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Lecture notes on

Introduction to Fluid Mechanics

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Contents

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CHAPTER 1 INTRODUCTION

(I) Fluid Mechanics

As we have learned from high school physics or chemistry, there are three phases of substances, solids, liquids and gases, depending on different values of pressure and temperature in the surroundings. The density of the solid and liquid are of the same order, and are in general of three order greater than that of the gas. However, the response of the liquid under the action of an applied shear stress is similar to that of the gas, and is fundamentally different from that of the solid. Therefore, the liquid and the gas are grouped together as a new category, the fluid.

Consider the case when a piece of substance (called a body) is subject to the action of an applied shear stress, as shown in figure 1-1. Unless the body is perfectly rigid, the body may deform due to the applied shear stress. If the substance is a solid, the deformation angle, γ, is proportional to the shear stress, τ. We may write

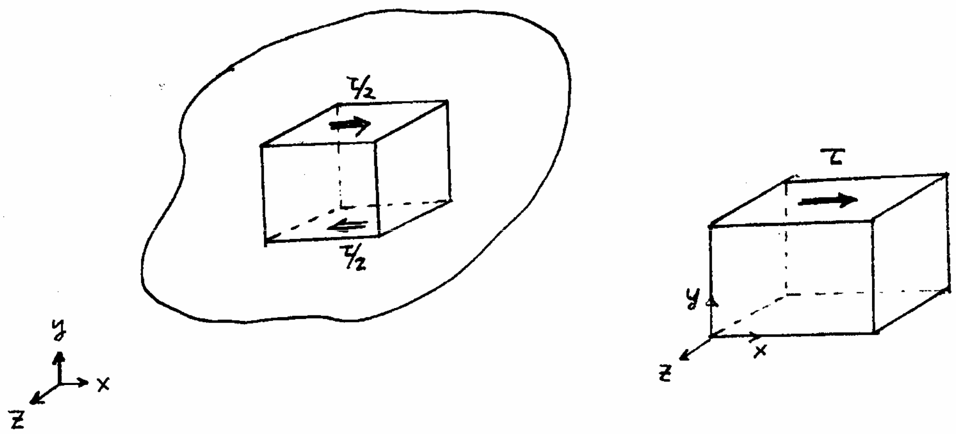
$$\tau = G\gamma \tag{1-1}$$

if the stress is below the proportional limit, where G is the shear modulus of elasticity which can be determined from experiment. The deformation angle, γ, is determined and stays at a constant value as long as τ remains constant (see figure 1-1(b)). However, if the substance is a fluid, it may deform continuously (i.e., flow, see figure 1-1(c)) under the action of the shear even if τ is kept constant and at a very small value. The deformation angle, γ, thus increases continuously with time as long as τ is applied. In such a case, we cannot relate τ to γ. Instead, we propose that

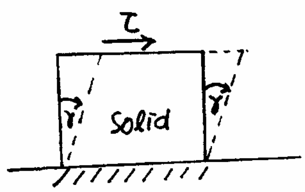
$$\tau \propto \dot{\gamma}, \tag{1-2}$$

where $\dot{\gamma}$ is the time rate of change of γ, the rate of deformation.

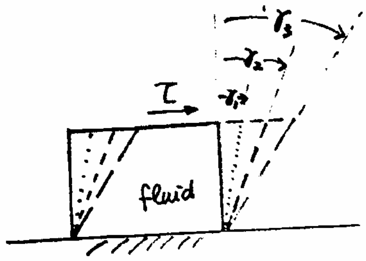
Therefore, a fluid is defined as a substance that deforms continuously when subjected to a shear stress, no matter how small that shear stress may be.



(a) Sketch for a piece of substance subject to a shear stress.



(b) Response of a solid under shear. γ is fixed for a given τ.



(c) Response of a fluid under shear. γ increases continuously (from 0 to γ_1 to γ_2 to γ_3 and etc.) as time increases for a given τ.

Figure 1-1

Fluid mechanics is a discipline for the study of the response of the fluid under the action of applied forces. The response of the fluid includes its translation, rotation and deformation. The applied forces are generalized forces associated with all kinds of the driving mechanisms, including all kinds of` forces such as that associated with the above shear stress, the buoyancy force resulting from an applied temperature contrast, the electric and magnetic forces if the fluid is a conductor, and other driving mechanisms.

(II) Microscopic versus macroscopic approach

Any substance, including the fluid, is composed of molecules, which is a basic concept in elementary physics or chemistry. However, many useful properties of the fluid, including the density, the velocity, the pressure and the temperature, are macroscopic properties. For example, if we measure the velocity at a location (denoted by a ‘point’ in space) in an air stream using a Laser Doppler Anemometry (LDA), we get a value at a ‘point’. The ‘point’ actually has a finite volume, which is approximately the measuring volume of the LDA, an ellipsoid with characteristic lengths of order of several hundreds micron (1 micron = 1 μm = $10^{-6}m$). We may have even a larger ‘point’ if the hot-wire anemometry or the Pitot tube is employed instead of the LDA. On the other hand, the mean free path, λ, of the air at standard condition is about 0.065 μm. The molecules move randomly in space with a velocity of order of the sound speed, and collide with other molecules in space. The mean free path is defined as the average distance traveled by a molecule before two succeeding collisions, and is in general of one order greater than the molecular spacing. Thus a measuring “point” is actually a very large space for a molecule to experience. Let L_m be the characteristic length scale of the measuring volume (i.e., the ‘measuring point’), we have

$$\frac{L_m}{\lambda} \sim \frac{650}{0.065} \sim 10^4, \qquad \text{or} \qquad \left(\frac{L_m}{\lambda}\right)^3 \sim 10^{12}.$$

There are about 10^{12} molecules within the above measuring ‘point’ if we take $L_m = 650\mu\text{m}$. The velocity of the fluid at a measuring ‘point’ is actually an average value of the velocity of the molecules contained in the ‘point’.

If we study the fluid mechanics problem from the molecular point of view (microscopic approach), we had to study the motion of many molecules governed by the law of dynamics (three degrees of freedom for a molecule if it is treated as a point mass). Statistical averaging processes are then carried out for obtaining various macroscopic properties based on the results of the molecular motion. Although a nonlinear integral-differential equation for the time development of the velocity distribution function, called the Boltzmann equation, can be derived under certain restrictions, the method of solution for the equation is very complicated. Here the concept of velocity distribution function is introduced to avoid the detailed evaluation of the motion of every individual molecule in the Boltzmann equation. One of the good references for the theories, the analyses, and the numerical simulations of the microscopic approach is the book "Molecular gas dynamics and the direct numerical simulation of gas flows" by G. A. Bird (Clarendon Press, Oxford, 1994).

Whether the microscopic (molecular) approach is necessary for studying fluid mechanics problems depends on the relative magnitude of the characteristic length scale of the problem (L) to the characteristic length scale of the molecules (S). For gas flow, the characteristic length scale of the molecules is the mean free path, λ. The Knudsen number, Kn, is defined as

$$Kn = \lambda / L. \tag{1-3}$$

We may ignore the molecular details if Kn is sufficiently less than unity. In practice, the macroscopic approach discussed below may be adopted if Kn is less than 0.1 (see Bird's book).

For liquid flow, the molecules are crowded together and constrained to move freely, which is different from that in gas flow. The concept of the mean free path does not apply for liquid, and thus the appropriate characteristic length scale of the molecules for liquid flow is the molecular spacing, S_l, which is in general of one order less than that for gas, S_g.

The macroscopic approach is a field approach, or a continuum approach. All of the macroscopic properties are considered as field variables, which are continuous functions in space and time, such that their variations can be studied via differential and integral calculus. A point in the field is called a fluid point, which should contain a sufficiently large number of molecules.

Consider a small volume of fluid ΔV containing a large number of molecules. Let Δm be the mass of an individual molecule. The density of the fluid under the continuum approach is defined as

$$\rho = \lim_{\Delta V \to \varepsilon} \left(\frac{\Sigma \Delta m}{\Delta V} \right), \tag{1-4}$$

where ε is a volume which is sufficiently small that $\varepsilon^{1/3}$ is small compared with the smallest significant length scale in the flow field and is sufficiently large that it contains a large number of molecules. The summation in (1-4) is taken over all the molecules contained within ΔV. Thus the continuum approach is valid if the relation

$$S^3 << \varepsilon << L^3 \tag{1-5}$$

is satisfied. Here $S = \lambda$ for gas flow, but $S = S_l$ for liquid flow. Figure 1-2 adopted from Batchelor's book (1967) is employed here for illustrating the above idea. If the sensitive volume for measurement of the density in the experiment is too small, the number of molecules contained within the volume fluctuates rapidly with time as the molecules enter or leaves the volume with a large characteristic speed, which is of order of the sound speed. The sensitive volume corresponding to the region with 'constant' value is consistent with the condition in (1-5), and the density for a continuum is thus defined. For sufficiently large sensitive volume, the measured density varies slowly from the constant value, in response to the spatial variation with the density. Other macroscopic variables such as the velocity, the temperature, and the pressure can be defined in a way similar to that in (1-4). For example, the velocity **u** is defined as

$$\rho \mathbf{u} = \lim_{\Delta V \to 0} \left(\frac{\sum \mathbf{v} \Delta m}{\Delta V} \right), \tag{1-6}$$

with **v** the velocity of the molecule. For most fluid mechanics problems, the condition for the validity of continuum approach, (1-5), is satisfied, and the scope of this course is limited to continuum fluid mechanics.

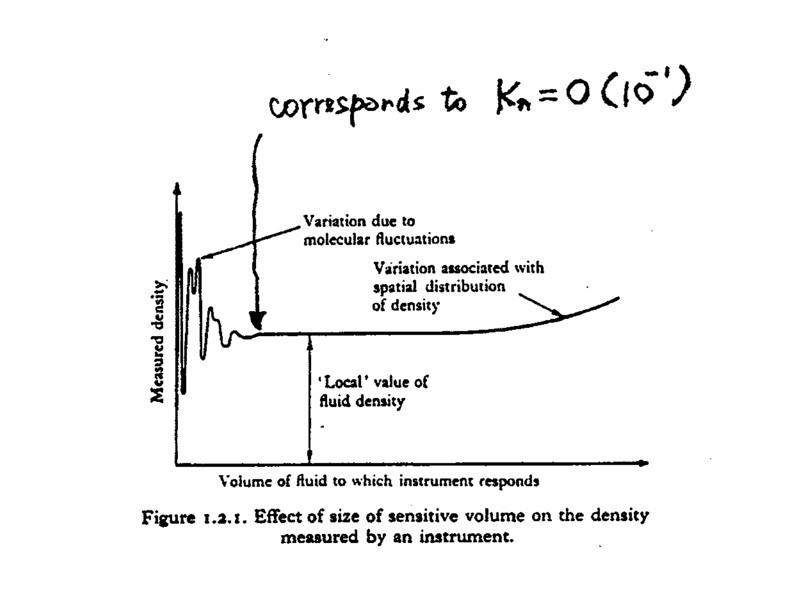


Figure 1-2: A figure for illustrating the condition for validity of continuum approach. This figure is adopted from Batchelor (1967)

Continuum approach is not valid when $Kn \geq O(1)$ for gas flow, which occurs when either λ is large or L is small. Examples for large λ can be found from the problem of vacuum pumping, and the flying of vehicle at the edge of the outer atmosphere. Examples for small L include the flying of the slider carrying the magnetic head above the disk in the hard disk drive of the computer system, and the study of the structure of a shock wave. For liquid flow, the continuum approach is valid even when the smallest characteristic length scale of the problem reaches the order of 10 nm. However, the problem for such small scale is usually complicated by other issues, such as the surface properties of the boundary, the electrostatic properties, and some intermolecular and surface forces, which are not considered in classical fluid mechanics problem. A good reference for the last issue is the book "Intermolecular and surface forces", 2nd ed., by Jacob N. Israelachvili, Academic Press, 1992.

