2}\theta\sin(2\theta) - \int \frac{1}{2}\sin(2\theta)d\theta]$$

$$= \frac{1}{4}\theta^2 + \frac{1}{4}\theta\sin(2\theta) + \frac{1}{8}\cos(2\theta) + C = \frac{1}{4}(\sin^{-1} x)^2 + \frac{1}{4}(\sin^{-1} x)\sin(2\sin^{-1} x) + \frac{1}{8}\cos(2\sin^{-1} x) + C,\ C\in\mathbb{R}$$

$$= \frac{1}{4}(\sin^{-1} x) + \frac{1}{2}(\sin^{-1} x)x\sqrt{1-x^2} - \frac{1}{4}x^2 + C', \quad C' = C + \frac{1}{8}$$

(ii) $\int \cos^2\theta\ d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)$.

Let $u=\theta;\ v = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)$. So we can get

$$\int \theta\cos^2\theta\ d\theta = (\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta))\theta - \int \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\ d\theta = \frac{1}{4}\theta^2 + \frac{1}{4}\theta\sin(2\theta) + \frac{1}{8}\cos(2\theta) + C,\ C\in\mathbb{R}$$

(iii) Let $x=\sin\theta$. So we can get

$$\int \sqrt{1-x^2}\ dx = \int \cos^2\theta\ d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) = \frac{1}{2}(\sin^{-1} x) + \frac{1}{2}x\sqrt{1-x^2}.$$

Let $u = \sin^{-1} x; v = \frac{1}{2}(\sin^{-1} x) + \frac{1}{2}x\sqrt{1-x^2}$.

Using integration by part,we can get

$$\int \sqrt{1-x^2}\sin^{-1} x\ dx = [\frac{1}{2}(\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2})](\sin^{-1} x) - \int (\frac{1}{2}(\sin^{-1} x) + \frac{1}{2}x\sqrt{1-x^2})\frac{1}{\sqrt{1-x^2}}\ dx$$

$$= \frac{1}{2}(\sin^{-1} x)^2 + \frac{1}{2}x\sqrt{1-x^2}\sin^{-1} x - \frac{1}{4}(\sin^{-1} x)^2 - \frac{1}{4}x^2 + C = \frac{1}{4}(\sin^{-1} x)^2 + \frac{1}{2}(\sin^{-1} x)x\sqrt{1-x^2} - \frac{1}{4}x^2 + C,\ C\in\mathbb{R}$$

(b)

Let $x = \tan^2\theta,\ dx = 2\tan\theta\sec^2\theta d\theta$

$$\int \log(\sqrt{x}+\sqrt{1+x})\ dx$$

$$= \int \log(\tan\theta + \sec x)2\tan\theta\sec^2\theta\ d\theta = \int \log(\tan\theta+\sec\theta)\ d(\sec^2\theta)$$

$$= \sec^2\theta\log(\tan\theta+\sec\theta) - \int \sec^3\theta\ d\theta = \sec^2\theta\log(\tan\theta+\sec\theta) - \frac{\sec\theta\tan\theta + \log(\sec\theta+\tan\theta)}{2} + C$$

$$= (\sec^2\theta - \frac{1}{2})\log(\sec\theta+\tan\theta) - \frac{1}{2}\sec\theta\tan\theta + C = (\tan^2\theta + \frac{1}{2})\log(\sec\theta+\tan\theta) - \frac{1}{2}\sec\theta\tan\theta + C$$

$$= (x+\frac{1}{2})\log(\sqrt{x}+\sqrt{1+x}) - \frac{1}{2}\sqrt{x}\sqrt{1+x} + C,\ C\in\mathbb{R}$$

3. (8%) Suppose that $f(x)$ is a polynomial whose coefficients are integers, and

$$\int_0^\infty \frac{f(x)}{(x+1)^2(4x^2+1)}dx = 2\ln 2 + 1.$$

Find $f(x)$.

Solution:

The degree of $f(x)$ must be less or equal than three.
However, if the degree of $f(x)$ equal three, and the value after the integral is divergent. Consequently, the degree of $f(x) = 2$ by estimation.

