1061微甲01-02班期末考解答和評分標準

1. (12 points) Find the limits.

   (a) (6 points) $\lim_{n\to+\infty}\left(\frac{n}{n^2+4\cdot 1^2}+\frac{n}{n^2+4\cdot 2^2}+\frac{n}{n^2+4\cdot 3^2}+\ldots+\frac{n}{5n^2}\right)=\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{n}{n^2+4i^2}$.

   (b) (6 points) $\lim_{h\to 0}\frac{1}{h}\int_{1-h}^{\sqrt[3]{1+h}}\sqrt{1+t^3}dt$.

Solution:

(a)

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+4i^2}&=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+4(\frac{i}{n})^2}\cdot\frac{1}{n}\\&=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+4x_i^2}\cdot\Delta x\\&=\int_0^1\frac{1}{1+4x^2}dx=\frac{1}{2}\tan^{-1}2\end{aligned}$$

where we take $a=0, b=1, \Delta x=\frac{b-a}{n}=\frac{1}{n}, x_i=a+i\Delta x=\frac{i}{n}$

Another approach:

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+4i^2}&=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+(\frac{2i}{n})^2}\cdot(\frac{1}{2}\cdot\frac{2}{n})\\&=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+x_i^2}\cdot\frac{1}{2}\Delta x\\&=\int_0^2\frac{1}{1+x^2}\cdot\frac{1}{2}dx=\frac{1}{2}\tan^{-1}2\end{aligned}$$

where we take $a=0, b=2, \Delta x=\frac{b-a}{n}=\frac{2}{n}, x_i=a+i\Delta x=\frac{2i}{n}$

(b) $\lim_{h\to 0}\frac{1}{h}\int_{1-h}^{\sqrt[3]{1+h}}\sqrt{1+t^3}\,dt$

$=\lim_{h\to 0}\frac{1}{1}\frac{d}{dh}\left[\int_{1-h}^{\sqrt[3]{1+h}}\sqrt{1+t^3}\,dt\right]$ $\quad\left[\frac{0}{0}\right]$L'H' (1 points)

$=\lim_{h\to 0}\sqrt{1+(\sqrt[3]{1-h})^3}\,\frac{d}{dh}[\sqrt[3]{1+h}]-\sqrt{1+(1-h)^3}\,\frac{d}{dh}[1-h]$ (2 points)

$=\lim_{h\to 0}\sqrt{1+(\sqrt[3]{1-h})^3}\,\frac{1}{3}[1+h]^{-2/3}-\sqrt{1+(1-h)^3}\,(-1)$ (2 points)

$=\sqrt{2}\,\frac{1}{3}\times(1)+\sqrt{2}$

$=\frac{4}{3}\sqrt{2}$ (1 points)

Note:

1. If you get one derivative wrong, you will get 1 point in the total of two points.
2. If you integrate $\int\sqrt{1+t^3}dt$, you will get no point.

2. (10 points) Evaluate the integrals.

(a) (5 points) $\int \tan x \ln(\cos x)\, dx$.

(b) (5 points) $\int \frac{\sin x - 1}{\sin x \cos x}\, dx$.

Solution:

(a) (Method I)

Let $t = \cos x \Rightarrow dt = -\sin x dx$

$\Rightarrow \int \tan x \ln(\cos x) dx (2\text{pts}) = -\int \frac{\ln t}{t} dt = -\frac{1}{2}(\ln t)^2 + C (2\text{pts}) = -\frac{1}{2}(\ln(\cos x))^2 + C$ (1pt)

(Method II) Let $u = \ln\cos x, dv = \tan x dx \Rightarrow du = -\tan x dx, v = -\ln\cos x$ (2pts)

$\Rightarrow \int \tan x \ln(\cos x) dx = -(\ln\cos x)^2 - \int \tan x \ln(\cos x) dx$ (2pts)

$\Rightarrow \int \tan x \ln(\cos x) dx = -\frac{1}{2}(\ln(\cos x))^2 + C$ (1pt)