(III) Eulerian and Lagrangian descriptions

The fluid motion under the continuum approach is governed by the conservation laws of mass, momentum and energy. In order to derive such conservation equations, a reference frame is required for the description of the fluid motion. There are two choices of reference frames, the Eulerian coordinates and the Lagrangian coordinates, in continuum fluid mechanics.

The basic element of the fluid under the continuum approach is a fluid point (or called a fluid particle), which is a material point composed of a large number of molecules. The fluid particles travel in the field as the fluid flows. Figure 1-3 shows the trajectories of several fluid particles which were at locations $\mathbf{r}_{01}, \mathbf{r}_{02}, \mathbf{r}_{03}, \ldots\ldots$ at time t = 0. These particles move to locations $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots\ldots$ at a later time, t. The motion of the fluid particles can be described by

$$\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t) \tag{1-7}$$

such that $\mathbf{r} = \mathbf{r}_0$ at t = 0. The function $\mathbf{r}(\mathbf{r}_0, t)$ may be regarded as a transformation or a mapping function between the coordinates $\mathbf{r}$ or (x, y, z) and $\mathbf{r}_0$ or (x_0, y_0, z_0). Inverse mapping

$$\mathbf{r}_0 = \mathbf{r}_0(\mathbf{r}, t) \tag{1-8}$$

exists if the mapping is one to one. The Jacobian for the transformation is

$$\mathbf{J} \equiv \begin{vmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} & \dfrac{\partial x}{\partial z_0} \\ \dfrac{\partial y}{\partial x_0} & \dfrac{\partial y}{\partial y_0} & \dfrac{\partial y}{\partial z_0} \\ \dfrac{\partial z}{\partial x_0} & \dfrac{\partial z}{\partial y_0} & \dfrac{\partial z}{\partial z_0} \end{vmatrix} = \det\left(\frac{\partial x_i}{\partial x_{0j}}\right) = \frac{\partial(x, y, z)}{\partial(x_{0,} y_{0,} z_0)}\,. \tag{1-9}$$

Physical meaning for the Jacobian is the ratio of an infinitesimal volume after (δV) the transformation to that before (δV_0) the transformation, i.e.,

$$\mathbf{J} = \frac{\delta V}{\delta V_0}\,. \tag{1-10}$$

By term-by-term differentiation, one can find

$$\frac{d\mathbf{J}}{dt} = \mathbf{J}\nabla \cdot \mathbf{u}, \tag{1-11}$$

where $\mathbf{u}$ is the local fluid velocity (defined later in (1-15)). Thus the physical meaning for $\nabla \cdot \mathbf{u}$ can be interpreted as

$$\nabla \cdot \mathbf{u} = \frac{1}{\mathbf{J}}\frac{d\mathbf{J}}{dt} = \frac{\delta V_0}{\delta V}\frac{d}{dt}\left(\frac{\delta V}{\delta V_0}\right) = \frac{\frac{d}{dt}\delta V}{\delta V}, \tag{1-12}$$

which says that $\nabla \cdot \mathbf{u}$ is the relative rate of change of volume of an infinitesimal volume element following the fluid motion. For incompressible fluid, $\delta V = \delta V_0$, J = 1, and thus $\nabla \cdot \mathbf{u}$ = 0.

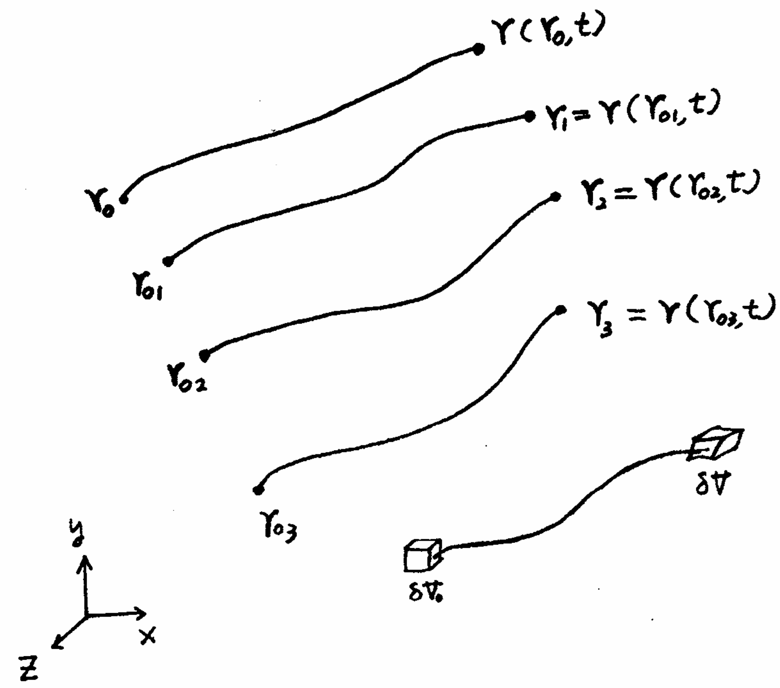


Figure 1-3: Trajectories of fluid particles in space. Infinitesimal material volume changes its shape and size from δV_0 to δV as it travels in the flow.

The variables **r** and *t*, or (x, y, z, *t*), are called the spatial variables or Eulerian variables; and $\mathbf{r_0}$ and *t*, or (x_0, y_0, z_0, t), are called the material variables or Lagrangian variables. The Eulerian and the Lagrangian terminology was introduced by Dirichlet in honor of the famous scientists Leonard Euler (1707-1783) and Joseph Louis Lagrange (1736-1813). However, Dirichlet made a mistake that Euler was actually the first to introduce the Lagrangian variables, and Lagrange was the first to employ the concept of Eulerian variables (see Ligget's book, p.40).

The macroscopic flow properties, such as the velocity, the pressure and the temperature, can be expressed in terms of both the Lagrangian and Eulerian variables. Any macroscopic property, F, may be prescribed by F($\mathbf{r}$, t) or F($\mathbf{r}_0$, t). The expression F($\mathbf{r}$, t) gives the value of F seen by an observer at a fixed point, $\mathbf{r}$, in space; the observer sees different fluid particles at different times, and thus the values of F observed are corresponding to different fluid particles. On the other hand, the expression F($\mathbf{r}_0$, t) gives the value of F seen by an observer riding on fluid particle which was at $\mathbf{r}_0$ at t = 0; the observer always sees the value of F for the same fluid particle. In most of the experiments, we place a sensor at a fixed location in the flow field, and record the macroscopic variables, such as the velocity, at different times. Thus such kind of experiment is carried out in an Eulerian frame. Both expressions for F in terms of the Eulerian and Lagrangian variables can be related by the mapping function and the inverse mapping function as

$$\mathrm{F}(\mathbf{r}, t) = \mathrm{F}(\mathbf{r}\,(\mathbf{r}_0, t), t) = \mathrm{F}(\mathbf{r}_0, t), \tag{1-13}$$

and

$$\mathrm{F}(\mathbf{r}_0, t) = \mathrm{F}(\mathbf{r}_0(\mathbf{r}, t), t) = \mathrm{F}(\mathbf{r}, t). \tag{1-14}$$

The mapping function can be evaluated using the definition of the velocity of a fluid particle. Equation (1-6) is an equation relating the fluid velocity of a fluid particle to the velocities of the neighboring molecules, which is not useful in the continuum approach. The velocity of a fluid particle in the flow, $\mathbf{u}(\mathbf{r}, t)$, is defined similarly to that of a solid particle in rigid body dynamics, i.e.,

$$\mathbf{u}(\mathbf{r}, t) \equiv \frac{d\mathbf{r}}{dt} \equiv \frac{\partial}{\partial t}\mathbf{r}\,(\mathbf{r}_0, t)\big|_{\mathrm{r}_0 = \text{fixed}}, \tag{1-15}$$

which is the time rate of change of the position vector of a chosen material point at location $\mathbf{r}$ and at time t. If the velocity field, $\mathbf{u}(\mathbf{r}, t)$, is given in spatial variables (i.e., the Eulerian variables), one can determine the mapping function $\mathbf{r}$ = $\mathbf{r}\,(\mathbf{r}_0, t)$ by solving

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}, t), \tag{1-16}$$

with initial condition $\mathbf{r} = \mathbf{r}_0$ at t = 0. The function F expressed in both the Eulerian and Lagrangian coordinates in (1-13) and (1-14) can then be related.

There are two kinds of time derivatives for F one can consider,

$$\frac{\partial \mathrm{F}}{\partial t} \equiv \frac{\partial}{\partial t} \mathrm{F}(\mathbf{r}, t)\,|_{\mathrm{r}\,=\,\mathrm{fixed}} \tag{1-17}$$

gives the rate of change of F seen by an observer at a fixed position **r**, and

$$\frac{d\mathrm{F}}{dt} \equiv \frac{\partial}{\partial t} \mathrm{F}(\mathbf{r}_{\mathbf{0}}, t)\,|_{\mathrm{r}_0\,=\,\mathrm{fixed}} \tag{1-18}$$

gives the rate of change of F seen by an observer following the motion of a fluid particle (i.e., $\mathbf{r}_0$ = fixed). The latter is usually called the material derivative. These two time derivatives for F are related by using the mapping function, the chain rule and the definition of velocity, as follows.

$$\frac{d\mathrm{F}}{dt} \equiv \frac{\partial}{\partial t}\left[\mathrm{F}(\mathbf{r}_0, t)\right]_{\mathrm{r}_0\,=\,\mathrm{fixed}} \qquad \text{(definition of the material derivative)}$$

$$= \frac{\partial}{\partial t}\left[\mathrm{F}(\mathbf{r}, t)\right]_{\mathrm{r}_0\,=\,\mathrm{fixed}} \qquad \text{(using the mapping function)}$$

$$= \frac{\partial \mathrm{F}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x}\frac{\partial x}{\partial t}\,|_{\mathrm{r}_0\,=\,\mathrm{fixed}} + \frac{\partial \mathrm{F}}{\partial y}\frac{\partial y}{\partial t}\,|_{\mathrm{r}_0\,=\,\mathrm{fixed}} + \frac{\partial \mathrm{F}}{\partial z}\frac{\partial z}{\partial t}\,|_{\mathrm{r}_0\,=\,\mathrm{fixed}} \qquad \text{(chain rule)}$$

$$= \frac{\partial \mathrm{F}}{\partial t} + \mathbf{u}(\mathbf{r}, t)\cdot\nabla \mathrm{F}, \tag{1-19}$$

The material time derivative (the time derivative in the Lagrangian frame), dF/dt, equals to the time derivative in the Eulerian frame, ∂F/∂t, plus an addition term, $\mathbf{u}(\mathbf{r}, t)\cdot\nabla \mathrm{F}$. The last term in (1-19) is called the convection term, which is associated with the net transport of F to the location of interest by the flow. If we take F = **u**, we have

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathrm{u}}{\partial t} + \mathbf{u}(\mathbf{r}, t)\cdot\nabla \mathbf{u}, \tag{1-20}$$

which is called the Euler’s acceleration formula. The acceleration of the fluid particle, d**u**/dt, includes two parts in the Eulerian frame, the temporal term, ∂**u**/∂t, and the convective term, $\mathbf{u}\cdot\nabla\mathbf{u}$.

