1061微甲01-02班期中考解答和評分標準

1. (15 points) Compute each of the following limits if it exists or explain why it doesn't exist.

   (a) (5 points) $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)\sin x$.

   (b) (5 points) $\lim_{x\to 0} \frac{\tan x}{\sqrt{1-\cos 3x}}$.

   (c) (5 points) $\lim_{x\to 0} (\cos x)^{\frac{2}{x^2}}$.

Solution:

(a) Method 1

(2 points) $-1 \le \sin\frac{1}{x^2} \le 1 \Rightarrow 0 \le |\sin\frac{1}{x^2}| \le 1 \Rightarrow 0 \le |\sin x \sin\frac{1}{x^2}| \le |\sin x|$

(2 points) $\lim_{x\to 0} \sin x = 0 \Rightarrow \lim_{x\to 0} |\sin x| = 0 = \lim_{x\to 0} 0$.

(1 point) By Squeeze Theorem, $\lim_{x\to 0} |\sin x \sin\frac{1}{x^2}| = 0 \Rightarrow \lim_{x\to 0} \sin x \sin\frac{1}{x^2} = 0$.

Method 2 To find the limit as x approaching 0, only to consider $x \in (-\pi, \pi)$.

(1 point) When $x > 0$, $-1 \le \sin\frac{1}{x^2} \le 1 \Rightarrow -\sin x \le \sin x \sin\frac{1}{x^2} \le \sin x$.

(1 point) $\lim_{x\to 0^+} -\sin x = 0 = \lim_{x\to 0^+} \sin x$.

(1 point) When $x < 0$, $-1 \le \sin\frac{1}{x^2} \le 1 \Rightarrow \sin x \le \sin x \sin\frac{1}{x^2} \le -\sin x$.

(1 point) $\lim_{x\to 0^-} \sin x = 0 = \lim_{x\to 0^-} -\sin x$.

(1 point) By Squeeze Theorem, $\lim_{x\to 0^+} \sin x \sin\frac{1}{x^2} = 0 = \lim_{x\to 0^-} \sin x \sin\frac{1}{x^2}$.

Hence, $\lim_{x\to 0} \sin x \sin\frac{1}{x^2} = 0$.

(b) (2 points) $\frac{\tan x}{\sqrt{1-\cos 3x}} = \frac{\tan x}{x}\frac{x}{\sqrt{1-\cos 3x}} = \left(\frac{1}{\cos x}\frac{x}{\sin x}\right)\left(\frac{x}{3|x|}\sqrt{\frac{(3x)^2}{1-\cos(3x)}}\right) = \frac{1}{3}\frac{1}{\cos x}\frac{\sin x}{x}\frac{x}{|x|}\sqrt{\frac{(3x)^2}{1-\cos(3x)}}$ for $x \ne 0$.

(1 point) $\lim_{x\to 0^+} \frac{\tan x}{\sqrt{1-\cos 3x}} = \lim_{x\to 0^+}\left(\frac{1}{3}\frac{1}{\cos x}\frac{\sin x}{x}\frac{x}{|x|}\sqrt{\frac{(3x)^2}{1-\cos(3x)}}\right) = \frac{1}{3}\cdot 1\cdot 1\cdot 1\cdot\sqrt{2} = \frac{\sqrt{2}}{3}$.

(1 point) $\lim_{x\to 0^-} \frac{\tan x}{\sqrt{1-\cos 3x}} = \lim_{x\to 0^-}\left(\frac{1}{3}\frac{1}{\cos x}\frac{\sin x}{x}\frac{x}{|x|}\sqrt{\frac{(3x)^2}{1-\cos(3x)}}\right) = \frac{1}{3}\cdot 1\cdot 1\cdot(-1)\cdot\sqrt{2} = -\frac{\sqrt{2}}{3}$.

(1 point) $\lim_{x\to 0^+} \frac{\tan x}{\sqrt{1-\cos 3x}} = \frac{\sqrt{2}}{3} \ne -\frac{\sqrt{2}}{3} = \lim_{x\to 0^-} \frac{\tan x}{\sqrt{1-\cos 3x}} \Rightarrow \lim_{x\to 0} \frac{\tan x}{\sqrt{1-\cos 3x}}$ doesn't exist.

(c) Observe that

$$\lim_{x\to 0}(\cos(x))^{2/x^2} = \lim_{x\to 0} e^{2\ln(\cos(x))/x^2} = e^{\lim_{x\to 0} 2\ln(\cos(x))/x^2} \textbf{ (1 pt)}.$$

Now since

$$\begin{aligned}\lim_{x\to 0}\frac{2\ln(\cos(x))}{x^2} &\overset{L}{=} \lim_{x\to 0}\frac{2\times -\sin(x)/\cos(x)}{2x} \textbf{ (2 pts)}\\ &= \lim_{x\to 0}\frac{-\sin(x)}{x}\times\frac{1}{\cos(x)}\\ &= -1 \textbf{ (1 pt)} \times 1\\ &= -1 \textbf{ (1 pt)},\end{aligned}$$

we conclude that $\lim_{x\to 0}(\cos(x))^{2/x^2} = e^{\lim_{x\to 0} 2\ln(\cos(x))/x^2} = e^{-1}$.

2. (10 points)

(a) (5 points) Compute the limit if it exists or explain why it doesn't exist.

$$\lim_{x\to+\infty} \sqrt{x+\sqrt{x}} - \sqrt{x-\sqrt{x}}.$$

(b) (5 points) Determine for what values of a, $0 < a < 1$, does $\lim_{x\to+\infty} f(x)$ exist, where $f(x) = \sqrt{x+x^a} - \sqrt{x-x^a}$ for $x \geq 1$.