(b) (Method I)

$\int \frac{\sin x - 1}{\sin x \cos x} dx = \int \frac{1}{\cos x} - \frac{1}{\sin x \cos x} dx (2\text{pts}) = \int \sec x - \frac{2}{\sin 2x} dx$

$= \ln|\sec x + \tan x| - \int 2\csc 2x dx = \ln|\sec x + \tan x| + \ln|\csc 2x + \cot 2x| + C$ (3pts)

(Method II)

$\int \frac{\sin x - 1}{\sin x \cos x} dx = \int \frac{1}{\cos x} - \frac{1}{\sin x \cos x} dx (2\text{pts}) = \int \sec x - \frac{\sec^2 x}{\tan x} dx$

$= \ln|\sec x + \tan x| - \ln|\tan x| + C$ (3pts)

(Method III)

$\int \frac{\sin x - 1}{\sin x \cos x} dx = \int \frac{1}{\cos x} - \frac{1}{\sin x \cos x} dx$ (2pts)

$= \int \sec x - \tan x - \cot x dx = \ln|\sec x + \tan x| + \ln|\cos x| - \ln|\sin x| + C$ (3pts)

(Method IV)

$\int \frac{\sin x - 1}{\sin x \cos x} dx = \int \frac{(\sin x - 1)\cos x}{\sin x \cos^2 x} dx$

Let $t = \sin x \Rightarrow dt = \cos x dx$

$\Rightarrow \int \frac{t-1}{t(1-t^2)} dt (2\text{pts}) = \int \frac{-1}{t(t+1)} dt = \int \frac{1}{t+1} - \frac{1}{t} dt$

$= \ln\left|\frac{t+1}{t}\right| + C (2\text{pts}) = \ln\left|\frac{\sin x + 1}{\sin x}\right| + C$ (1pt)

(Method V)

Let $t = \tan\frac{x}{2} \Rightarrow dx = \frac{2}{1+t^2} dt$

$\Rightarrow \int \frac{\left(\frac{2t}{1+t^2}\right) - 1}{\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{1+t^2} dt (2\text{pts}) = \int \frac{2t - (1+t^2)}{t(1-t^2)} dt = \int \frac{(t-1)^2}{t(t+1)(t-1)} dt = \int \frac{2}{t+1} - \frac{1}{t} dt$

$= \ln\left|\frac{(t+1)^2}{t}\right| + C (2\text{pts}) = \ln\left|\frac{\left(\tan\frac{x}{2} + 1\right)^2}{\tan\frac{x}{2}}\right| + C$ (1pt)

(a) $\left(\begin{array}{ll} \ln|\csc 2x + \cot 2x| & = \ln|1 + \cos 2x| - \ln|\sin 2x| \\ & = \ln|2\cos^2 x| - \ln|2\sin x \cos x| = \ln|\cos x| - \ln|\sin x| \end{array}\right)$

(b) $\left(\ln|\sec x + \tan x| + \ln|\cos x| - \ln|\sin x| = \ln\left|\frac{1}{\sin x} + 1\right| = \ln\left|\frac{1 + \sin x}{\sin x}\right|\right)$

(c) $\left(\ln\left|\frac{\left(\tan\frac{x}{2} + 1\right)^2}{\tan\frac{x}{2}}\right| = \ln\left|\frac{\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1}{\tan\frac{x}{2}}\right| = \ln\left|\frac{2}{\sin x} + 2\right| = \ln\left|\frac{1 + \sin x}{\sin x}\right| + \ln 2\right)$

3. (16 points) Evaluate the integrals.

(a) (8 points) $\int \frac{x\,dx}{\sqrt{25-8x+x^2}}$.

(b) (8 points) $\int \frac{e^{2x}}{16-8e^x+e^{2x}}\,dx$.