A number of fluid particles in the field can be grouped together and identified as a fluid element. There are line elements (called material lines), surface elements (called material surfaces) and volume elements (called material volume) in the study of continuum fluid mechanics. Figure 1-4 shows that both the shape and the size of a volume element at time *t* are changed in general from those at its initial position. However, the volume element in figure 1-4 is a constant mass (or controlled mass) system, which is employed for studying the Newton's second law (conservation of momentum) and the first law of thermodynamics (conservation of energy). The constant mass system is related to the Lagrangian description, and thus the Lagrangian variables will be employed for deriving the governing equations. However, the problem can be solved more easily in general in the Eulerian frame, and thus the Eulerian variables will be employed for solving the equations. Note also that most of the experiments are carried out in the Eulerian frame.

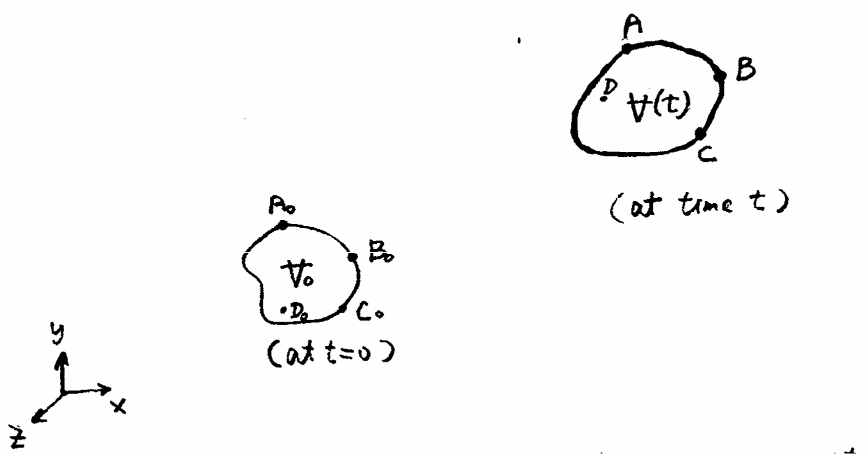


Figure 1-4: A material volume changes its shape and size as it moves in the fluid, but always contains the same fluid particles.

(IV) Calculation of the mapping function $\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t)$

If the velocity field, $\mathbf{u}(\mathbf{r}, t)$, is given in Eulerian frame, we can evaluate the mapping function using (1-15), i.e., to solve

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}, t) \tag{1-21}$$

subject to the initial condition

$$\mathbf{r} = \mathbf{r}_0 \quad \text{at} \quad t = 0. \tag{1-22}$$

The result of (1-21) and (1-22) depicts a pathline. An example is given as follows. The velocity field,

$$\mathbf{u} \equiv u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = ky\mathbf{i}\,, \tag{1-23}$$

is given in a Cartesian coordinates, with (**i, j, k**) the unit vectors along the corresponding axes, and *k* a given constant. Equation (1-21) implies

$$\frac{dx}{dt} = u = ky\,, \qquad \text{with} \qquad x = x_0 \qquad \text{at} \qquad t = 0\,, \tag{1-24a}$$

$$\frac{dy}{dt} = v = 0\,, \qquad \text{with} \qquad y = y_0 \qquad \text{at} \qquad t = 0\,, \tag{1-24b}$$

$$\frac{dw}{dt} = w = 0 \qquad \text{with} \qquad z = z_0 \qquad \text{at} \qquad t = 0, \tag{1-24c}$$

where (x, y, z) and (x_0, y_0, z_0) are the components of $\mathbf{r}$ and $\mathbf{r}_0$, respectively. The solutions of (1-24a)-(1-24c) are

$$x = ky_0 t + x_0\,, \tag{1-25a}$$

$$y = y_0\,, \tag{1-25b}$$

$$z = z_0\,, \tag{1-25c}$$

which are also the mapping functions. The inverse mapping functions are obtained by expressing (x_0, y_0, z_0) in terms of (x, y, z) using (1-25a) – (1-25c). They are

$$x_0 = x - kyt\,, \tag{1-26a}$$

$$y_0 = y\,, \tag{1-26b}$$

$$z_0 = z\,. \tag{1-26c}$$

The properties between the Lagrangian and the Eulerian coordinates can be related using (1-13) or (1-14) once we know the mapping or the inverse mapping functions.

(V) Reynolds transport theorem

We have related any flow property, F, and its rate of change with respect to time between the Lagrangian and the Eulerian coordinates in (1-13) or (1-14) and (1-19), respectively. In order to apply the Newton's second law and the first law of thermodynamics to derive the governing equations for fluid mechanics, we also need to consider the time rate of change of total F within a material volume, $\frac{d}{dt}\iiint\limits_{V(t)} \mathrm{F}(\mathbf{r},t)dV$, following the fluid motion. Here *V(t)* is the material volume shown in figure (1-4), which is a fixed mass system although its magnitude, its shape, and its position change with time in general. As *V(t)* varies with time, the order of differentiation and integration cannot be interchanged. Hence, we first transform the domain of integration into V_0 (recall the definition for d/dt in (1-18)), and then interchange the order of differentiation and integration. The procedures are as follows.

$$\frac{d}{dt}\iiint\limits_{V(t)} \mathrm{F}(\mathbf{r},t)dV = \frac{d}{dt}\iiint\limits_{V_0} \mathrm{F}(\mathbf{r}(\mathbf{r}_0,t),t)\mathbf{J}dV_0 \quad \text{(using (1-10) and (1-13))}$$

$$= \iiint\limits_{V_0} \frac{d}{dt}\left(\mathrm{F}(\mathbf{r}_0,t)\mathbf{J}\right)dV_0 \quad \text{(since } V_0 \text{ is independent of } t\text{)}$$

$$= \iiint\limits_{V_0} \left(\frac{d\mathrm{F}}{dt}\mathbf{J} + \mathrm{F}\frac{d\mathbf{J}}{dt}\right)dV_0 \quad \text{(recall } \mathrm{F} = \mathrm{F}(\mathbf{r}_0,t)\text{)}$$

$$= \iiint\limits_{V_0} \left(\frac{d\mathrm{F}}{dt} + \mathrm{F}\nabla\cdot\mathbf{u}\right)\mathbf{J}dV_0. \quad \text{(using (1-11))}$$

With the application of (1-10) and (1-14), we may transform the domain of integration back to *V(t)*, and have

$$\frac{d}{dt}\iiint\limits_{V(t)} \mathrm{F}(\mathbf{r},t)dV = \iiint\limits_{V(t)} \left(\frac{d\mathrm{F}}{dt} + \mathrm{F}\nabla\cdot\mathbf{u}\right)dV = \iiint\limits_{V(t)} \left(\frac{\partial\mathrm{F}}{\partial t} + \nabla\cdot(\mathrm{F}\mathbf{u})\right)dV$$

$$= \iiint\limits_{V(t)} \frac{\partial\mathrm{F}}{\partial t}dV + \iint\limits_{S(t)} \mathbf{n}\cdot\mathbf{u}\mathrm{F}dS, \quad (1\text{-}27)$$

where (1-19) and the divergence theorem have been employed for deriving (1-27). Here *S(t)* is the surface enclosing *V(t)*, and **n** is the unit outward normal vector of *S(t)*. All the three forms expressed in (1-27) may be employed, and are called the Reynolds Transport theorem. The above derivation is according to the lecture notes of Professor T. S. Lundgren of the University of Minnesota. A more intuitive derivation of (1-27) can be found from Ligget's book (p.3-7).

If *V(t)* is a control volume (same definition as in the undergraduate fluid mechanics) fixed in space at time *t*, (1-27) states that the instantaneous time rate of change of total F inside *V(t)* seen by an observer moving with the fluid (i.e., in Lagrangian frame) is expressed in two terms in the Eulerian frame, as shown on the right hand side of the equation. One is associated with the temporal change of F within the control volume, and the other is associated with the net influx of F into the control volume.

(VI) Applications of fluid mechanics

As we live in air and cannot survive without water, it is no doubt for the importance of fluid mechanics. In fact, there always exist many interesting phenomenon in our daily live. It is suggested that the students should try to give some examples and provide discussion.

The rapid development of fluid mechanics in the last twentieth century is motivated mainly by two topics, the aeronautical and aerospace engineering, and the geophysical fluid dynamics. Recently, some interesting fluid mechanics problems in micro-scale appear in the microelectromechanical systems (MEMS). Some applications of these three topics can be found from the associated powerpoint file of this chapter, which is available in the IAM300 server. It is hoped that we can find some new areas, and the research in fluid mechanics grows continuously in this century.

Homework 1

(1) Given the velocity field

$$u = \frac{x}{1+t}, \qquad v = -\frac{y}{1+t}, \qquad w = 0,$$

which satisfies $\nabla \cdot \mathbf{u} = 0$.

(a) Calculate the mapping function, $\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t)$, and the inverse mapping function, $\mathbf{r}_0 = \mathbf{r}_0(\mathbf{r}, t)$.

(b) Calculate the Jacobian, J, and show that it is unity.

(c) Calculate the acceleration in two ways, from Euler's acceleration formula and from the material derivative of u.

(2) Show that $\dfrac{d\mathbf{J}}{dt} = \mathbf{J}\nabla \cdot \mathbf{u}$.

(3) The density of the fluid in a pail does not change at any point fixed with respect to the pail, and at time t = 0 the density distribution is given by $\rho = \rho_0 - \beta z_0$, in which ρ_0 and β are constants, and z_0 increases in the direction of the vertical. The coordinates x, y, and z are fixed in space. The pail moves with a vertical velocity w = - gt, g being the gravitational acceleration. Calculate, at any t > 0, the quantities $\partial\rho/\partial t$, $\partial\rho/\partial z$ and $d\rho/dt$. (From Yih's book, Problem 1 of chapter 1)

CHAPTER 2 PHYSICAL AND MATHEMATICAL FORMULATIONS OF FLUID MECHANICS

(I) Conservation of mass --- Continuity equation

The mass of material inside the material volume, *V*(*t*), in Figure 1-4 can be expressed as

$$m = \oiiint_{V(t)} \rho(\mathbf{r},t)dV \ , \tag{2-1}$$

where $\rho(\mathbf{r},t)$ is the density. Conservation of mass implies that m = constant, or

$$\frac{dm}{dt} = 0 \,. \tag{2-2}$$

On substituting (2-1) into (2-2), and applying the Reynolds transport theorem, (1-27), we have

$$\oiiint_{V(t)} \left(\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} \right) dV = 0 \,. \tag{2-3}$$

Since *V(t)* is arbitrary, it is required from (2-3) that

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0, \tag{2-4}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{2-5}$$

if (1-19) is employed. Equation (2-4) or (2-5) expresses the local conservation of mass in Eulerian frame, and is called the continuity equation in the literature.