Solution:

(a)

$$\lim_{x\to+\infty} \sqrt{x+\sqrt{x}} - \sqrt{x-\sqrt{x}} = \lim_{x\to+\infty} \frac{(x+\sqrt{x})-(x-\sqrt{x})}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}} \textbf{ (2 pts)}$$

$$= \lim_{x\to+\infty} \frac{2\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}} = \lim_{x\to+\infty} \frac{2}{\sqrt{1+\frac{\sqrt{x}}{x}}+\sqrt{1-\frac{\sqrt{x}}{x}}}$$

$$= \lim_{x\to+\infty} \frac{2}{\sqrt{1+\frac{1}{\sqrt{x}}}+\sqrt{1-\frac{1}{\sqrt{x}}}} \textbf{ (2 pts)} = \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{1+1} = 1 \textbf{ (1 pt)}$$

(b) $f(x) = \frac{2x^a}{\sqrt{x+x^a}+\sqrt{x-x^a}} = \frac{2x^{a-\frac{1}{2}}}{\sqrt{1+x^{a-1}}+\sqrt{1-x^{a-1}}}$

$\because a-1<0 \quad \therefore x^{a-1} \to 0$ as $x \to 0$

Hence $\sqrt{1+x^{a-1}} + \sqrt{1-x^{a-1}} \to 2$ as $x \to 0$

If $a > \frac{1}{2}$, then $x^{a-\frac{1}{2}} \to \infty$ as $x \to 0$, which implies that $\lim_{x\to\infty} f(x) = \infty$

If $a = \frac{1}{2}$, then $f(x) = \frac{2}{\sqrt{1+\frac{1}{\sqrt{x}}}+\sqrt{1-\frac{1}{\sqrt{x}}}} \to 1$ as $x \to \infty$

If $a < \frac{1}{2}$, then $\lim_{x\to\infty} x^{a-\frac{1}{2}} = 0$ and .t $\lim_{x\to\infty} f(x) = 0$

Therefore, the answer is $0 < a \leq \frac{1}{2}$.

3. (15 points) Differentiate the following functions.

(a) (5 points) $f(x) = \frac{\operatorname{arcsec}(e^x)}{1+x^e}$.

(b) (5 points) $f(x) = \log_2 \sqrt{x} + \tan^{-1}(x^3)$

(c) (5 points) $f(x) = x^{\cos x}$.

Solution:

(a) $f'(x) = \frac{(1+x^e)\frac{e^x}{e^x\sqrt{e^{2x}-1}} + \sec^{-1}(e^x)\cdot e\cdot x^{e-1}}{(1+x^e)^2} = \frac{1+x^e+e\cdot x^{e-1}\sqrt{e^{2x}-1}\cdot\sec^{-1}(e^x)}{(1+x^e)^2\sqrt{e^{2x}-1}}$

(b) (Method 1) Simplify $f(x)$ as

$$f(x) = \overbrace{\frac{\ln x}{2\ln 2}}^{1\text{pt}} + \tan^{-1}(x^3)$$

Then

$$f'(x) = \overbrace{\frac{1}{(2\ln 2)x}}^{1\text{pt}} + \overbrace{3x^2}^{1\text{pt}} \cdot \overbrace{\frac{1}{1+x^6}}^{1\text{pt}}$$

(All correct +1pt.)

(Method 2) Differentiate $f(x)$ directly

$$f'(x) = \overbrace{\frac{1}{(\ln 2)\sqrt{x}}}^{1\text{pt}} \cdot \overbrace{\frac{1}{2\sqrt{x}}}^{1\text{pt}} + \overbrace{3x^2}^{1\text{pt}} \cdot \overbrace{\frac{1}{1+x^6}}^{1\text{pt}}$$

(All correct +1pt.)

(c) (Method 1) Write $f(x)$ as

$$\overbrace{f(x) = e^{\cos x \ln x}}^{1\text{pt}}$$

Then differentiate $f(x)$

$$f'(x) = \overbrace{e^{\cos x\ln x}(\cos x\ln x)'}^{1\text{pt}} = e^{\cos x\ln x}\overbrace{(-\sin x\ln x + \frac{\cos x}{x})}^{1\text{pt}}$$

(All correct +2pts.)

(Method 2) Use logarithmic differentiation. Write $f(x)$ as

$$\overbrace{\ln f(x) = \cos x\ln x}^{1\text{pt}}$$

Differentiate

$$\overbrace{\frac{f'(x)}{f(x)}}^{1\text{pt}} = \overbrace{(-\sin x\ln x + \frac{\cos x}{x})}^{1\text{pt}}$$

Thus,

$$f'(x) = x^{\cos x}(-\sin x\ln x + \frac{\cos x}{x})$$

(All correct +2pts.)

Remark 計算錯誤至少扣2分，答案正確但沒有計算過程或說明扣1分，關鍵過程大致正確但抄寫錯誤扣1分。

4. (12 points) Let $f(x) = \begin{cases} x^{\frac{4}{3}} \cos\left(\frac{1}{x}\right), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0. \end{cases}$

(a) (3 points) Is $f(x)$ continuous at $x = 0$?

(b) (6 points) Compute $f'(x)$ for $x \neq 0$ and $f'(0)$.

(c) (3 points) Is $f'(x)$ continuous at $x = 0$?

Solution:

(a) Because the cosine of any number lies between -1 and 1, we can write.