Solution:

(a) First notice that

$$\int \frac{x\,dx}{\sqrt{25-8x+x^2}} = \underbrace{\int \frac{(x-4)\;dx}{\sqrt{9+(x-4)^2}}}_{=:I_1} + \underbrace{\int \frac{4\,dx}{\sqrt{9+(x-4)^2}}}_{=:I_2} \;.$$

To calculate I_1, substitute $u=(x-4)^2$ into the first integral on the right-hand-side to obtain

$$I_1 := \int \frac{(x-4)\;dx}{\sqrt{9+(x-4)^2}} = \int \frac{du}{2\sqrt{9+u}} = \sqrt{9+u}+C_1 = \sqrt{9+(x-4)^2}+C_1$$

(Method + Answer: 3+1 points)

(-0.5 points for no constant of integration or not expressing the answer in terms of x.)

To calculate I_2, let $x-4=3\tan\theta$, where $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$, then $dx=3\sec^2\theta\,d\theta$ and $\sec\theta\geq 0$. One then also has $\sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+\left(\frac{x-4}{3}\right)^2}=\frac{1}{3}\sqrt{25-8x+x^2}$. It follows that

$$\begin{aligned} I_2 := \int \frac{4\,dx}{\sqrt{9+(x-4)^2}} &= 4\int \frac{3\sec^2\theta\,d\theta}{3\sqrt{1+\tan^2\theta}} \\ &= 4\int |\sec\theta|\,d\theta \overset{(\text{as } \sec\theta\geq 0)}{=} 4\int \sec\theta\,d\theta \\ &= 4\ln|\sec\theta+\tan\theta|+C' \\ &= 4\ln\left|\sqrt{25-8x+x^2}+x-4\right|+C_2\;. \end{aligned}$$

where $C_2 := C'-4\ln 3$. (Method + Answer: 3+1 points)

(-0.5 points for no constant of integration or not expressing the answer in terms of x.)

Therefore,

$$\int \frac{x\,dx}{\sqrt{25-8x+x^2}} = \sqrt{25-8x+x^2}+4\ln\left|\sqrt{25-8x+x^2}+x-4\right|+C\;,$$

where $C := C_1+C_2$.

(No deduction even without explaining relation between different constants of integration.)

(b) Let $u=e^x$, then $du=e^x dx$. It follows that

$$\begin{aligned} \int \frac{e^{2x}\,dx}{16-8e^x+e^{2x}} &= \int \frac{u\,du}{16-8u+u^2} = \int \frac{(u-4)+4\,du}{(u-4)^2} = \int \frac{du}{u-4}+\int \frac{4\,du}{(u-4)^2} \\ &= \ln|u-4|-\frac{4}{u-4}+C = \ln|e^x-4|-\frac{4}{e^x-4}+C\;. \end{aligned}$$

(Method (substitution + partial fraction decomposition): 3+3 points)

(Answer (2 antiderivatives): 1+1 points)

(-0.5 points for no constant of integration or not expressing the answer in terms of x.)

4. (10 points) Find the value of the constant c for which the integral

$$\int_0^\infty \frac{x^2+8}{x^3+8} - \frac{c}{\sqrt{x^2+1}}\,dx \text{ converges.}$$

Evaluate the integral for this value of c.

Solution:

(2 pts) $\dfrac{x^2+8}{x^3+8} = \dfrac{1}{x+2} + \dfrac{2}{x^2-2x+4}$

(1 pts) $\displaystyle\int \frac{2}{x^2-2x+4}dx = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C_1$

(1 pts) $\displaystyle\int \frac{1}{\sqrt{x^2+1}}dx = \ln\left|\sqrt{x^2+1}+x\right| + C_2$

(1 pts) $\displaystyle\int_0^b \frac{x^2+8}{x^3+8} - \frac{c}{\sqrt{x^2+1}}\,dx = \ln|b+2| - \ln 2 + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{b-1}{\sqrt{3}}\right) + \frac{\pi}{3\sqrt{3}} - c\ln\left|\sqrt{b^2+1}+b\right|$

(1 pts) $\displaystyle\int_0^\infty f(x)dx = \lim_{b\to\infty}\int_0^b f(x)dx$

(1 pts) $\displaystyle\int_0^\infty \frac{x^2+8}{x^3+8} - \frac{c}{\sqrt{x^2+1}}\,dx = \frac{4\pi}{3\sqrt{3}} - \ln 2 + \lim_{b\to\infty}\ln\left|\frac{b+2}{\left(\sqrt{b^2+1}+b\right)^c}\right|$

(2 pts) The integral converges only if $c = 1$.