If we replace F(**r**, t) in (1-27) by ρ(**r**, t) F(**r**, t), we have

$$\frac{d}{dt}\oiiint_{V(t)}\rho \mathrm{F} dV = \oiiint_{V(t)}\left[\frac{d(\rho \mathrm{F})}{dt} + (\rho \mathrm{F})\nabla\cdot \mathrm{u}\right]dV = \oiiint_{V(t)}\left[\rho\frac{d\mathrm{F}}{dt} + \mathrm{F}\frac{d\rho}{dt} + (\mathrm{F}\rho)\nabla\cdot \mathbf{u}\right]dV$$

The sum of the last two terms in the integral is zero according to (2-4), the continuity equation. Thus

$$\frac{d}{dt}\oiiint_{V(t)}\rho \mathrm{F} dV = \oiiint_{V(t)}\rho\frac{d\mathrm{F}}{dt}dV\ , \tag{2-6}$$

which is another form of the Reynolds transport theorem, and will be employed later in this chapter.

(II) Conservation of momentum

The equation for the conservation or momentum is a generalization of the Newton's second law to continuum, and is simply called the momentum equation. Consider again the material volume, *V(t)*, in Figure (1-4). The linear momentum of material in *V(t)* is $\oiiint_{V(t)}\rho(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)dV$. We postulate

$$\frac{d}{dt}\oiiint_{V(t)}\rho\mathbf{u}dV = \oiiint_{V(t)}\rho\mathbf{f}dV + \oiint_{S(t)}\mathbf{t}dS\ , \tag{2-7}$$

which express that the rate of change of linear momentum within *V(t)* equals the total force acting on it according to Newton's second law. Here $\mathbf{f} = \mathbf{f}(\mathbf{r},t)$ is the force per unit mass of material, called the body force; and $\mathbf{t} = \mathbf{t}(\mathbf{r},t;\mathbf{n})$ is the traction force per unit area, called the surface force. Examples for the body force are the gravitational force and the electromagnetic force. The traction force is the force that the material on the outside of *V(t)* exerts on the fluid inside the volume, which depends on the position, **r**, the time, t, and the orientation of the surface element. Here **n** is the local unit normal of *S*(*t*).

It will be shown that **t** is a linear function of **n**, i.e.,

$$\mathbf{t} = \mathbf{n}\cdot\mathbf{T}(\mathbf{r},t)\ , \tag{2-8}$$

where **T(r,** ***t*****)** is called the stress tensor, which is a second order tensor. In component form, we may write

$$t_i = n_j T_{ji}, \tag{2-9}$$

where t_i, n_j and T_{ji} (with i, j = 1, 2, 3) are the component of **t**, **n** and **T**, respectively. Equation (2-8) or (2-9) is called the Cauchy stress principle, which will be derived later in this section. Consider the perfect fluid as an example. **t** = - p**n** and **T** = - p**I**, where p is the thermodynamic pressure, and **I** is the unit tensor. The perfect fluid is an inviscid fluid, which is a good approximation of the real fluid for understanding certain hydrodynamic and aerodynamic problems.

With the Reynolds transport theorem, (2-6), and the Cauchy stress principle, (2-8), (2-7) becomes

$$\oiiint_{V(t)} \rho \frac{d\mathbf{u}}{dt} dV = \oiiint_{V(t)} \rho \mathbf{f} dV + \oiint_{S(t)} \mathbf{n} \cdot \mathbf{T} dS .$$

With the application of the divergence theorem, the surface integral in the above equation is transformed into a volume integral. Then

$$\oiiint_{V(t)} \rho \frac{d\mathbf{u}}{dt} dV = \oiiint_{V(t)} \rho \mathbf{f} dV + \oiiint_{V(t)} \nabla \cdot \mathbf{T} dV . \tag{2-10}$$

Since $V(t)$ is arbitrary, (2-10) implies that

$$\rho \frac{d\mathbf{u}}{dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{T}, \tag{2-11}$$

or written in index form as

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho f_i + \frac{\partial T_{ji}}{\partial x_j}, \tag{2-12}$$

where f_j is the component of **f**. Equation (2-11) or (2-12) is called the momentum equation. It is understood that the repeated indices stand for summation, and i, j = 1, 2 and 3. For example, the x-component of (2-11) in Cartesian coordinates is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho f_x + \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z}, \tag{2-13}$$

where (*u, v, w*) is the component of **u**, and f_x is the x-component of **f**. For perfect fluid, we have $\mathbf{T} = -p\mathbf{I}$. Thus $\nabla \cdot \mathbf{T} = -\nabla p$, and the momentum equation becomes

$$\rho \frac{d\mathbf{u}}{dt} = \rho\mathbf{f} - \nabla p \,. \qquad \text{(2-14)}$$

Equation (2-14) is called the Euler's equation, which expresses the conservation of momentum for the flow of a perfect fluid.

To show the Cauchy Stress principle in (2-8), we first rewrite (2-7) as

$$\oiiint_{V(t)} \rho \frac{d\mathbf{u}}{dt} dV = \oiiint_{V(t)} \rho\mathbf{f}\, dV + \oiint_{S(t)} \mathbf{t}\, dS \qquad \text{(2-15)}$$

by using the Reynolds transport theorem, (2-6). Then apply (2-15) to a small material volume with length scale, *l*, as shown in Figure 2-1(a). The volume and surface integral in (2-15) are proportional to l^3 and l^2, respectively. Thus the volume integrals are negligible in compare with the surface integral, and (2-15) reduces to

$$\oiint_{S(t)} \mathbf{t}\, dS = 0 \qquad \text{(2-16)}$$

as $l \to 0$, i.e., the traction **t** is in static balance for a small material volume. Next we apply (2-16) to a tetrahedron shown in Figure 2-1(b). The volume of the tetrahedron is sufficiently small such that not only (2-16) holds, also $\mathbf{t}(\mathbf{r}, t; \mathbf{n}) \cong \mathbf{t}(\mathbf{r}^*, t; \mathbf{n})$, where **r*** is the location of the center of the tetrahedron. For simplicity, we write $\mathbf{t}(\mathbf{r}^*, t; \mathbf{n})$ as $\mathbf{t}(\mathbf{n})$, which represents the traction on surface with normal **n** at position **r*** at time *t*, in the following derivation. It follows from (2-16) that

$$\mathbf{At(n)} + [\mathbf{An \cdot i}]\mathbf{t(\text{-}i)} + [\mathbf{An \cdot j}]\mathbf{t(\text{-}j)} + [\mathbf{An \cdot k}]\mathbf{t(\text{-}k)} = 0,$$

or

$$\mathbf{t(n)} = [-\mathbf{n \cdot i}]\mathbf{t(\text{-}i)} + [-\mathbf{n \cdot j}]\mathbf{t(\text{-}j)} + [-\mathbf{n \cdot k}]\mathbf{t(\text{-}k)} \qquad \text{(2-17)}$$

after the area, A, has been divided from both sides. In a similar way, we also apply (2-16) to a thin disk as shown in Figure 2-1(c). The thickness of the disk is small enough that the side area of the disk is negligible in compare with the area of the top and bottom surfaces. Thus

$$\mathbf{t(n)} + \mathbf{t(-n)} = 0\,,$$

or

$$\mathbf{t(-n)} = -\mathbf{t(n)}\,. \tag{2-18}$$

With (2-18), (2-17) becomes

$$\mathbf{t(n)} = [\mathbf{n}\cdot\mathbf{i}]\mathbf{t(i)} + [\mathbf{n}\cdot\mathbf{j}]\mathbf{t(j)} + [\mathbf{n}\cdot\mathbf{k}]\mathbf{t(k)}\,, \tag{2-19}$$

or written in index form as

$$\mathbf{t(n)} = \mathbf{n}\cdot\mathbf{e}_k\,\mathbf{t}(\mathbf{e}_k) \equiv \mathbf{n}\cdot\mathbf{T}\,, \tag{2-20}$$

where $\mathbf{e}_k$ stands for the unit vectors **i, j** and **k**, and

$$\mathbf{T} = \mathbf{e}_k\,\mathbf{t}(\mathbf{e}_k) \tag{2-21}$$

is the stress tensor at the location of interest at time *t*. The component of the stress tensor, T_{ij}, is calculated from

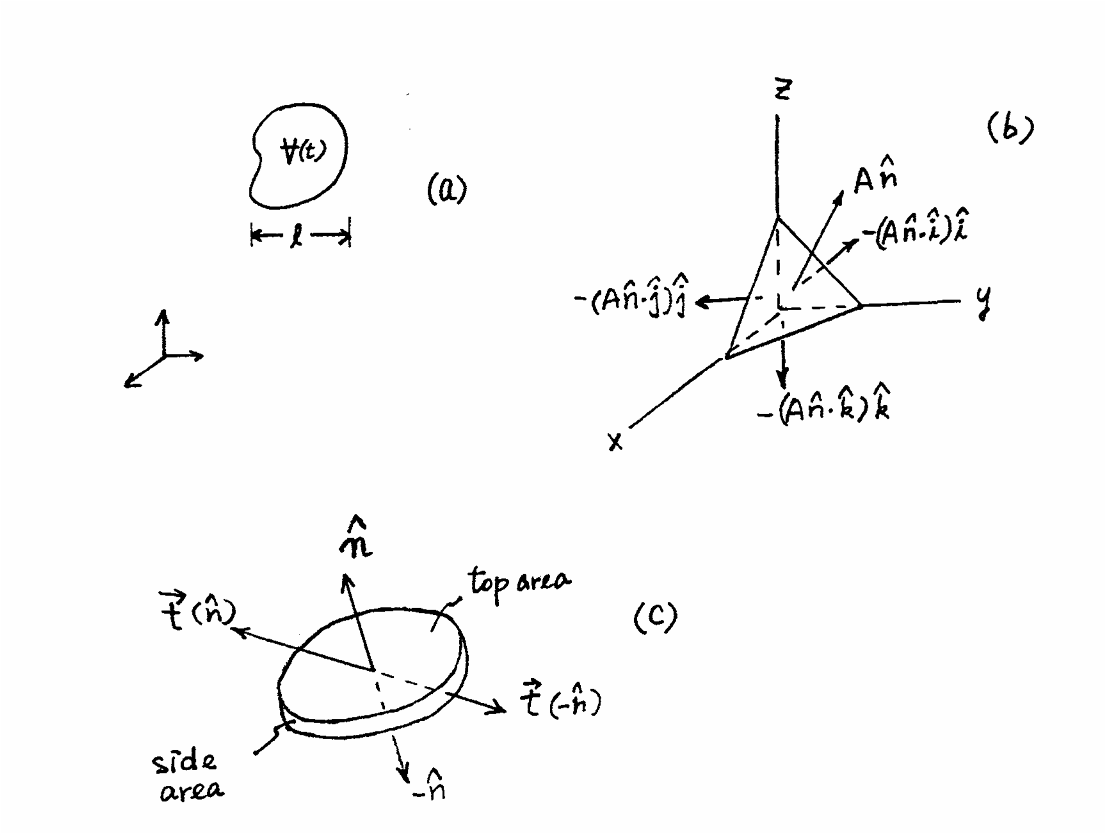


Figure 2-1: Sketch for illustrating the Cauchy stress principle.