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

Any inequality remains true when multiplied by a positive number. We know that $x^{\frac{4}{3}} \geq 0$ for all x and so, multiplying each side of the inequalities by $x^{\frac{4}{3}}$, we get

$$-x^{\frac{4}{3}} \leq x^{\frac{4}{3}} \cos\left(\frac{1}{x}\right) \leq x^{\frac{4}{3}}$$

We know that

$$\lim_{x\to 0} x^{\frac{4}{3}} = \lim_{x\to 0} -x^{\frac{4}{3}} = 0$$

By Squeeze Theorem, we obtain

$$\lim_{x\to 0} x^{\frac{4}{3}} \cos\left(\frac{1}{x}\right) = 0$$

Therefore, $f(x)$ is continuous at $x = 0$.

(b) By definition of differential and pinching theorem,

$$f'(0) = \lim_{x\to 0} \frac{x^{\frac{4}{3}} \cos\left(\frac{1}{x}\right) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^{\frac{4}{3}} \cos\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} x^{\frac{1}{3}} \cos\left(\frac{1}{x}\right) = 0$$

On the other hand, we consider $x \neq 0$. By the Product Rule, $f'(x) = \left(x^{\frac{4}{3}}\right)' \cos\left(\frac{1}{x}\right) + x^{\frac{4}{3}} \{\cos\left(\frac{1}{x}\right)\}'$. Now by Chain Rule, $\{\cos\left(\frac{1}{x}\right)\}' = \sin\left(\frac{1}{x}\right) \cdot \frac{1}{x^2}$. Therefore, we obtain

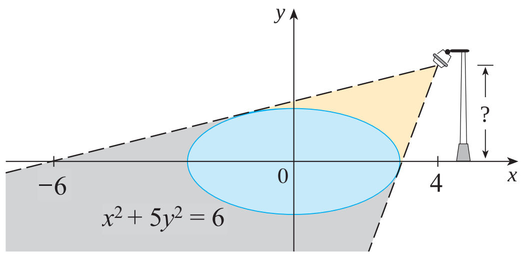
$$f'(x) = \frac{4}{3} x^{\frac{1}{3}} \cos\left(\frac{1}{x}\right) + x^{\frac{-2}{3}} \sin\left(\frac{1}{x}\right)$$

(c) Because $\limsup_{x\to 0} f'(x) = \infty$ and $\liminf_{x\to 0} f'(x) = -\infty$, $\lim_{x\to 0} f'(x)$ does not exist. Therefore, we can deduce that $f'(x)$ is not continuous at $x = 0$.

[Remark]

In question (b), suppose you know the definition of $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$. You can get 2 points. Suppose you write $f'(0) = \lim_{x\to 0} f'(x)$. You can not get any point in (b).

5. (10 points) The figure shows a lamp located 4 units to the right of the y−axis and a shadow created by the elliptical region $x^2 + 5y^2 \le 6$. If the point $(-6, 0)$ is on the edge of the shadow, how far above the x−axis is the lamp located?



Solution:

1. (Method 1)
By implicit differentiation, we have $2x + 10yy' = 0$, or, $y' = -\frac{x}{5y}$ (**5%**)
Suppose the point of tangency is (x_o, y_o), then the tangent line is given by $y = -\frac{x_o}{5y_o}(x - x_o) + y_o$
Plug in $(-6, 0)$, we have $0 = -\frac{x_o}{5y_o}(-6 - x_o) + y_o$, or, $x_o^2 + 5y_o^2 = -6x_o$.
Also, $x_o^2 + 5y_o^2 = 6$, so $x_o = -1$, $y_o = 1$(**3%**)
Then the tangent line is $y = \frac{1}{5}x + \frac{6}{5}$, so $y\Big|_{x=4} = 2$(**2%**)

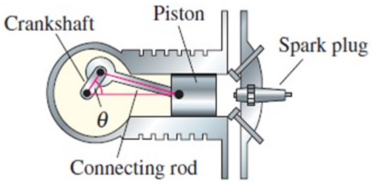
2. (Method 2)
Suppose the lamp is located at $(4, h)$. Then the tangent line is given by $y = \frac{h}{10}(x+6)$(**4%**)
Since it's tangent to the ellipse, the equation

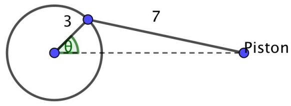
$$\begin{cases} y = \frac{h}{10}(x+6) \\ x^2 + 5y^2 = 6 \end{cases}$$

should have only one zero(repeated roots), or equivalently, the discriminant of $x^2 + 5\left(\frac{h}{10}(x+6)\right)^2 = 6$ sholud be zero.(**4%**)
Thus, $36h^4 - (20 + h^2)(36h^2 - 120) = 0$, we have $h = 2$(**2%**)

3. (Method 3) Suppose the pint of tangency is $(x_o, \sqrt{\frac{6-x_o^2}{5}})$, and the lamp is located at $(4, h)$
Then we have $\frac{\sqrt{\frac{6-x_o^2}{5}}-0}{x_o-(-6)} = \frac{h-0}{4-(-6)}$, or $h = \frac{2\sqrt{30-5x_o^2}}{x_o+6}$(**4%**)
We can find h by using the condition $\frac{dh}{dx} = 0$(why?)(**4%**)
Thus, $0 = \frac{\frac{-10x_o}{\sqrt{30-5x_o^2}}\cdot(x_o+6) - 2\sqrt{30-5x_o^2}\cdot 1}{(x_o+6)^2} = \frac{-10x_o^2 - 60x_o - 60 + 10x_o^2}{(x_o+6)^2\sqrt{30-5x_o^2}}$
Hence, $x_o = -1$, $y_o = 1$, and $h = 2$(**2%**)

6. (10 points) In the engine, a 7-inch connecting rod is fastened to a crank of radius 3 inches. The crankshaft(機軸) rotates counterclockwise at a constant rate of 100 revolutions per minute. Find the velocity of the piston(活塞) when $\theta = \frac{\pi}{3}$. (Reminder: the angular velocity of a circular motion at a constant speed of 1 revolution per minute is 2π rad/min.)