(1 pts) $\displaystyle\lim_{b\to\infty}\left|\frac{b+2}{\sqrt{b^2+1}+b}\right| = -\ln 2$, the value of the integral is $\dfrac{4\pi}{3\sqrt{3}} - 2\ln 2$.

Students, who correctly discuss the value of c making the integral converges only, get 2 points.

5. (18 points) Let R be the region bounded above by the curve $y = \tan^2 x$, left by $x = 0$, below by $y = 0$, and right by $x = \pi/4$. Let $\tilde{R}$ be the region bounded above by the curve $y = \tan^p x$, left by $x = 0$, below by $y = 0$, and right by $x = \pi/2$, where $p > 0$ is a constant.

(a) (6 points) Rotate R about the x-axis. Find the resulting volume.

(b) (6 points) Rotate R about the y-axis. Find the resulting volume.

(c) (6 points) Rotate $\tilde{R}$ about the x-axis. Find the values of p such that the resulting volume is finite. (Hint: You may use the inequality $(\frac{\pi}{2} - x) \cdot \tan x < 2$, for $\frac{\pi}{4} \le x < \frac{\pi}{2}$.)

Solution:

(a) Volume $V = \int_0^{\frac{\pi}{4}} \pi y^2 dx = \pi \int_0^{\frac{\pi}{4}} \tan^4 x dx$

$$\int (\tan x)^4 dx = \int (\sec^2 x - 1)\tan^2 x dx = \int \sec^2 x \tan^2 x dx - \int \tan^2 x dx \tag{1}$$

For first integral in (1), let $u = \tan x$, $du = \sec^2 x dx$. Then

$$\int \sec^2 x \tan^2 x dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C \tag{2}$$

And for second integral in (1)

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C \tag{3}$$

Substitute (2) and (3) into (1) we get

$$\int (\tan x)^4 dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

Hence the volumn V is

$$\begin{aligned} V &= \pi\Big[\frac{1}{3}\tan^3 x - \tan x + x\Big]_0^{\frac{\pi}{4}} = \pi\Big(\frac{1}{3}\tan^3\frac{\pi}{4} - \tan\frac{\pi}{4} + \frac{\pi}{4}\Big) \\ &= \frac{\pi}{3} - \pi + \frac{\pi^2}{4} \\ &= -\frac{2\pi}{3} + \frac{\pi^2}{4} \end{aligned}$$

Grading criteria:

- Write down the formula of volumn V (either cylinder method or shell method) get 2 points.
- There are two part in the integration of $\tan^4 x$, get 3 points if you calculate both part correctly, get 1 points if you calculate only one of those correctly.
- If both part of integration are wrong, you can get 1 points if you try to simplify the integration of $\tan^4 x$
- Write down answer correctly get 1 points.

(b)

By the shell method, the resulting volume is

$$V_R = 2\pi\int_0^{\frac{\pi}{4}} x\tan^2 xdx.$$

Setting

$$u = x, \qquad dv = \tan^2 xdx = (\sec^2 x - 1)dx,$$

we have

$$du = dx, \quad v = \tan x - x.$$

Therefore,

$$V_R = 2\pi\left\{\left[x(\tan x - x)\right]_0^{\pi/4} - \int_0^{\frac{\pi}{4}} \tan x - xdx\right\}$$

$$= 2\pi\left[x\left(\tan x - x\right) - \left(\ln\left|\sec x\right| - \frac{x^2}{2}\right)\right]_0^{\pi/4}$$

$$= \frac{\pi^2}{2} - \frac{\pi^3}{16} - \pi\ln 2$$

(1 pt: volume formula.)