By using partial fractions method,
$\frac{f(x)}{(x+1)^2(4x^2+1)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{Cx+D}{4x^2+1}$. (3 points)
We integrate both sides,
$2\ln 2 + 1 = \lim_{p\to\infty}\int_0^p \frac{A}{(x+1)^2}dx + \lim_{p\to\infty}\int_0^p \frac{B}{x+1}dx + \lim_{p\to\infty}\int_0^p \frac{Cx+D}{4x^2+1}dx.$
$= \lim_{p\to\infty}[\frac{-A}{x+1} + B\ln|x+1| + \frac{C}{8}\ln|4x^2+1| + \frac{D}{2}\tan^{-1}(2x)]_0^p$ (1 points)
By using the limits must be exist, i.e. the natural log term must be equal to 2 ln 2 , and we found $C = -4B$.

By campring the coefficents with $(2\ln 2 + 1)$,
We found that $A = 1, B = -2, C = 8$ and $D = 0$.
(Because of A, B, C and D are all integers in the problem, the values of A, B, C and D which we found are correct.)

Therefore,
$f(x) = [\frac{1}{(x+1)^2} + \frac{-2}{x+1} + \frac{8x}{4x^2+1}] \times [(x+1)^2(4x^2+1)].$
$= 12x^2 + 6x - 1.$ (4 points)

4. (16%) Let $M_n = \sum_{i=1}^{n} \frac{1}{n + \sqrt{n(i - \frac{1}{2})}}$.

(a) Recognize M_n as the Midpoint approximation of a definite integral and compute $I = \lim_{n\to\infty} M_n$.

(b) Write down the Trapezoidal approximation of the integral, T_n, and the right point approximation, R_n.

(c) For any value of n, list M_n, T_n, R_n, and I in increasing order.

Solution:

(a)

$$I = \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1 + \sqrt{\frac{i-\frac{1}{2}}{n}}} \frac{1}{n}$$

$$= \int_0^1 \frac{1}{1+\sqrt{x}} dx \ (2pt) \quad (\text{ Let } \sqrt{x} = u,\ dx = 2udu\ )$$

$$= \int_0^1 \frac{2u}{1+u} du = \int_0^1 2 - \frac{2}{1+u} du = (2u - 2\ln|1+u|)\Big|_0^1$$

$$= 2 - 2\ln 2 \ (2pt)$$

(b)

$$T_n = \frac{1}{2n} \sum_{i=1}^{n} \left(\frac{1}{1+\sqrt{\frac{i}{n}}} + \frac{1}{1+\sqrt{\frac{i-1}{n}}} \right) \ (4pt)$$

$$R_n = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{1+\sqrt{\frac{i}{n}}} \right) \ (4pt)$$

(c)

$R_n < M_n < I < T_n$ $(4pt)$

5. (8%) A sphere of radius 1 overlaps a smaller shpere of radius r $(0 < r < 1)$ in such a way that their intersection is a circle of radius r, i.e. a great circle of the small sphere. Find r so that the volume inside the small sphere and outside the large sphere is as large as possible.

Solution:

Let C_1 be a circle with center $(0,0)$ and radius 1,
and let C_2 be a circle with center $(\sqrt{1-r^2},0)$ and radius r.

Then the desired volume $V(r)$ is equal to the volume of the solid obtained by rotating the region inside C_2 and outside C_1 about the x-axis, so

$$V(r) = \frac{1}{2}\cdot\frac{4}{3}\pi r^3 - \int_{\sqrt{1-r^2}}^{1} \pi\left(\sqrt{1-x^2}\right)^2 dx \quad \textbf{(4pts)}$$

Applying the Fundamental Theorem of Calculus, we can obtain

$$\begin{aligned} \frac{dV}{dr} &= 2\pi r^2 + \pi\left(1 - \left(\sqrt{1-r^2}\right)^2\right)\cdot\frac{d}{dr}\sqrt{1-r^2} \\ &= \pi r^2\left(2 - \frac{r}{\sqrt{1-r^2}}\right) \quad \textbf{(2pts)} \end{aligned}$$

The derivative $\dfrac{dV}{dr}$ is 0 when $2 = \dfrac{r}{\sqrt{1-r^2}}$, that is, $r = \dfrac{2}{\sqrt{5}}$. **(1pt)**