Solution:

First, we know $\frac{d\theta}{dt} = 100 \cdot 2\pi = 200\pi$ (1 point)

(Idea I)

$\because 7^2 = 3^2 + x^2 - 2 \cdot 3 \cdot x\cos\theta$ (3 points)

$\therefore 0 = 2x\frac{dx}{dt} - 6(\frac{dx}{dt}\cos\theta - x\sin\theta\frac{d\theta}{dt})$

$\Rightarrow (6\cos\theta - 2x)\frac{dx}{dt} = 6x\sin\theta\frac{d\theta}{dt} \Rightarrow \frac{dx}{dt} = \frac{6x\sin\theta}{6\cos\theta - 2x}\frac{d\theta}{dt}$ (3 points)

When $\theta = \frac{\pi}{3}$, then $x = 8$. $\Rightarrow \frac{dx}{dt} = -\frac{4800\sqrt{3}}{13}\pi$. (2 points)

(Idea II)

$\because 7^2 = 3^2 + x^2 - 2 \cdot 3 \cdot x\cos\theta \Rightarrow x^2 - 6\cos\theta - 40 = 0$ (3 points)

$\therefore x = 3\cos\theta + \sqrt{9\cos^2\theta + 40}$ (for $x > 0$) $\Rightarrow \frac{dx}{dt} = -3\sin\theta\frac{d\theta}{dt} + \frac{1}{2} \cdot \frac{-18\cos\theta\sin\theta}{\sqrt{9\cos^2\theta + 40}} \cdot \frac{d\theta}{dt}$ (3 points)

when $\theta = \frac{\pi}{3}$, we have $\frac{dx}{dt} = \left[-3 \cdot \left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\frac{18 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}}{\sqrt{\frac{9}{4} + 40}}\right] \cdot 200\pi = -\frac{4800\sqrt{3}}{13}\pi$ (2 points)

(Idea III)

$\because \cos\theta = \frac{3^2 + x^2 - 7^2}{2 \cdot 3 \cdot x}$ $\left(\Rightarrow x = 3\cos\theta + \sqrt{9\cos^2\theta + 40}\ (\text{for } x > 0)\right)$ (3 points)

$\therefore -\sin\theta\frac{d\theta}{dt} = \frac{2x \cdot 6x\frac{dx}{dt} - 6(x^2 - 40)\frac{dx}{dt}}{36x^2} \Rightarrow -\sin\theta\frac{d\theta}{dt} = \frac{x^2 + 40}{6x^2}\frac{dx}{dt}$ (3 points)

When $\theta = \frac{\pi}{3}$, then $x = 8$. $\Rightarrow \frac{dx}{dt} = -\sin\theta\frac{d\theta}{dt} \cdot \frac{6x^2}{x^2 + 40}\bigg|_{x=8, \theta=\frac{\pi}{3}} = -\frac{4800\sqrt{3}}{13}\pi$. (2 points)

(Idea IV)

$\because x = 3\cos\theta + \sqrt{49 - (3\sin\theta)^2}$ $\left(= 3\cos\theta + \sqrt{40 + 9(1 - \sin^2\theta)} = 3\cos\theta + \sqrt{9\cos^2\theta + 40}\right)$ (3 points)

$\therefore \frac{dx}{dt} = -3\sin\theta\frac{d\theta}{dt} + \frac{1}{2} \cdot \frac{-18\sin\theta\cos\theta}{\sqrt{49 - 9\sin^2\theta}} \cdot \frac{d\theta}{dt}$ (3 points)

when $\theta = \frac{\pi}{3}$, we have $\frac{dx}{dt} = \left[-3 \cdot \left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\frac{18 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}}{\sqrt{49 - \frac{27}{4}}}\right] \cdot 200\pi = -\frac{4800\sqrt{3}}{13}\pi$ (2 points)

(Idea V)

$7\sin\phi = 3\sin\theta \Rightarrow 7\cos\phi\frac{d\phi}{dt} = 3\cos\theta\frac{d\theta}{dt}$

$\Rightarrow \frac{d\phi}{dt} = \frac{3\cos\theta}{7\cos\phi}\frac{d\theta}{dt} = \frac{3\cos\theta}{\sqrt{49 - 49\sin^2\phi}}\frac{d\theta}{dt} = \frac{3\cos\theta}{\sqrt{49 - 9\sin^2\theta}}\frac{d\theta}{dt}$ (2 points)

$\because x = 3\cos\theta + 7\cos\phi$ $\left(= 3\cos\theta + 7\sqrt{1 - \frac{9}{49}\sin^2\theta} = 3\cos\theta + 7\sqrt{49 - 9\sin^2\theta}\right)$ (1 point)