(2 pts: integration by part.)

(2 pts: process.).

(1 pt: right volume.)

(c)

The volume of the given solid of revolution is given by the improper integral.

$$V = \pi\int_0^{\frac{\pi}{2}} \tan^{2p} xdx$$

$$= \pi \lim_{\theta\to(\frac{\pi}{2})^-}\int_0^{\theta} \tan^{2p} xdx$$

$$= \pi\int_0^{\frac{\pi}{4}} \tan^{2p} xdx + \pi \lim_{\theta\to(\frac{\pi}{2})^-}\int_{\frac{\pi}{4}}^{\theta} \tan^{2p} xdx.$$

Note that the volume is finite if and only if the latter integral is convergent.

(1.5 pts: correct impretation of the improper integral and convergence criterion.)

The inequality $\left(\frac{\pi}{2} - x\right)\tan x < 2$ for $\frac{\pi}{4} \le x < \frac{\pi}{2}$

implies that

$$0 < \tan^{2p} x < \left(\frac{2}{\frac{\pi}{2} - x}\right)^{2p} \quad \text{for } \frac{\pi}{4} \le x < \frac{\pi}{2}.$$

Since,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2^{2p}}{(\frac{\pi}{2} - x)^{2p}}dx = \lim_{\theta\to(\frac{\pi}{2})^-}\int_{\frac{\pi}{4}}^{\theta} \frac{2^{2p}}{(\frac{\pi}{2} - x)^{2p}}xdx$$

$$= \lim_{\theta \to (\frac{\pi}{2})^-} 2^{2p} \left[\frac{1}{(2p-1)(\frac{\pi}{2}-x)^{2p-1}} \right]_{\frac{\pi}{4}}^{\theta}$$

$$= \left\{ \frac{1}{(2p-1)(\frac{\pi}{2}-\theta)^{2p-1}} - \frac{1}{(2p-1)(\frac{\pi}{4})^{2p-1}} \right\}$$

$$= \begin{cases} (\frac{\pi}{4})^{1-2p} \frac{1}{1-2p} & \text{when } p < \frac{1}{2} \\ +\infty & \text{when } p > \frac{1}{2} \end{cases}.$$

It follows by the comparison test that $\int_0^{\frac{\pi}{2}} \tan^{2p} x dx$ is convergent when $p < \frac{1}{2}$.

(2 pts: showing that volume is convergent when $p < \frac{1}{2}$.)

Note also that, when $p = \frac{1}{2}$,

$$V = \pi \int_0^{\frac{\pi}{2}} \tan x dx = \lim_{\theta \to (\frac{\pi}{2})^-} \int_0^{\theta} \tan x dx = \lim_{\theta \to (\frac{\pi}{2})^-} \left[\ln|\sec x| \right]_{\frac{\pi}{4}}^{\theta} = \infty$$

(1 pt: showing that volume is divergent when $p = \frac{1}{2}$.)

when $p > \frac{1}{2}$, the volume is

$$V = \pi \int_0^{\frac{\pi}{4}} \tan^{2p} x dx + \pi \lim_{\theta \to (\frac{\pi}{2})^-} \int_{\frac{\pi}{4}}^{\theta} \tan^{2p} x dx > \pi \int_0^{\frac{\pi}{4}} \tan x dx + \pi \lim_{\theta \to (\frac{\pi}{2})^-} \int_{\frac{\pi}{4}}^{\theta} \tan\ x dx$$

Since $\tan^{2p} x > \tan x$ for $\frac{\pi}{4} \le x < \frac{\pi}{2}$,

the comparision test then assumes that the volume in question is finite if and only if

$0 < p < \frac{1}{2}$.

(1.5 pts: showing that volume is divergent when $p > \frac{1}{2}$.)

6. (12 points) Consider the differential equation

$$y''(t) + 2y'(t) + 10y(t) = f(t),\ y(0) = 0,\ y'(0) = 1,$$

where $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 13\sin 2t, & t \ge \pi. \end{cases}$

(a) (2 points) Write down the general solution of its related homogeneous equation.