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{T} \cdot \mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_k \mathbf{t}(\mathbf{e}_k) \cdot \mathbf{e}_j = \delta_{ik} \mathbf{t}(\mathbf{e}_k) \cdot \mathbf{e}_j = \mathbf{t}(\mathbf{e}_i) \cdot \mathbf{e}_j , \qquad (2\text{-}22)$$

which express the j^{th} component of the traction on an element of surface having normal in the direction $\mathbf{e}_i$ **.** For example, in Cartesian coordinates, T_{xy} is the y-component of traction on surface element with normal in the x-direction. The traction on the surface with unit vector **i** can be expressed as

$$\mathbf{t}(\mathbf{i}) = T_{xx}\mathbf{i} + T_{xy}\mathbf{j} + T_{xz}\mathbf{k} \; . \qquad (2\text{-}22a)$$

Note that the traction force and the stress tensor are functions of spatial coordinates and time in the Eulerian frame.

(III) Conservation of angular momentum

Conservation of angular momentum will not be employed directly for solving the fluid mechanics problem in general. However, we may employ it to prove that the stress tensor is symmetric, i.e., $T_{ij} = T_{ji}$, provided that there exists no body moment. Body moment may exist for some special material such as the ferromagnetic fluid.

Consider again the material volume in Figure 1-4. The angular momentum of the total material within $V(t)$ with respect to the origin of the coordinates is $\oiiint_{V(t)} \mathbf{r} \times \rho(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)dV$. Conservation of angular momentum about the origin is

$$\frac{d}{dt}\oiiint_{V(t)} \mathbf{r} \times \rho\mathbf{u}dV = \oiiint_{V(t)} \mathbf{r} \times \rho\mathbf{f}dV + \oiint_{S(t)} \mathbf{r} \times \mathbf{t}dS \; . \qquad (2\text{-}23)$$

The first and the second terms on the right hand side of (2-23) are the moments due to the body and surface forces, respectively. With the Reynolds transport theorem, (2-6), the term on the left hand side of (2-23) can be written as

$$\frac{d}{dt}\oiiint_{V(t)} \mathbf{r} \times \rho\mathbf{u}dV = \oiiint_{V(t)} \rho\frac{d}{dt}(\mathbf{r} \times \mathbf{u})dV = \oiiint_{V(t)} \rho(\mathbf{r} \times \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{u})dV$$

$$= \oiiint_{V(t)} \rho(\mathbf{r} \times \frac{d\mathbf{u}}{dt} + \mathbf{u} \times \mathbf{u})dV = \oiiint_{V(t)} \rho\mathbf{r} \times \frac{d\mathbf{u}}{dt}dV$$

since $\mathbf{u} \times \mathbf{u} = 0$. Thus we have

$$\oiiint_{V(t)} \rho \mathbf{r} \times \frac{d\mathbf{u}}{dt} dV = \oiiint_{V(t)} \mathbf{r} \times \rho \mathbf{f} dV + \oiint_{S(t)} \mathbf{r} \times \mathbf{t} dS \,. \qquad (2\text{-}24)$$

As in the derivation of the Cauchy stress principle, (2-24) reduces to

$$\oiint_{S(t)} \mathbf{r} \times \mathbf{t} dS \cong 0 \,, \qquad (2\text{-}25)$$

if the conservation of angular momentum is applied to a sufficiently small material volume. Consider a small cubic material volume in Figure 2-2. The volume is small enough that both (2-25) holds, and the stress tensor, T_{ij}, is approximately constant within the cube. Let's place the origin of the coordinates at the center of the cube in Figure 2-2. Equation (2-25) is a vector equation, which can be expressed as three scalar equations. Applying the y-component of (2-25) to the cube leads to

$$T_{zx} = T_{xz} \qquad (2\text{-}26a)$$

if (2-22a) and other similar equations have been employed. Similarly, the *x*- and *z*-components of (2-25) lead to

$$T_{yz} = T_{zy} \quad \text{and} \quad T_{xy} = T_{yx} \,, \qquad (2\text{-}26b)$$

respectively. A more elegant method for deriving (2-26a) and (2-26b) is as follows. On substituting the momentum equation, (2-11), into (2-24), we have

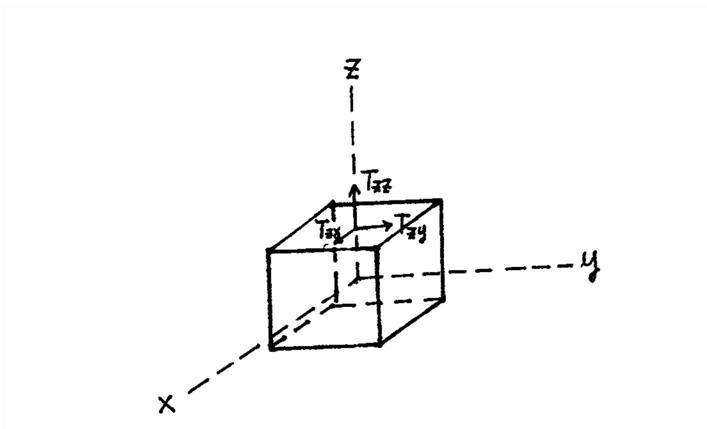


Figure 2-2: Sketch for illustrating the conservation of angular momentum. The stress components on other five surfaces are not shown for clarity.

$$\oiiint_{V(t)} \mathbf{r} \times (\rho \mathbf{f} + \nabla \cdot \mathbf{T}) dV = \oiiint_{V(t)} \mathbf{r} \times \rho \mathbf{f} dV + \oiint_{S(t)} \mathbf{r} \times \mathbf{t} dS \,.$$

The first term on the left hand side equals that on the right hand side, thus the above equation reduces to

$$\oiiint_{V(t)} \mathbf{r} \times \nabla \cdot \mathbf{T} dV = \oiint_{S(t)} \mathbf{r} \times \mathbf{t} dS \,. \qquad \text{(2-27)}$$

The i^{th}-component of (2-27) is

$$\oiiint_{V(t)} e_{ijk} x_j \frac{\partial T_{\alpha k}}{\partial x_\alpha} dV = \oiint_{S(t)} e_{ijk} x_j t_k dS \,, \qquad \text{(2-28)}$$

where e_{ijk} is the permutation symbol, x_j or (x_α), $T_{\alpha k}$ and t_k are components of **r**, **T** and **t**, respectively. The left hand side of (2-28) can be written as

$$\oiiint_{V(t)} e_{ijk} x_j \frac{\partial T_{\alpha k}}{\partial x_\alpha} dV = \oiiint_{V(t)} \left\{ e_{ijk} \left[\frac{\partial}{\partial x_\alpha} \left(x_j T_{\alpha k} \right) - \frac{\partial x_j}{\partial x_\alpha} T_{\alpha k} \right] \right\} dV$$
$$= e_{ijk} \oiint_{S(t)} n_\alpha x_j T_{\alpha k} dS - \oiiint_{V(t)} e_{ijk} T_{jk} dV \,,$$

where the divergence theorem and $\partial x_j / \partial x_\alpha = \delta_{j\alpha}$ have been employed, with n_α the component of the unit vector of $S(t)$, and $\delta_{j\alpha}$ the Kronecker delta. Also with (2-9), the right hand side of (2-28) can be written as

$$\oiint_{S(t)} e_{ijk} x_j t_k dS = \oiint_{S(t)} e_{ijk} x_j (n_\alpha T_{\alpha k}) dS \,.$$

By substituting the above two equations into (2-28), we obtain

$$\oiiint_{V(t)} e_{ijk} T_{jk} dV = 0 \,. \qquad \text{(2-29)}$$

As $V(t)$ is arbitrary, (2-29) implies

$$e_{ijk} T_{jk} = 0 \,,$$

or

$$T_{jk} = T_{kj} \,. \tag{2-30}$$

Note that there are only six (out of nine) independent components since the stress tensor is symmetric.

(IV) Kinematics of deformation

As we learned from elementary Physics, the conservation of energy involves the work done and the net heat input into a system with fixed mass. The work done on the material volume by the forces always induces the deformation of the volume, and the study of the heat input involves the knowledge of thermodynamics. Therefore, we will first discuss the kinematics of deformation and review thermodynamics in this and the next sections, respectively, before we study the conservation of energy.

Consider two neighboring fluid points in space, located at **r** and **r**+*d***r**, and having velocities **u**(**r**, *t*) and **u**(**r**+*d***r**, *t*), respectively. The velocity difference, *d***u** may be written as

$$d\mathbf{u} = \mathbf{u}(\mathbf{r} + d\mathbf{r}, t) - \mathbf{u}(\mathbf{r}, t) = d\mathbf{r} \cdot \nabla \mathbf{u} \,, \tag{2-31}$$

where $\nabla \mathbf{u}$ is the velocity gradient evaluated at **r**. $\nabla \mathbf{u}$ is a second order tensor having nine components, $\partial u_j / \partial x_i$ (with i, j = 1,2,3), and can be decomposed as

$$\nabla \mathbf{u} = \mathbf{D} + \mathbf{\Omega} \,, \tag{2-32}$$

where $\mathbf{D}$ is a symmetric tensor and $\mathbf{\Omega}$ is an anti-symmetric (or skew-symmetric) tensor. In component form,

$$\frac{\partial u_j}{\partial x_i} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) \equiv D_{ij} + \Omega_{ij} \,. \tag{2-33}$$

Note that $D_{ij} = D_{ji}$ (symmetric) and $\Omega_{ij} = -\Omega_{ji}$ (anti-symmetric). $\mathbf{D}$ is called "the rate of strain tensor" or "deformation tensor", and $\mathbf{\Omega}$ is called the "vorticity tensor".

(i) Physical meaning of the deformation tensor

(a) Stretching

Consider two neighboring material points located at $\mathbf{r}_1$ and $\mathbf{r}_2(=\mathbf{r}_1+d\mathbf{r})$, respectively, as shown in Figure 2-3(a). $d\mathbf{r}$ may be regarded as an infinitesimal material line element. We have

$$\frac{d}{dt}(d\mathbf{r})=\frac{d}{dt}(\mathbf{r}_2-\mathbf{r}_1)=\mathbf{u}_2-\mathbf{u}_1=\mathbf{u}(\mathbf{r}+d\mathbf{r})-\mathbf{u}(\mathbf{r})=d\mathbf{u}=d\mathbf{r}\cdot\nabla\mathbf{u}\,. \qquad (2\text{-}34)$$