$\therefore \frac{dx}{dt} = -3\sin\theta\frac{d\theta}{dt} - 7\sin\phi\frac{d\phi}{dt} = -3\sin\theta\frac{d\theta}{dt} - 3\sin\theta\left(\frac{3\cos\theta}{\sqrt{49 - 9\sin^2\theta}}\frac{d\theta}{dt}\right)$ (3 points)

when $\theta = \frac{\pi}{3}$, we have $\frac{dx}{dt} = \left[-3 \cdot \left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\frac{18 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}}{\sqrt{49 - \frac{27}{4}}}\right] \cdot 200\pi = -\frac{4800\sqrt{3}}{13}\pi$ (2 points)

(Idea VI)

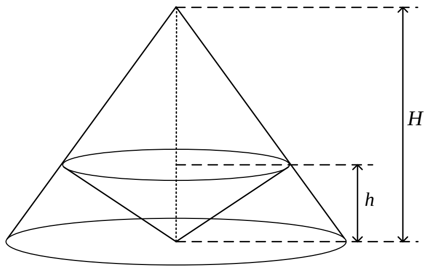
$7\sin\phi = 3\sin\theta \Rightarrow 7\cos\phi\frac{d\phi}{dt} = 3\cos\theta\frac{d\theta}{dt}$, when $\theta = \frac{\pi}{3} \Rightarrow \sin\phi = \frac{3\sqrt{3}}{14}$, $\cos\phi = \frac{13}{14}$

$\Rightarrow \frac{d\phi}{dt} = \frac{3\cos\theta}{7\cos\phi}\frac{d\theta}{dt} = \frac{3\cos\theta}{\sqrt{49 - 49\sin^2\phi}}\frac{d\theta}{dt} = \frac{3\cos\theta}{\sqrt{49 - 9\sin^2\theta}}\frac{d\theta}{dt}$ (2 points)

$\because \frac{x}{\sin(\pi - \phi - \theta)} = \frac{7}{\sin\theta} \Rightarrow x = 7\frac{\sin(\pi - \phi - \theta)}{\sin\theta} = 7\frac{\sin(\theta + \phi)}{\sin\theta}$ (1 point)

$\therefore \frac{dx}{dt} = 7\frac{\cos(\theta+\phi)\cdot(\frac{d\theta}{dt}+\frac{d\phi}{dt})\sin\theta-\cos\theta\sin(\theta+\phi)\frac{d\theta}{dt}}{\sin^2\theta}$ (3 points)
When $\theta = \frac{\pi}{3}$,
we have $\frac{dx}{dt} = 7\frac{\left[(\frac{1}{2})(\frac{13}{14})-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{3\sqrt{3}}{14}\right)\right]\cdot\left[\frac{\frac{3}{2}}{\sqrt{49-\frac{27}{4}}}+1\right]\frac{\sqrt{3}}{2}-\frac{1}{2}\left[\left(\frac{\sqrt{3}}{2}\right)(\frac{13}{14})+(\frac{1}{2})\left(\frac{3\sqrt{3}}{14}\right)\right]}{\frac{3}{4}}\cdot 200\pi$
$\Rightarrow \frac{dx}{dt} = 7\left[\left(\frac{4}{28}\right)\left(\frac{16}{13}\right)\frac{\sqrt{3}}{2} - \frac{16\sqrt{3}}{56}\right]\cdot\frac{800}{3}\pi = 7\cdot\frac{-9\cdot 16\cdot 800\sqrt{3}\pi}{3\cdot 56\cdot 13} = -\frac{4800\sqrt{3}}{13}\pi$ (2 points)
That is the velocity of the piston is $\frac{4800\sqrt{3}}{13}\pi$ inch/min. with the direction to the left. (1 point)

7. (10 points) A right circular cone is inscribed in a larger right circular cone so that its vertex is at the center of the base of the larger one. Denote the height of the large cone by H and the height of the small one by h. When the large cone is fixed, find h that maximizes the volume of the small cone and find out this maximum volume in terms of the volume of the large cone. (Hint: The volume of a right circular cone with height h and base radius r is $\frac{1}{3}\pi r^2 h$.)



Solution:

We denote the radii of the bases of the larger and smaller cones as R and r, respectively. Then we have the relation

$$\frac{H-h}{H} = \frac{r}{R}.$$

This gives us

$$r = \frac{R}{H}(H-h). \text{ (2 points)}$$

Hence the volume of the small cone is

$$\begin{aligned} V_{small} &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi h \cdot \frac{R^2}{H^2}(H-h)^2 \end{aligned}$$

for $0 \le h \le H$ (2 points). In order to compute the maximum of V_{small}, we compute

$$\frac{dV_{small}}{dh} = \frac{\pi R^2}{3H^2}(H-h)(H-3h). \text{ (2 points)}$$

By setting $\frac{dV_{small}}{dh} = 0$ we have $h = H$ or $h = H/3$ (2 points). Since

$$\begin{aligned} V(0) &= 0, \\ V(H) &= 0, \\ V(\frac{H}{3}) &= \frac{4}{81}\pi R^2 H > 0, \end{aligned}$$

we know that when $h = H/3$, the maximum of V_{small} is

$$V_{small} = \frac{4}{81}\pi R^2 H = \frac{4}{27}V_{large}, \text{ (2 points)}$$

where $V_{large} = \frac{1}{3}\pi R^2 H$ is the fixed volume of the larger cone.

註1: $h = \frac{H}{3}$ and $V_{small} = \frac{4}{27}V_{large}$ should both be answered. Not answering both of the two will cost you 2 points.

註2: Assuming $H = 2R$ will cost you 2 points. □

8. (18 points) Let $f(x) = \frac{\ln|x|}{x}$, $x \neq 0$. Answer the following questions by filling each blank below and give your reasons (including computations). Put **None** in the blank if the item asked does *not* exist.