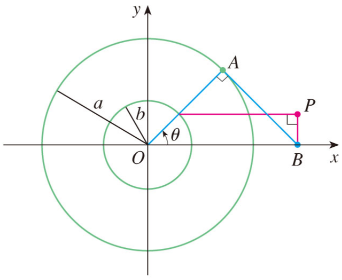
Since

$$\lim_{r\to 0^+} V(r) = 0, \quad \lim_{r\to 1^-} V(r) = 0, \quad V\left(\frac{2}{\sqrt{5}}\right) = 2\pi\left(\frac{1}{\sqrt{5}} - \frac{1}{3}\right) > 0$$

we have the maximum value $V(r) = 2\pi\left(\dfrac{1}{\sqrt{5}} - \dfrac{1}{3}\right)$ when $r = \dfrac{2}{\sqrt{5}}$. **(1pt)**

6. (16%) Let a and b be fixed numbers.

(a) Find parametric equations for the curve C that consists of all possible positions of the point P in the figure using the angle θ as the parameter.



(b) What are the horizontal asymptotes of the curve C?

(c) Let R be the region bounded by C and its horizontal asymptotes. Find the area of R if it is finite..

(d) The region is R rotated about the y-axis. Find the volume of the resulting solid if it is finite.

Solution:

(a) $Q = (b\cos\theta, b\sin\theta)$ (1分)
$B = (a\sec\theta, 0)$ (1分)
$\rightarrow P = (a\sec\theta, b\sin\theta)$ (2分)
$\theta \in [0, 2\pi]$不包含$\{\frac{\pi}{2}, \frac{3\pi}{2}\}$ (少寫不扣分)

(b) 作法一: Use. $(\frac{a}{x})^2 + (\frac{y}{b})^2 = 1$ (2分) and $x \to \pm\infty, y = \pm b$ (1分)
作法二: $y = \pm b\sqrt{1 - (\frac{a}{x})^2}, |x| \geq$ (2分) $\Rightarrow \lim_{x\to\infty} y = \pm b$ (1分)
作法三: P的x座標$a\sec\theta \to \pm\infty \Leftrightarrow \theta \to \frac{\pi}{2}$ or $\frac{3\pi}{2}$ (2分), then y座標 $b\sin\theta \to \pm b$ (1分)
作法四: 作圖 $\rightarrow y = \pm b$ (3分)

(c) (列式2分; 計算3分)
$R = 4\{ab + \int_a^\infty b - b\sqrt{1 - (\frac{a}{x})^2}dx\}$ (列式2分)
$\int \sqrt{1 - (\frac{a}{x})^2}dx = \int \frac{\sqrt{x^2 - a^2}}{x^2}dx \overset{u=\sqrt{x^2-a^2}}{=} \int \frac{u^2}{u^2 + a^2}du = \sqrt{x^2 - a^2} - a\tan^{-1}(\sqrt{x^2 - a^2}a)$
$= 4\{ab + b(x - \sqrt{x^2 - a^2} + a\tan^{-1}(\frac{\sqrt{x^2 - a^2}}{a}))_a^\infty\} = 2\pi ab$
OR
$R = 4[ab + \int_a^\infty (b - y)dx] = 2\pi ab$ (列式2分)
$\int_0^{\pi/2} (b - b\sin\theta)a\tan\theta\sec\theta d\theta = ab\int_0^{\pi/2} (\tan\theta\sec\theta - \theta^2\theta)d\theta$
$= ab(\sec\theta - \tan\theta + \theta)_9^{\pi/2} = ab(\frac{\pi}{2} - 1)$

(d) (計算2分; 答案2分)
$x = \frac{a}{\sqrt{1 - (\frac{y}{b})^2}} \Rightarrow$ Volume $= 2\int_0^b \pi x^2 dy = 2\pi a^2 b^2 \int_0^b \frac{dy}{b^2 - y^2}$ diverges
OR
Volume$=2\pi_0^{\frac{\pi}{2}} a^2 \sec^2\theta b\cos\theta d\theta = 2a^2 b\int_0^{\frac{\pi}{2}} \sec\theta d\theta = 2a^2 b\ln|\sec\theta + \tan\theta|_0^{\frac{\pi}{2}}$ diverges

7. (16%)

(a) Sketch the curve $r = 1 + 2\cos\theta$.

(b) Compute the area of the region that is inside the larger loop of the curve $r = 1 + 2\cos\theta$ and outside the smaller loop of the curve $r = 1 + 2\cos\theta$.