(b) (5 points) Derive the algebraic equation that $Y(s)$, the Laplace transform of the solution $y(t)$, satisfies.

(c) (5 points) Solve this differential equation.

(Hint: $\mathfrak{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}\mathfrak{L}\{f(t)\}$, where $\mathcal{U}(t)$ is the unit step function.)

Solution:

(a)

$m^2 + 2m + 10 = 0 \rightarrow m = -1 \pm i3$

$e^{-t}(c1\sin 3t + c2\cos 3t)$

(b)

$$[s^2\mathrm{Y(s)} - \mathrm{s}\cdot\mathrm{y(0)} - \mathrm{y'(0)}] - 2[\mathrm{Y(s)} - 0] + 10\mathrm{Y(s)} = 13\cdot\frac{2}{s^2+4}e^{-\pi s}$$

$$\rightarrow \mathrm{Y(s)} = \frac{26}{(s^2+4)(s^2+2s+10)}e^{-\pi s} + \frac{1}{s^2+2s+10}$$

(c)

$$\mathrm{Y(s)} = \frac{26}{(s^2+4)(s^2+2s+10)}e^{-\pi s} + \frac{1}{s^2+2s+10}$$

$$= \left[\frac{-s+3}{(s^2+4)} + \frac{\mathrm{s}-1}{(s^2+2s+10)}\right]e^{-\pi s} + \left[\frac{1}{s^2+2s+10}\right]$$

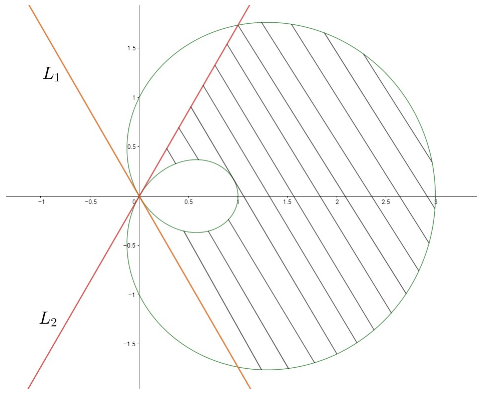
$$= \left[-\frac{s}{s^2+2^2} + \frac{3}{2}\cdot\frac{2}{s^2+2^2} + \frac{s+1}{(s+1)^2+3^2} - \frac{2}{3}\cdot\frac{3}{(s+1)^2+3^2}\right]e^{-\pi s}$$

$$+ \left[\frac{1}{s^2+2s+10}\right]$$

$$\rightarrow \mathrm{y(t)} = \left\{-\cos 2(t-\pi) + \frac{3}{2}\sin 2(t-\pi)\right.$$

$$\left. + e^{-(t-\pi)}\left[\sin 3(t-\pi) - \frac{2}{3}\cos 3(t-\pi)\right]\right\}\mathrm{U}(t-\pi) + \frac{1}{3}e^{-t}\sin 3t$$

7. (10 points) The curve $C: \ r = 1 + 2\cos\theta$ and its two tangent lines, L_1 and L_2, at the pole are shown in the graph.



(a) (6 points) Find the area of the shaded region.

(b) (4 points) Now consider another curve $\tilde{C}: \ r = -1 - 2\cos(\theta - \frac{\pi}{6})$. How is the curve $\tilde{C}$ related to the curve C?