(a) (3 points) Find all asymptote(s) of the curve $y = f(x)$.
Vertical asymptote(s):________________________.
Horizontal asymptote(s):________________________.
Slant saymptote(s):________________________.

(b) (4 points) $f(x)$ is increasing on the interval(s) ________________________.
$f(x)$ is decreasing on the interval(s) ________________________.

(c) (2 points) Find all local extreme values of $f(x)$.
Local maximum point(s): $(x, f(x)) =$ ________________________.
Local minimum point(s): $(x, f(x)) =$ ________________________.

(d) (4 points) $f(x)$ is concave upward on the interval(s) ________________________.
$f(x)$ is concave downward on the interval(s) ________________________.

(e) (2 points) List the inflection point(s) of the curve $y = f(x)$: $(x, f(x)) =$ ________________________.

(f) (3 points) Sketch the graph of f, and indicate all asymptotes, extreme values, and inflection points.

Solution:

(a) 1. Vertical asymptote:
Answer : $x = 0$ (y-axis). Correctness : 0.5 point ; Explanation : 0.5 point.

Solution:

$x > 0$: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(\frac{\ln(x)}{x}\right) = \frac{-\infty}{0} = -\infty$
(means $f(x) \to -\infty$ when $x \to 0^+$).

$x < 0$: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(-\frac{\ln(-x)}{-x}\right) = \frac{+\infty}{0} = +\infty$
(means $f(x) \to +\infty$ when $x \to 0^-$).

Note: We can't use L'Hospital's Rule to solve this question.

2. Horizontal asymptote:
Answer : $y = 0$ (x-axis). Correctness : 0.5 point ; Explanation : 0.5 point.

Solution:

$x > 0$: $\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \left(\frac{\ln(x)}{x}\right) = \lim_{x\to +\infty} \left(\frac{\frac{1}{x}}{1}\right) = \frac{0}{1} = 0$
($\frac{+\infty}{+\infty}$ type, use L'Hospital's Rule.)
(means $f(x) \to 0$ when $x \to +\infty$).

$x < 0$: $\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \left(-\frac{\ln(-x)}{-x}\right) = \lim_{x\to -\infty} \left(\frac{\frac{1}{x}}{1}\right) = \frac{0}{1} = 0$
($\frac{+\infty}{+\infty}$ type, use L'Hospital's Rule.)
(means $f(x) \to 0$ when $x \to -\infty$).

3. Slant asymptote:
Answer : *None*. Correctness : 0.5 point ; Explanation : 0.5 point.

Solution1:

If a slant asymptote exists, the slope of a slant asymptote can be expressed as $\lim_{x\to \pm\infty} \frac{f(x)}{x}$.

$\underline{x > 0}$: $\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty}\left(\frac{\ln(x)}{x^2}\right) = \lim_{x\to+\infty}\left(\frac{\frac{1}{x}}{2\cdot x}\right) = \frac{0}{+\infty} = 0$

($\frac{+\infty}{+\infty}$ type, use L'Hospital's Rule.)

(This result is in contradiction, therefore a slant asymptote doesn't exist).

$\underline{x < 0}$: $\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty}\left(\frac{\ln(-x)}{x^2}\right) = \lim_{x\to-\infty}\left(\frac{\frac{1}{x}}{2\cdot x}\right) = \frac{0}{-\infty} = 0$

($\frac{+\infty}{+\infty}$ type, use L'Hospital's Rule.)

(This result is in contradiction, therefore a slant asymptote doesn't exist).

Solution2:

A slant asymptote of $f(x)$ only occurs when $x \to \pm\infty$, but $\lim_{x\to\pm\infty} f(x) = 0$ from the above, therefore a slant asymptote doesn't exist.

(b) Answer : increasing interval:$(0, e) \cup (-e, 0)$, decreasing interval:$(e, +\infty) \cup (-\infty, -e)$.
Correctness : 1 point per question;
Determine $f'(x)$ (1 point) + Illustrate $f'(x) > 0$ and $f'(x) < 0$ (1 point).

Solution:

$\underline{x > 0}$: $f(x) = \frac{\ln(x)}{x}$, $f'(x) = \frac{1-\ln(x)}{x^2}$ (from the Quotient Rule).

$\underline{f'(x) > 0}$: $1 - \ln(x) > 0 \Rightarrow \ln(x) < 1 \Rightarrow 0 < x < e$
(increasing interval).

$\underline{f'(x) < 0}$: $1 - \ln(x) < 0 \Rightarrow \ln(x) > 1 \Rightarrow x > e$
(decreasing interval).

$\underline{x < 0}$: $f(x) = -\frac{\ln(-x)}{-x}$, $f'(x) = \frac{1-\ln(-x)}{x^2}$ (from the Quotient Rule).

$\underline{f'(x) > 0}$: $1 - \ln(-x) > 0 \Rightarrow \ln(-x) < 1 \Rightarrow -e < x < 0$
(increasing interval).

$\underline{f'(x) < 0}$: $1 - \ln(-x) < 0 \Rightarrow \ln(-x) > 1 \Rightarrow x < -e$
(decreasing interval).

(c) Answer : local maximum point:$(e, \frac{1}{e})$, local minimum point:$(-e, \frac{-1}{e})$.
Correctness : 0.5 point per question; Explanation : 1 point.

Solution:

Local maximum: Because $f(x)$ is increasing in $(0, e)$ and decreasing in $(e, +\infty)$, local maximum point occurs at $(e, f(e)) = (e, \frac{1}{e})$ $(f'(e) = 0)$.