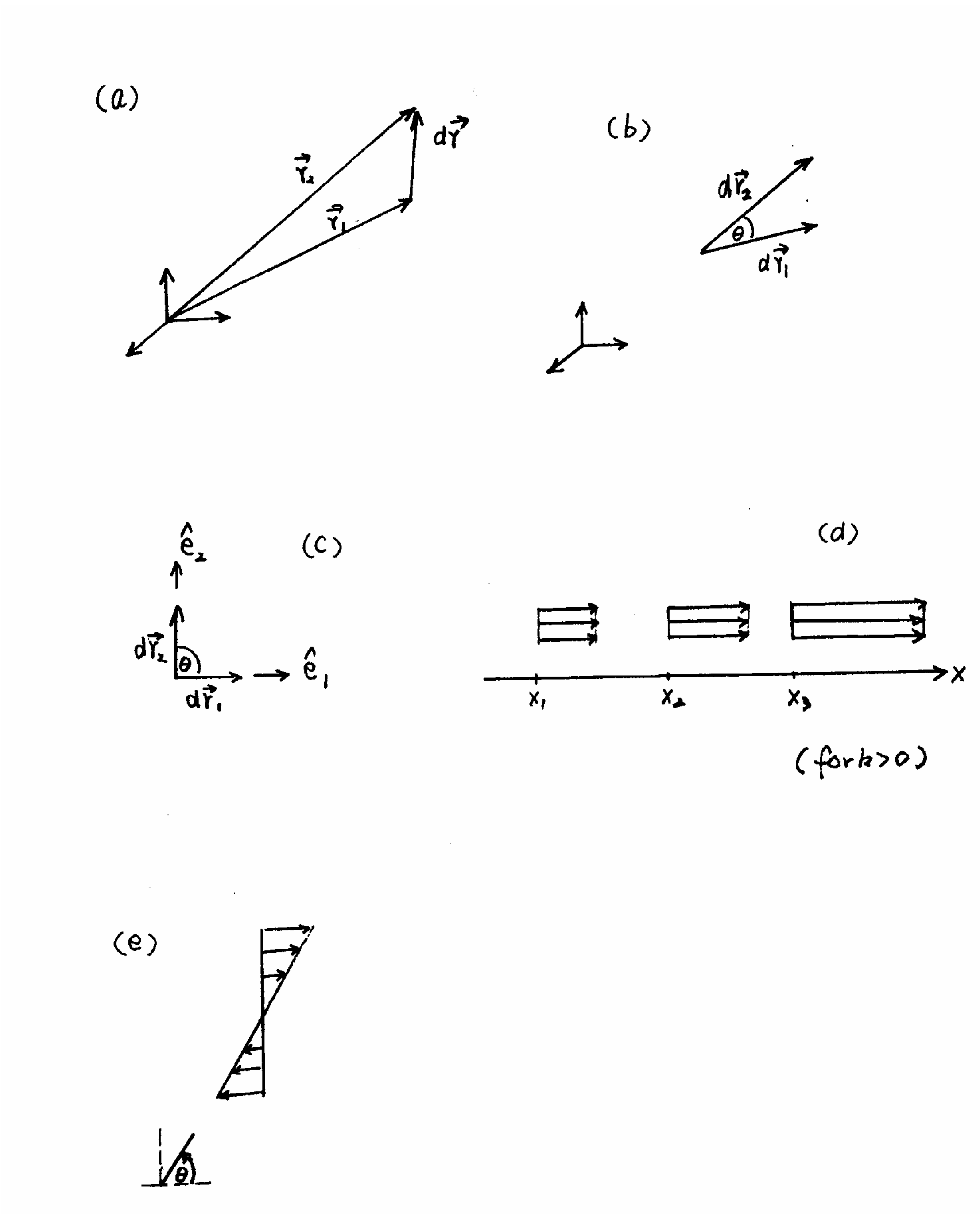


Figure 2-3: Sketch for illustrating the deformation of fluid motion.

On the other hand, let ds be the magnitude of $d\mathbf{r}$, i.e., the distance between the two material points. We have $(ds)^2 = d\mathbf{r}\cdot d\mathbf{r}$. Then

$$\frac{d}{dt}(ds)^2 = 2ds\frac{d}{dt}(ds) = \frac{d}{dt}(d\mathbf{r}\cdot d\mathbf{r}) = 2d\mathbf{r}\cdot\frac{d}{dt}(d\mathbf{r}) = 2d\mathbf{r}\cdot(d\mathbf{r}\cdot\nabla\mathbf{u}) = 2d\mathbf{r}\cdot\nabla\mathbf{u}\cdot d\mathbf{r}\ ,$$

where (2-34) has been employed for deriving the fifth equality in the above equation. Equating the second and the sixth terms, we have

$$\frac{1}{(ds)}\frac{d}{dt}(ds) = \frac{d\mathbf{r}}{ds}\cdot\nabla\mathbf{u}\cdot\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{ds}\cdot\mathbf{D}\cdot\frac{d\mathbf{r}}{ds} + \frac{d\mathbf{r}}{ds}\cdot\mathbf{\Omega}\cdot\frac{d\mathbf{r}}{ds}\ , \qquad \text{(2-35)}$$

where the definition for the tensor decomposition, (2-32), has been employed . Let $\mathbf{n} = d\mathbf{r}/ds$ be the unit vector along the line element $d\mathbf{r}$. The last term on the right hand side of (2-35) can be expanded as

$$\begin{aligned}\frac{d\mathbf{r}}{ds}\cdot\mathbf{\Omega}\cdot\frac{d\mathbf{r}}{ds} &= \mathbf{n}\cdot\mathbf{e}_i\Omega_{ij}\mathbf{e}_j\cdot\mathbf{n} = n_i\Omega_{ij}n_j \\ &= n_1\Omega_{12}n_2 + n_2\Omega_{21}n_1 + n_2\Omega_{23}n_3 + n_3\Omega_{32}n_2 + n_3\Omega_{31}n_1 + n_1\Omega_{13}n_3 \\ &= n_1\Omega_{12}n_2 - n_2\Omega_{12}n_1 + n_2\Omega_{23}n_3 - n_3\Omega_{23}n_2 + n_3\Omega_{31}n_1 - n_1\Omega_{31}n_3 \\ &= 0\ , \end{aligned} \qquad \text{(2-36)}$$

where $\mathbf{e}_i$ and $\mathbf{e}_j$ are the unit vectors along the *i*- and *j*-directions, respectively, n_i is the component of $\mathbf{n}$ along the *i*-direction, and the anti-symmetric condition for $\mathbf{\Omega}$, $\Omega_{ij} = -\Omega_{ji}$, has been employed. Thus (2-35) reduces to

$$\frac{1}{(ds)}\frac{d}{dt}(ds) = \frac{d\mathbf{r}}{ds}\cdot\mathbf{D}\cdot\frac{d\mathbf{r}}{ds} = \mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n}\ . \qquad \text{(2-37)}$$

We call $\mathbf{D}$ the rate of strain tensor because the left hand side of (2-37) represents the rate of strain of a material line element, $d\mathbf{r}$. If the line element lies along the x-direction with unit vector $\mathbf{e}_1$, then

$$\frac{1}{(ds)}\frac{d}{dt}(ds) = \mathbf{e}_1\cdot\mathbf{D}\cdot\mathbf{e}_1 = D_{11} = \frac{1}{dx}\frac{d}{dt}(dx)\ . \qquad \text{(2-37a)}$$

The component D_{11} equals the rate of strain of the material line element. In the same way, the other diagonal components, D_{22} and D_{33} , show similar stretching behavior.

(b) Shear

Consider two material line elements, $d\mathbf{r}_1$ and $d\mathbf{r}_2$, with the angle between them, θ, as shown in Figure 2-3(b). Let ds_1 and ds_2 be the length of $d\mathbf{r}_1$ and $d\mathbf{r}_2$, respectively. We have

$$\frac{d}{dt}(d\mathbf{r}_1 \cdot d\mathbf{r}_2) = \frac{d}{dt}(ds_1 ds_2 \cos\theta)$$
$$= \frac{1}{ds_1}\left(\frac{d}{dt}ds_1\right)[ds_1 ds_2 \cos\theta] + \frac{1}{ds_2}\left(\frac{d}{dt}ds_2\right)[ds_1 ds_2 \cos\theta] - ds_1 ds_2 \sin\theta \frac{d\theta}{dt}. \quad \text{(2-38a)}$$

On the other hand,

$$\frac{d}{dt}(d\mathbf{r}_1 \cdot d\mathbf{r}_2) = d\mathbf{r}_1 \cdot \frac{d}{dt}(d\mathbf{r}_2) + d\mathbf{r}_2 \cdot \frac{d}{dt}(d\mathbf{r}_1)$$
$$= d\mathbf{r}_1 \cdot (d\mathbf{r}_2 \cdot \nabla\mathbf{u}) + d\mathbf{r}_2 \cdot (d\mathbf{r}_1 \cdot \nabla\mathbf{u}) \qquad \text{(using (2-34))}$$
$$= d\mathbf{r}_2 \cdot \nabla\mathbf{u} \cdot d\mathbf{r}_1 + d\mathbf{r}_1 \cdot \nabla\mathbf{u} \cdot d\mathbf{r}_2$$
$$= d\mathbf{r}_2 \cdot (\mathbf{D} + \mathbf{\Omega}) \cdot d\mathbf{r}_1 + d\mathbf{r}_1 \cdot (\mathbf{D} + \mathbf{\Omega}) \cdot d\mathbf{r}_2 \qquad \text{(using (2-32))}$$
$$= 2d\mathbf{r}_2 \cdot \mathbf{D} \cdot d\mathbf{r}_1, \qquad \text{(2-38b)}$$

where the symmetric property of **D** and the anti-symmetric property of **Ω** have been employed. By combining (2-38a) and (2-38b), and dividing both sides by $ds_1 ds_2$, we obtain

$$\frac{1}{ds_1}\left(\frac{d}{dt}ds_1\right)\cos\theta + \frac{1}{ds_2}\left(\frac{d}{dt}ds_2\right)\cos\theta - \sin\theta\frac{d\theta}{dt} = 2\frac{d\mathbf{r}_2}{ds_2} \cdot \mathbf{D} \cdot \frac{d\mathbf{r}_1}{ds_1}. \qquad \text{(2-39)}$$

In order to have a more clear physical picture of (2-39), let $d\mathbf{r}_1 / ds_1 = \mathbf{e}_1$ and $d\mathbf{r}_2 / ds_2 = \mathbf{e}_2$, with $\mathbf{e}_1 \perp \mathbf{e}_2$ instantaneously as shown in Figure 2-3(c), then $\theta = 90^o$, cos θ = 0, and sin θ = 1. Thus (2-39) becomes

$$\frac{d\theta}{dt} = -2e_2 \cdot \mathbf{D} \cdot e_1 = -2D_{21}, \qquad \text{or} \qquad D_{12} = D_{21} = -\frac{1}{2}\frac{d\theta}{dt}, \qquad \text{(2-39a)}$$

which represent a pure shear motion. Similarly, we can find the same behavior for other off diagonal components, D_{13} and D_{23}.

Example 1: Given the velocity field expressed in component form in Cartesian coordinates (see Figure 2-3(d)),

$$u = kx \qquad v = 0, \qquad w = 0.$$

The components of the deformation tensor calculated according to (2-33) are

$$D_{11} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial x} = k \ ,$$
$$D_{22} = D_{33} = D_{12} = D_{23} = D_{31} = 0 \ ,$$

which is a pure straining (if $k > 0$) or shrinking (if $k < 0$) motion in the x-direction.

Example 2 : Given the velocity field expressed in component form in Cartesian coordinates (see Figure 2-3(e)),

$$u = ky \ , \qquad v = 0 \ , \qquad w = 0 \ .$$

The components of the deformation tensor calculated according to (2-33) are

$$D_{12} = D_{21} = -\frac{1}{2}\frac{d\theta}{dt} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}k \ ,$$
$$D_{11} = D_{22} = D_{33} = D_{23} = D_{31} = 0 \ ,$$

which is a pure shearing motion in the xy-plane. $-d\theta / dt = k$ is called the shear rate. The angle, θ, becomes smaller and smaller in figure 2-5(e) as time increases for positive value of k.