Solution:

(a) $L_1 : \theta = \frac{2\pi}{3}$ **(1%)**, $L_2 : \theta = \frac{4\pi}{3}$ **(1%)**.

$$\text{Area} = 2\left[\underbrace{\int_0^{\frac{\pi}{3}}}_{(1\%)} \underbrace{\frac{1}{2}(1+2\cos\theta)^2 \mathrm{d}\theta}_{(1\%)} - \underbrace{\int_{\frac{2\pi}{3}}^{\pi}}_{(1\%)} \frac{1}{2}(1+2\cos\theta)^2 \mathrm{d}\theta\right]$$

$$\left(\text{or } 2\left[\int_{\frac{5\pi}{3}}^{2\pi} \frac{1}{2}(1+2\cos\theta)^2 \mathrm{d}\theta - \int_{\pi}^{\frac{4\pi}{3}} \frac{1}{2}(1+2\cos\theta)^2 \mathrm{d}\theta\right]\right)$$

$= 4\sqrt{3}$ **(1%)**

(b) Reflect through the pole **(2%)** and rotate $\frac{\pi}{6}$ (rad) counter-clockwise **(2%)**.

8. (12 points)

(a) (4 points) There are two polar curves C_1 and C_2 with equations $r = r_1(\theta)$ and $r = r_2(\theta)$. Suppose that they intersect orthogonally at the point with polar coordinates (r_0, θ_0). Show that $r_1'(\theta_0) \cdot r_2'(\theta_0) = -r_0^2$ which means that $\left(\frac{1}{r_1}\frac{dr_1}{d\theta}\right) \cdot \left(\frac{1}{r_2}\frac{dr_2}{d\theta}\right)\Big|_{\theta=\theta_0} = -1$.
(Hint: Compute the slopes of the tangent lines of C_1 and C_2 at the intersection.)

(b) (3 points) Consider the family of cardioids $r = r_c(\theta) = c(1 + \cos\theta)$, where c is an arbitrary constant. Derive the differential equation that $r_c(\theta)$ satisfies for all c.

(c) (5 points) Find the orthogonal trajectories of the family of cardioids $r = c(1 + \cos\theta)$.

Solution:

(a) **(1 pts)** $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

(1 pts) $\because \perp$ at (r_0, θ_0), $\therefore \dfrac{dy_1}{dx_1} \cdot \dfrac{dy_2}{dx_2} = -1$ at (r_0, θ_0).

(1 pts) $\dfrac{r_1'\sin\theta_0 + r_1\cos\theta_0}{r_1'\cos\theta_0 - r_1\sin\theta_0} \cdot \dfrac{r_2'\sin\theta_0 + r_2\cos\theta_0}{r_2'\cos\theta_0 - r_2\sin\theta_0} = -1$

$$\Longrightarrow \frac{r_1'r_2'\sin^2\theta_0 + (r_1'r_2 + r_2'r_1)\cos\theta_0\sin\theta_0 + r_1r_2\cos^2\theta_0}{r_1'r_2'\cos^2\theta_0 - (r_1'r_2 + r_2'r_1)\cos\theta_0\sin\theta_0 + r_1r_2\sin^2\theta_0} = -1$$

(1 pts) $\Longrightarrow r_1'r_2'\left(\sin^2\theta_0 + \cos^2\theta_0\right) = -r_1r_2\left(\cos^2\theta_0 + \sin^2\theta_0\right) \Longrightarrow r_1'(\theta_0)r_2'(\theta_0) = -r_0^2.$

(b)

$$r_c'(\theta) = -c\sin\theta\,, \text{c} = \frac{r_c}{1+\cos\theta} \rightarrow \frac{r'_c}{r_c} = \frac{-\sin\theta}{1+\cos\theta}$$

(c)

$$\frac{r'_T}{r_T} = \frac{1+\cos\theta}{\sin\theta} \rightarrow \ln r_T = \int \frac{1+\cos\theta}{\sin\theta}d\theta = \int \csc\theta + \cot\theta\, d\theta$$

$$\rightarrow \int \csc\theta + \cot\theta\, d\theta = \ln|\csc\theta - \cot\theta| + \ln|\sin\theta| + C \rightarrow r_T = A(1-\cos\theta)$$

或

$$\rightarrow \int \csc\theta + \cot\theta\, d\theta = -\ln|\csc\theta + \cot\theta| + \ln|\sin\theta| + C \rightarrow r_T = A\frac{sin^2\theta}{\cos\theta + 1}$$

A ∈ constant