Local minimum: Because $f(x)$ is decreasing in $(-\infty, -e)$ and increasing in $(-e, 0)$, local minimum point occurs at $(-e, f(-e)) = (-e, \frac{-1}{e})$ $(f'(-e) = 0)$.

(d) (4 points) $f(x)$ is concave upward on the interval(s) $(-e^{\frac{3}{2}}, 0) \bigcup (e^{\frac{3}{2}}, \infty)$.
$f(x)$ is concave downward on the interval(s) $(0, e^{\frac{3}{2}}) \bigcup (-\infty, -e^{\frac{3}{2}})$.

$$\begin{cases} f''(x) & = x > 0, \Rightarrow \frac{\frac{-1}{x}x^2 - (1-\ln x)2x}{x^4} = \frac{x(-3+2\ln x)}{x^4} \\ f''(x) & = x < 0, \Rightarrow \frac{\frac{-1}{x}x^2 - (1-\ln(-x))2x}{x^4} = \frac{x(-3+2\ln(-x))}{x^4} \end{cases} \tag{1}$$

$f''(x) > 0 \Rightarrow$ (concave upward)

$$\begin{cases} \frac{x(-3+2\ln x)}{x^4} > 0, x > 0 \Rightarrow -3 + 2\ln x > 0 \Rightarrow \ln x > \frac{3}{2} \Rightarrow x > e^{\frac{3}{2}} \Rightarrow \textit{choose } x > e^{\frac{3}{2}} \\ \frac{x(-3+2\ln(-x))}{x^4} > 0, x < 0 \Rightarrow -3 + 2\ln(-x) < 0 \Rightarrow \ln(-x) > \frac{3}{2} \Rightarrow x > -e^{\frac{3}{2}} \\ \Rightarrow \textit{choose } -e^{\frac{3}{2}} < x < 0 \end{cases} \tag{2}$$

$f''(x) < 0 \Rightarrow$ (concave downward)

$$\begin{cases} \frac{x(-3+2\ln x)}{x^4} > 0, x > 0 \Rightarrow -3+2\ln x > 0 \Rightarrow \ln x > \frac{3}{2} \Rightarrow x > e^{\frac{3}{2}} \\ \Rightarrow \textit{choose } 0 < x < e^{\frac{3}{2}} \\ \frac{x(-3+2\ln(-x))}{x^4} > 0, x < 0 \Rightarrow -3+2\ln(-x) < 0 \Rightarrow \ln(-x) > \frac{3}{2} \Rightarrow x > -e^{\frac{3}{2}} \\ \Rightarrow \textit{choose } x < -e^{\frac{3}{2}} \end{cases} \quad (3)$$

score: answer 0.5 point separately, right concept 2 points. (notify: if the graph of f lies above all of its tangents on an interval I, then called concave upward. ex: $f''(x) > 0 \Rightarrow$ concave upward)

if $\bigcup$ write $\bigcap$ lose 0.5 point

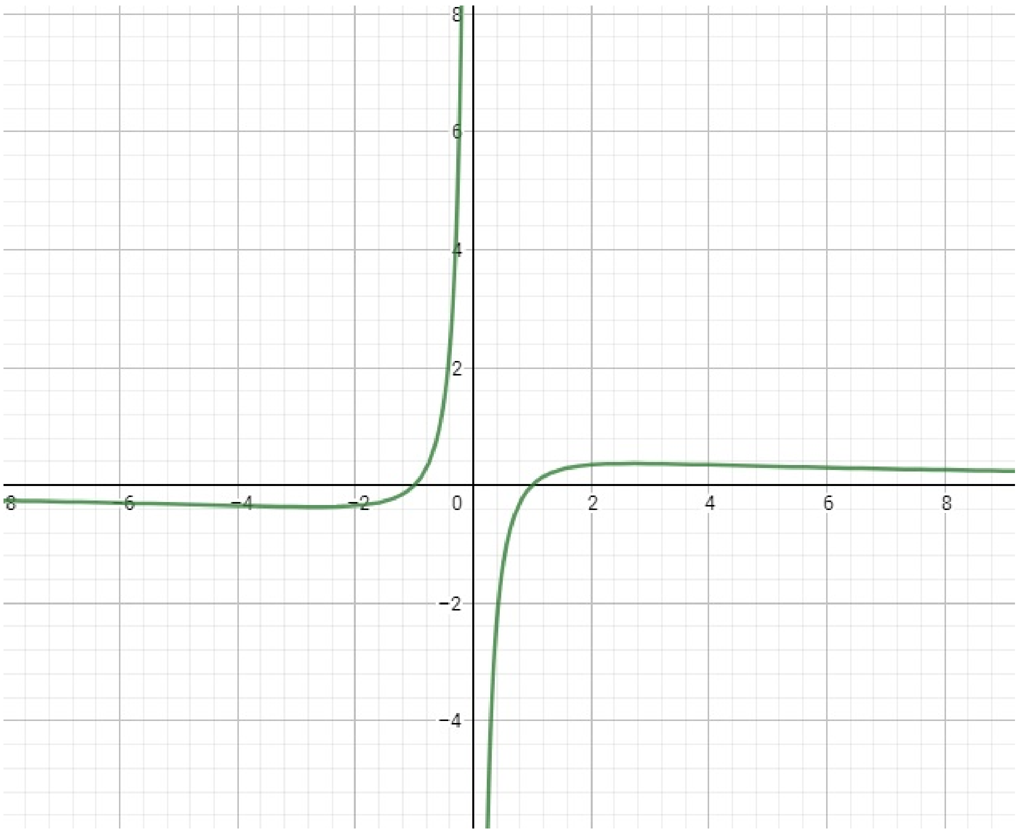
(e) (2 points) List the inflection point(s) of the curve $y = f(x) : (x, f(x)) =$ $\underline{(e^{\frac{3}{2}}, \frac{3}{2e^{\frac{3}{2}}}), (-e^{\frac{3}{2}}, -\frac{3}{2e^{\frac{3}{2}}})}$