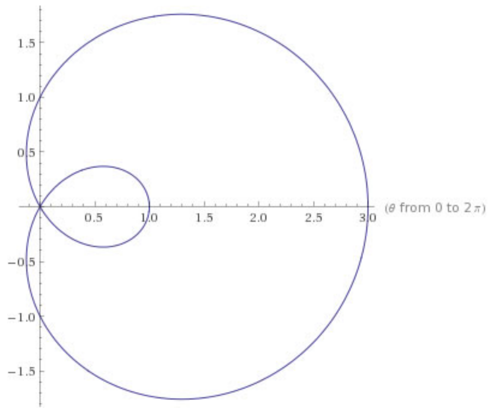
(c) Let C be the smaller loop of $r = 1 + 2\cos\theta$, and rotate C about the x-axis. Find the area of the resulting surface.

Solution:

(a) $0 = 1 + 2\cos\theta \Rightarrow \cos\theta = \frac{-1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

Figure: (2 points)

The figure for $r = 1 + 2\cos\theta$.



(b)$A = A_{total} - A_{inner\ circle}$

$= 2[\frac{1}{2}\int_0^{\frac{2\pi}{3}} (1 + 2\cos\theta)^2 d\theta \ - \ \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (1 + 2\cos\theta)^2 d\theta]$ (4 points)

$= \int_0^{\frac{2\pi}{3}} (1 + 4\cos\theta + 4\cos^2\theta)d\theta \ - \ \int_{\frac{2\pi}{3}}^{\pi} (1 + 4\cos\theta + 4\cos^2\theta)d\theta$

$= (3\theta + 4\sin\theta + \sin 2\theta)\,|_0^{\frac{2\pi}{3}} \ - \ (3\theta + 4\sin\theta + \sin 2\theta)\,|_{\frac{2\pi}{3}}^{\pi}$

$= (2\pi + \frac{3\sqrt{3}}{2}) - (\pi - \frac{3\sqrt{3}}{2})$

$= \pi + 3\sqrt{3}$. (3points)

(c) $A = \int 2\pi y ds$

$= 2\pi \int_a^b r\sin\theta\sqrt{r^2 + r'^2}\ d\theta$ (2 points)

There are two ranges could describe this figure, i.e. from π to $\frac{4\pi}{3}$ or from $\frac{2\pi}{3}$ to π.

However, if we choose from $\frac{2\pi}{3}$ to π, and the area would be negative because of $r\sin\theta \le 0$. Therefore, we choose the range which is from π to $\frac{4\pi}{3}$.

$A = 2\pi \int_{\pi}^{\frac{4\pi}{3}} (1 + 2\cos\theta)\sin\theta\sqrt{(1 + 2\cos\theta)^2 + (2\sin\theta)^2}\ d\theta$

$= 2\pi \int_{\pi}^{\frac{4\pi}{3}} (1 + 2\cos\theta)\sin\theta\sqrt{5 + 4\cos\theta}\ d\theta$

Let $u = \cos\theta$, and $du = -\sin\theta\ d\theta$.

θ from π to $\frac{4\pi}{3} \Rightarrow$ u from -1 to -0.5.

$= 2\pi \int_{-1}^{-0.5} (1 + 2u)\sqrt{5 + 4u}\ (-du)$. (2 points)

Then, Let $t = 5 + 4u$, and $dt = 4du$

u from -1 to -0.5 $\Rightarrow$ t from 1 to 3.

$= 2\pi \int_1^3 (1 + 2 \cdot \frac{t - 5}{4})\sqrt{t}\ (-\frac{1}{4})dt$.

$= \frac{\pi}{5}(3\sqrt{3} - 2)$. (3 points)

8. (16%) A candle is located at the origin O, a bug, P, crawls on the plane so that the angle between its velocity and the vector $\overrightarrow{PO}$ is always $\frac{\pi}{6}$.

(a) Suppose that the bug crawls at a curve with polar equation $r = f(\theta)$. Derive the differential equation that $f(\theta)$ satisfies.

(b) If the Cartesian coordinates for the bug's initial position are $(1, 0)$, solve for the curve $r = f(\theta)$.

(c) Compute the arc length function $s(\theta)$ and find $\lim_{\theta\to\infty} s(\theta)$.