(c) *Principal axes*

If $\mathbf{D} = 0$, the motion is called rigid, i.e., the distance between any two fluid particles remains unchanged during the fluid motion. An example is the steady state fluid motion within a cylindrical tank, which is rotating about its principal axis at a constant rate. If $\mathbf{D} \neq 0$, it is always possible to find three perpendicular directions (at a point) along which there is straining and such that the angles between them are instantaneously rigid, the axes along these directions are called the principal axes. We want to find directions, $\mathbf{e}_1'$, $\mathbf{e}_2'$ and $\mathbf{e}_3'$, such that in these coordinates, $D_{ij}' = 0$ unless ij = 11, 22 and 33, i.e.,

$$\mathbf{D} = D_{11}'\mathbf{e}_1'\mathbf{e}_1' + D_{22}'\mathbf{e}_2'\mathbf{e}_2' + D_{33}'\mathbf{e}_3'\mathbf{e}_3' \ . \tag{2-40}$$

If this is the case, we have

$$\mathbf{D}\cdot\mathbf{e}_1' = D_{11}'\mathbf{e}_1',$$
$$\mathbf{D}\cdot\mathbf{e}_2' = D_{22}'\mathbf{e}_2',$$
$$\mathbf{D}\cdot\mathbf{e}_3' = D_{33}'\mathbf{e}_3'.$$

Therefore, we are looking for directions $\mathbf{e}$ (stands for $\mathbf{e}_1'$, $\mathbf{e}_2'$ and $\mathbf{e}_3'$) such that

$$\mathbf{D}\cdot\mathbf{e} = \lambda\mathbf{e}, \tag{2-41a}$$

or

$$(\mathbf{D} - \lambda\mathbf{I})\cdot\mathbf{e} = 0, \tag{2-41b}$$

where λ is called the eigenvalue of the tensor **D**, which is also the principal strain rate. Equation (2-41b) consists of three algebraic equations for three unknowns, the three components of **e**. The solution is nontrivial if

$$\begin{vmatrix} D_{11}-\lambda & D_{12} & D_{13} \\ D_{12} & D_{22}-\lambda & D_{23} \\ D_{13} & D_{23} & D_{33}-\lambda \end{vmatrix} = 0, \tag{2-41c}$$

which is a cubic algebraic equation for λ. As **D** is a symmetric second order tensor, the three roots of (2-41c) are real. For a given root, λ_i, we may find a corresponding principal direction, $\mathbf{e}_i'$, using (2-41b). Here i = 1, 2, and 3. The three axes defined by $\mathbf{e}_1'$, $\mathbf{e}_2'$ and $\mathbf{e}_3'$ are perpendicular to one another, and are called the principal axes. In the principal coordinates,

$$\mathbf{D} = \lambda_1\mathbf{e}_1'\mathbf{e}_1' + \lambda_2\mathbf{e}_2'\mathbf{e}_2' + \lambda_3\mathbf{e}_3'\mathbf{e}_3'. \tag{2-42}$$

The readers may review the eigenvalue problem of a square matrix in the textbook of undergraduate engineering mathematics for evaluating the principal strain rates and the principal axes.

(ii) Vorticity tensor and vorticity vector

The vorticity tensor,

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right),$$

has the property

$$e_{ijk}\Omega_{jk} = e_{ijk}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k}\right) = \frac{1}{2}(\nabla\times\mathbf{u})\cdot\mathbf{i} + \frac{1}{2}(\nabla\times\mathbf{u})\cdot\mathbf{i} = (\nabla\times\mathbf{u})\cdot\mathbf{i} = \omega_{\mathrm{i}}, \quad (2\text{-}43)$$

where ω_i is the component of the vorticity vector, $\boldsymbol{\omega}$, which is defined as

$$\boldsymbol{\omega} = (\nabla\times\mathbf{u}). \quad (2\text{-}44)$$

The three independent components of the vorticity tensor are directly related to the three components of the vorticity vector through (2-43). To see this further, we dot (2-43) by e_{ilm} from both side,

$$e_{ilm}e_{ijk}\Omega_{jk} = e_{ilm}\omega_{\mathrm{i}}.$$

With $e_{ilm}e_{ijk} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}$,

$$e_{ilm}\omega_{\mathrm{i}} = (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj})\Omega_{jk} = \Omega_{lm} - \Omega_{ml} = 2\Omega_{lm}, \quad (2\text{-}45)$$

or in matrix form,

$$\begin{pmatrix} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\omega_3 & -\frac{1}{2}\omega_2 \\ -\frac{1}{2}\omega_3 & 0 & \frac{1}{2}\omega_1 \\ \frac{1}{2}\omega_2 & -\frac{1}{2}\omega_1 & 0 \end{pmatrix}, \quad (2\text{-}46)$$

which provides a clear relation between the components of the vorticity tensor and the vorticity vector. As

$$\frac{d}{dt}(d\mathbf{r}) = d\mathbf{u} = d\mathbf{r}\cdot\nabla\mathbf{u} = d\mathbf{r}\cdot\mathbf{D} + d\mathbf{r}\cdot\boldsymbol{\Omega},$$

the part of motion associated with $\boldsymbol{\Omega}$ is

$$\begin{aligned} d\mathbf{r}\cdot\boldsymbol{\Omega} &= dx_i\mathbf{e}_i\cdot\mathbf{e}_l\Omega_{lm}\mathbf{e}_m \\ &= dx_i\mathbf{e}_i\cdot\mathbf{e}_l\frac{1}{2}\omega_i\mathbf{e}_{ilm}\mathbf{e}_m \qquad \text{(using (2-45))} \\ &= dx_l\frac{1}{2}\omega_i e_{ilm}\mathbf{e}_m \\ &= \frac{1}{2}\boldsymbol{\omega}\times d\mathbf{r}, \end{aligned}$$

which is a solid body rotating with angular velocity $\frac{1}{2}\boldsymbol{\omega}$.

Example 3: Solid body rotation may be described by the velocity field in the cylindrical coordinates (r,θ,z) as $v_\theta = r\Delta$, $v_r = 0$, and $v_z = 0$, with Δ the angular speed of the rotation. The vorticity vector

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & \mathrm{r}\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ v_r & rv_\theta & v_z \end{vmatrix} = 2\Delta\mathbf{e}_z,$$

and the components of the vorticity tensor,

$$\Omega_{r\theta} = -\Omega_{\theta r} = \Delta, \qquad \text{other components = 0.}$$

(iii) The general local motion

We have learned from the above discussion that the rate of strain tensor, D, is related to the deformation (stretching and shearing) and the vorticity tensor is related to the rotation of the fluid element. The instantaneous general local motion of a fluid at time *t* can be written as

$$\begin{aligned} \mathbf{u}(\mathbf{r} + d\mathbf{r}) &= \mathbf{u}(\mathbf{r}) + d\mathbf{u} \\ &= \mathbf{u}(\mathbf{r}) + d\mathbf{r} \cdot \nabla\mathbf{u} \\ &= \mathbf{u}(\mathbf{r}) + d\mathbf{r} \cdot \mathbf{D} + d\mathbf{r} \cdot \boldsymbol{\Omega} \\ &= \mathbf{u}(\mathbf{r}) + dx_1'\lambda_1\mathbf{e}_1' + dx_2'\lambda_2\mathbf{e}_2' + dx_3'\lambda_3\mathbf{e}_3' + \frac{1}{2}\boldsymbol{\omega} \times d\mathbf{r}, \end{aligned} \qquad (2\text{-}47)$$

which consists of a translation with uniform velocity $\mathbf{u}(\mathbf{r})$, plus a local stretching along the principal axes, and a rotation with angular velocity ω/2.

(V) A brief review in thermodynamics

Classical thermodynamics deals with systems in equilibrium states; all the processes involved are sufficiently slow processes. Strictly speaking, a fluid system is in equilibrium only when it is at rest. Therefore, we shall consider first the thermodynamics of a uniform fluid at rest.

A fluid cannot withstand shear forces without flowing according to its definition. Thus if the fluid is at rest, the traction must be of form $\mathbf{t} = -p(\mathbf{r},t)\mathbf{n}$, and $\mathbf{T} = -p\mathbf{I}$, where p is the thermodynamic pressure. p is one of the thermodynamic variables, other important variables include the temperature T, the density ρ, the internal energy e, the enthalpy h, the entropy s, the specific volume v (= $1/\rho$), and etc.. For fluid, the local thermodynamic state is fixed by any two of these intensive thermodynamic variables, provided there are no chemical reactions. The rest variables can be expressed in terms of the specified two variables. For example, if we choose ρ and T as independent variables, we may write

$$p = p(\rho,T)\,, \tag{2-48a}$$

$$e = e(\rho,T)\,, \tag{2-48b}$$

$$s = s(\rho,T)\,, \tag{2-48c}$$

and etc., which are called the equations of states. A familiar equation of state that we have learned before in elementary Physics and Chemistry is

$$p = \rho RT\,, \tag{2-49}$$

which describes the equation of state for a perfect gas, and is regarded as a special case of (2-48a). Here R is the gas constant. The variables in the above equations are also called the state variables. The plane constructed by the two independent variables, say, ρ and T, can be used to describe the variations of the thermodynamic states of a system (see Figure 2-4). Any point in the plane is referred to a state. The lines joining two points in the plane are called the processes. There are two kinds of processes one may consider, the reversible process and the irreversible process. An example of the reversible process is the infinitesimal work done on a compressible fluid by the pressure, and that for the irreversible process is the work done by the friction force.

The changes of state variables are related by the first and second law of thermodynamics, which can be written as

$$de = \delta Q + \delta W \tag{2-50a}$$

and

$$ds = \frac{\delta Q}{T} + d_i s\,, \tag{2-50b}$$

respectively. Here de and ds are the changes of e and s between two neighboring states, δQ and δW are the heat addition and the work done, respectively, on the system, and $d_i s$ is the change of entropy associated with the irreversible processes. Note that the heat addition and work done processes in general are irreversible. For a reversible process,

$$d_i s = 0, \tag{2-51a}$$

and

$$dW = -pd\text{v} = -pd(1/\rho). \tag{2-51b}$$

Here we use the symbol "dW" instead of "δW" to represent that the work is done for a (particular) reversible process.