$f''(x) = 0 \Rightarrow x = e^{\frac{3}{2}}$ and $x = -e^{\frac{3}{2}}$

$f''(e^{\frac{3}{2}}) = \frac{3}{2}e^{\frac{-3}{2}}$ and $f''(-e^{\frac{3}{2}}) = -\frac{3}{2}e^{\frac{-3}{2}}$

score: answer 0.5 points separately, right concept 1 points. (notify $f''(x) = 0, x = -e^{\frac{3}{2}}$ and $e^{\frac{-3}{2}}$)

(f) (3 points) Plot the f(x) and indicate asymptote, local point and infection point.



vertical asymptote 0.5point, horiaontal asymptote 0.5point, infection points 0.5point separately, local points 0.5point spearately.

only mark points(all points must be correct) get 1 point.

only draw image (image must be correct) get 1.5 points

9. (10 points) $f(x)$ is a differentiable function defined on $\mathbb{R}$. Let $g(x) = f(x) \cdot |f(x)|$.

(a) (3 points) Find the domain of $g'(x)$ and compute $g'(x)$. (Hint: To compute $g'(x_0)$ you may need to discuss the cases $f(x_0) > 0$, $f(x_0) < 0$, and $f(x_0) = 0$ seperately.)

(b) (4 points) Suppose that $f'(x) > 0$ on the interval (a, b). Show that $g(x)$ has at most one critical point on (a, b).

(c) (3 points) Suppose that $f'(x) > 0$ on the interval (a, b). Show that $g(x_1) < g(x_2)$ for all $a \le x_1 < x_2 \le b$.

Solution:

(a) For x_0 s.t. $f(x_0) > 0$, because $f(x)$ is continuous so that there is some open interval I containing x_0 s.t. $f(x) > 0$ on I.
Hence, $g(x) = f(x) \mid f(x) \mid = f^2(x)$ on I.
$g'(x) = 2f(x) \cdot f'(x)$ on I. $\quad g'(x_0) = 2f(x_0) \cdot f'(x_0)$.

For x_0 s.t. $f(x_0) < 0$, $\quad g(x) = -f^2(x)$.
$g'(x) = -2f(x) \cdot f'(x)$ on I. $\quad g'(x_0) = -2f(x_0) \cdot f'(x_0)$.

For x_0 s.t. $f(x_0) = 0$, $\quad g(x_0) = 0$.
$g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x)|f(x)|}{x - x_0} = \lim_{x \to x_0} \mid f(x) \mid \frac{f(x) - f(x_0)}{x - x_0}$
$= \mid f(x_0) \mid f'(x_0) = 0$
$\mid f(x_0) \mid \quad \because f(x)$ is continous.
$f'(x_0) \quad \because f'(x_0)$ is differentiable.
Therefore, $g'(x)$ is defined on $\mathbb{R}$, and $g'(x) = 2 \mid f(x) \mid f'(x)$.

Other Sol.
$g(x) = f(x)(f^2(x))^{\frac{1}{2}}$ when $f(x) = 0$
$g'(x) = f'(x) \cdot \mid f(x) \mid + f(x) \frac{2f(x) \cdot f'(x)}{2|f(x)|} = 2 \mid f(x) \mid f'(x) \quad$ when $f(x) \ne 0$

(b) Because g is differentiable on (a,b). Hence the critical numbers of g are numbers on $g' = 0$.
Suppose that $f'(x) > 0$ on (a, b).
Then on (a, b),
$g'(x) = 2 \mid f(x) \mid \cdot f'(x) = 0 \iff f(x) = 0 \quad (\because f'(x) > 0 \text{ on} (a, b))$.

Assume that g has more than one critical point on (a, b).
Then $\exists c_1, c_2 \in (a, b)$ s.t. $g'(c_1) = g'(c_2) = 0$.
$g'(c_1) = 0 \to f(c_1) = 0$
$g'(c_2) = 0 \to f(c_2) = 0$
$\because f$ is continuous on $[c_1, c_2]$, f is also differentiable on (c_1, c_2), and $f(c_1) = f(c_2)$
$\therefore$ By Rolle's Theorem, $\exists$ some $c \in (c_1, c_2) \subset (a, b) \Rightarrow$, which is contradiction the above assumption.

Other sol.
$\because f'(x) > 0$ on (a, b).
$\therefore f(x)$ is strictly increasing on (a, b) (, and it is 1-to-1).
Therefore, there is at most one point on (a, b) s.t. $f(c) = 0$.

(c) Case 1: g has no critical point on (a, b). (1 point)
The $g'(x) = 2 \mid f(x) \mid f'(x) > 0$ on (a, b).
By the Incresing Test, $g(x_1) < g(x_2)$, $\quad \forall x_1 < x_2, \quad x_1, x_2 \in (a, b)$.

Case 2: g has one critical point c on (a, b). (1 point)
Discuss $g(x_1) < g(c) < g(x_2)$ at different boundaries. (1 point)