Solution:

(a) (8pts)

Let ϕ be the angle between the tangent line at P and the x-axis.

Then

$$\begin{aligned}
\pm\tan\frac{\pi}{6} &= \tan(\phi-\theta) = \frac{\tan\phi - \tan\theta}{1+\tan\phi\tan\theta} = \frac{\frac{dy}{dx} - \tan\theta}{1+\frac{dy}{dx}\cdot\tan\theta} \\
&= \frac{\frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} - \frac{\sin\theta}{\cos\theta}}{1+\frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}} = \frac{(r'\sin\theta + r\cos\theta)\cos\theta - (r'\cos\theta - r\sin\theta)\sin\theta}{(r'\cos\theta - r\sin\theta)\cos\theta + (r'\sin\theta + r\cos\theta)\sin\theta} = \frac{r}{r'} \\
&\Rightarrow \begin{cases} f'(\theta) = \sqrt{3}f(\theta) \\ f'(\theta) = -\sqrt{3}f(\theta) \end{cases}
\end{aligned}$$

(b) Case 1. If $f'(\theta) = \sqrt{3}f(\theta)$, then

$$\begin{aligned}
\int \frac{1}{f}df &= \int \sqrt{3}\theta d\theta \\
\Rightarrow \ln|f| &= \sqrt{3}\theta + C
\end{aligned}$$

and so

$$f(\theta) = C'e^{\sqrt{3}\theta} \quad \textbf{(2pts)}$$

Since $y(1) = 0 \Rightarrow f(0) = 1$, we have

$$1 = f(0) = C'e^{\sqrt{3}\cdot 0} = C'$$

Therefore, the solution is

$$f(\theta) = e^{\sqrt{3}\theta} \quad \textbf{(1pt)}$$

Case 2. If $f'(\theta) = -\sqrt{3}f(\theta)$, then, similarly, the solution is

$$f(\theta) = e^{-\sqrt{3}\theta}$$

(c) Case 1. If $f(\theta) = e^{\sqrt{3}\theta}$, then

$$\begin{aligned}
s(\theta) &= \int_0^\theta \sqrt{\left(f(t)\right)^2 + \left(\frac{d}{dt}f(t)\right)^2}\,dt \quad \textbf{(2pts)} \\
&= \int_0^\theta \sqrt{\left(e^{\sqrt{3}t}\right)^2 + \left(\frac{d}{dt}e^{\sqrt{3}t}\right)^2}\,dt \\
&= \frac{2}{\sqrt{3}}\left(e^{\sqrt{3}\theta} - 1\right) \quad \textbf{(2pts)}
\end{aligned}$$

Therefore,

$$\lim_{\theta\to\infty} s(\theta) = \infty \quad \textbf{(1pt)}$$

Case 2. If $f(\theta) = e^{-\sqrt{3}\theta}$, then

$$s(\theta) = \frac{2}{\sqrt{3}}\left(1 - e^{-\sqrt{3}\theta}\right), \quad \lim_{\theta\to\infty} s(\theta) = \frac{2}{\sqrt{3}}$$

9. (8%) Solve the differential equation

$$x^2 y' - y = 2x^3 e^{-\frac{1}{x}},\ y(1) = 1.$$

Solution:

$x^2 y' - y = 2x^3 e^{-\frac{1}{x}}$

$y' - \dfrac{1}{x^2} = 2xe^{-\frac{1}{x}}$ $(2pt)$

$I(x) = e^{\int -\frac{1}{x^2} dx} = e^{\frac{1}{x}}$ $(1pt)$

$$\begin{aligned} y(x) &= \frac{1}{I(x)}[\int I(x)r(x)dx + \text{c}] \\ &= e^{-\frac{1}{x}}[\int e^{\frac{1}{x}} 2xe^{-\frac{1}{x}} dx + \text{c}] \\ &= e^{-\frac{1}{x}}[\int 2xdx + \text{c}] \\ &= e^{-\frac{1}{x}}(x^2 + \text{c})(2pt) \end{aligned}$$

and $y(1) = 1$

$\Rightarrow 1 = e^{-1}(1 + \text{c})$

$\Rightarrow \text{c} = e - 1$ $(1pt)$

$\therefore y = e^{-\frac{1}{x}}(x^2 + e - 1)$ $(2pt)$