The entropy, s, is a useful parameter in fluid mechanics primary because it may be taken as constant in many flows under certain plausible assumption. The following is to discuss how to evaluate the entropy. Since the entropy is a function of state of the fluid, changes in entropy do not depend on the particular process. Therefore, we can use a reversible process to find the entropy, and (2-50a) and (2-50b) reduce to

$$de = \delta Q - pd(1/\rho) \tag{2-52a}$$

and

$$ds = \frac{\delta Q}{T}, \tag{2-52b}$$

with the aid of (2-51a) and (2-51b). By combining (2-52a) and (2-52b), we can find

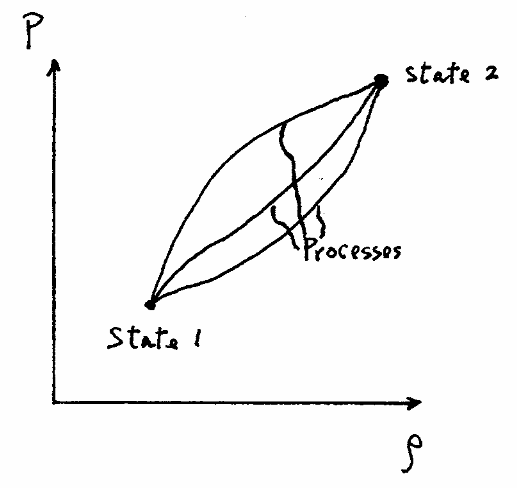


Figure 2-4: Sketch for illustrating thermodynamic states and processes.

$$Tds = de + pd(1/\rho)\,, \tag{2-53a}$$

which is called the Gibb's equation. If we can integrate (2-53a), we can obtain $s = s(p,\rho)$ (see Example 4 later). As the enthalpy, *h*, is defined through

$$h = e + p/\rho\,,$$

then

$$dh = de + pd(1/\rho) + \frac{1}{\rho}dp\,,$$

and the Gibb's equation may be written in another form as

$$Tds = dh - \frac{1}{\rho}dp\,. \tag{2-53b}$$

We may also integrate (2-53b) to express *s* in terms of other thermodynamic variables. In the fluid mechanics problem, the operator " *d* " in the Gibb's equation may stand for the material derivative " *d/dt* ", the spatial gradient operator "∇", or the time derivative "$\partial/\partial t$", i.e., the Gibb's equation implies the following equations:

$$T\frac{ds}{dt} = \frac{de}{dt} + p\frac{d}{dt}\left(\frac{1}{\rho}\right), \tag{2-54a}$$

$$T\nabla s = \nabla e + p\nabla\left(\frac{1}{\rho}\right), \tag{2-54b}$$

and

$$T\frac{\partial s}{\partial t} = \frac{\partial e}{\partial t} + p\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right), \tag{2-54c}$$

which will be employed later in the text.

The fluid is in motion for most fluid mechanics problem. One of the basic assumptions in fluid mechanics is that the equilibrium thermodynamics is locally applicable to fluids in motion. This is valid when the time required for a given fluid change associated with marcoscopic phenomenon is much greater than the relaxation time associated with a distributed quantity to regain its equilibrium value in the molecular level (see P. A. Thompson, "Compressible-fluid dynamics," McGraw-Hill, 1972 Chapter 2).

Three are two types of specific heat that one may consider, the specific heat at constant volume, C_v, and the specific heat at constant pressure, C_p. C_v is the heat added per unit mass per unit temperature change during a constant volume process, i.e.,

$$\delta Q = C_v dT \,. \qquad \text{(at constant volume)}$$

For a constant volume process, $pd(1/\rho) = pd\mathrm{v} = 0$, thus

$$de = \delta Q = Tds \qquad \text{(at constant volume)}$$

according to (2-52a) and (2-53a). It follows that

$$C_v = \left.\frac{\partial e}{\partial T}\right|_\rho , \qquad \text{with} \quad e = e(T,\rho) \,. \qquad \text{(2-55a)}$$

On the other hand, C_p is the heat added per unit mass per unit temperature change during a constant pressure process, i.e.,

$$\delta Q = C_p dT \,. \qquad \text{(at constant pressure)}$$

For a constant pressure process, we have

$$dh = \delta Q = Tds \qquad \text{(at constant pressure)}$$

according to (2-52b) and (2-53b). Thus

$$C_p = \left.\frac{\partial h}{\partial T}\right|_p , \qquad \text{with} \quad h = h(T,p) \,. \qquad \text{(2-55b)}$$

Although $e = e(T,\rho)$ in general, we may show that $e = e(T)$ only for a perfect gas (see Example 5). With $p = \rho RT$, $h = e + p/\rho = e + RT$, we also have $h = h(T)$. Thus (2-55a) and (2-55b) reduce to

$$C_v = \frac{de}{dT} \qquad \text{and} \qquad C_p = \frac{dh}{dT} \,,$$

or

$$C_p = \frac{dh}{dT} = \frac{d}{dT}(e + RT) = C_v + R \,. \qquad \text{(2-56a)}$$

for a perfect gas. If we define,

$$\gamma = \frac{C_p}{C_v}, \tag{2-56b}$$

we may obtain

$$C_v = \frac{R}{\gamma - 1} \qquad \text{and} \qquad C_p = \frac{\gamma R}{R - 1}. \tag{2-56c}$$

by solving (2-56a) and (2-56b) for C_v and C_p.

Example 4: *The entropy for a perfect gas.*

We have $p = \rho RT$, $e = e(T)$ and $de = C_v(T)dT$ for perfect gas, then the Gibb's equation, (2-53a), gives

$$ds = \frac{C_v(T)}{T}dT + \rho R d(1/\rho).$$

Integration gives

$$s = \int C_v(T) d(\ln T) + R \ln(1/\rho) + \text{constant}.$$

If $C_v(T) = \text{constant}$, then

$$s = C_v \ln T + R \ln(1/\rho) + \text{constant},$$

or

$$\begin{aligned}
\frac{s}{C_v} &= \ln T + (\gamma - 1)\ln(1/\rho) + \text{constant} && \text{(using (2-56c))} \\
&= \ln\left(\frac{T}{\rho^{\gamma-1}}\right) + \text{constant} \\
&= \ln\left(\frac{p}{\rho^{\gamma} R}\right) + \text{constant} && \text{(using } p = \rho RT \text{)} \\
&= \ln\left(\frac{p}{\rho^{\gamma}}\right) - \ln R + \text{constant} \\
&= \ln\left(\frac{p}{\rho^{\gamma}}\right) + \text{constant}. && (R \text{ is also a constant})
\end{aligned}$$

Therefore,

$$\frac{s - s_R}{C_v} = \ln\left(\frac{p}{\rho^{\gamma}}\right) - \ln\left(\frac{p_R}{\rho_R{}^{\gamma}}\right) = \ln\frac{P/P_R}{(\rho/\rho_R)^{\gamma}} \tag{2-57}$$

for a reference state with the related properties denoted by subscript "*R*".

Example 5: *Prove that* $e = e(T)$ *for perfect gas.*

For a perfect gas, Gibb's equation, (2-53a), gives

$$Tds = de + pd\mathsf{v} = de + \rho RTd\mathsf{v} \; . \qquad (2\text{-}58)$$

If $e = e(T, \mathsf{v})$, then

$$de = \left.\frac{\partial e}{\partial \mathsf{v}}\right|_T d\mathsf{v} + \left.\frac{\partial e}{\partial T}\right|_{\mathsf{v}} dT \; .$$

By substituting the above equation into (2-58), we have

$$ds = \left(\frac{1}{T} \left.\frac{\partial e}{\partial \mathsf{v}}\right|_T + \rho R \right) d\mathsf{v} + \frac{1}{T} \left.\frac{\partial e}{\partial T}\right|_{\mathsf{v}} dT \; . \qquad (2\text{-}59a)$$

On the other hand, for $s = s(T, \mathsf{v})$, we have

$$ds = \left.\frac{\partial s}{\partial \mathsf{v}}\right|_T d\mathsf{v} + \left.\frac{\partial s}{\partial T}\right|_{\mathsf{v}} dT \; . \qquad (2\text{-}59b)$$

By comparing (2-59a) and (2-59b), we obtain

$$\left.\frac{\partial s}{\partial \mathsf{v}}\right|_T = \frac{1}{T} \left.\frac{\partial e}{\partial \mathsf{v}}\right|_T + \rho R \qquad \text{and} \qquad \left.\frac{\partial s}{\partial T}\right|_{\mathsf{v}} = \frac{1}{T} \left.\frac{\partial e}{\partial T}\right|_{\mathsf{v}} \; .$$

After differentiation, we have (recall that the two independent variables for s are T and v)

$$\frac{\partial^2 s}{\partial T \partial \mathsf{v}} = -\frac{1}{T^2} \frac{\partial e}{\partial \mathsf{v}} + \frac{1}{T} \frac{\partial^2 e}{\partial \mathsf{v} \partial T} \qquad \text{and} \qquad \frac{\partial^2 s}{\partial \mathsf{v} \partial T} = \frac{1}{T} \frac{\partial^2 e}{\partial \mathsf{v} \partial T} \; .$$

Thus

$$-\frac{1}{T^2} \frac{\partial e}{\partial \mathsf{v}} = 0 \, ,$$

or $e \neq e(\mathsf{v})$, or $e = e(T)$ alone.

(VI) Energy equation

The first law of thermodynamics states that the rate of increase of energy in a closed system (fixed mass) is equal to the rate at which heat is added to the system plus the rate at which work is done on the system. Consider the material volume in Figure 2-5. The material volume contains the same mass all the time, and is thus a closed system. Let **q** be the heat flux vector, which is defined as the energy flux relative to the moving material. $-\mathbf{q}\cdot\mathbf{n}dS$ is the heat flux through an element of surface dS to material inside. The energy of the system is $\oiiint_{V(t)} \rho e_0 dV$, where e_0 is the total energy density (i.e., energy per unit mass), is made up of two parts : **e** and $\mathbf{u}\cdot\mathbf{u}/2$. **e** is the internal energy per unit mass, which includes the energy of molecules relative to the mean motion and strain energy etc.. $\mathbf{u}\cdot\mathbf{u}/2$ is the kinetic energy per unit mass, which is a result of the fluid motion. By applying the first law of thermodynamics to the material volume in Figure 2-5, we have

$$\frac{d}{dt}\oiiint_{V(t)} \rho(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u})dV = \oiint_{S(t)} -\mathbf{q}\cdot\mathbf{n}dS + \oiiint_{V(t)} \rho\mathbf{f}\cdot\mathbf{u}dV + \oiint_{S(t)} \mathbf{t}\cdot\mathbf{u}dS\,. \quad (2\text{-}60)$$

The first, the second and the third terms on the right hand of (2-60) are the total heat flux into the system, the rate of work done by the body force, and the rate of work done by the surface traction on the system, respectively. With the Reynolds transport theorem, the Cauchy stress principle and the divergence theorem, (2-60) becomes

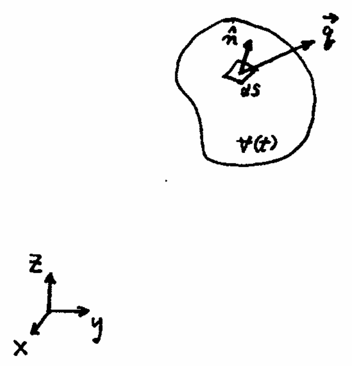


Figure 2-5: Sketch for illustrating the heat flux vector out of a local surface element of a material volume.

$$\oiiint_{V(t)} \rho \frac{d}{dt}(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u})dV = \oiiint_{V(t)} -\nabla\cdot\mathbf{q}dV + \oiiint_{V(t)} \rho\mathbf{f}\cdot\mathbf{u}dV + \oiiint_{V(t)} \nabla\cdot(\mathbf{T}\cdot\mathbf{u})dV \,. \quad (2\text{-}61)$$

Since $V(t)$ is arbitrary, (2-61) implies that

$$\rho\frac{d}{dt}(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}) = -\nabla\cdot\mathbf{q} + \rho\mathbf{f}\cdot\mathbf{u} + \nabla\cdot(\mathbf{T}\cdot\mathbf{u})\,, \quad (2\text{-}62)$$

which is the direct consequence of the first law of thermodynamics, and is called the “total energy equation”.

On the other hand, we may derive a kinetic (mechanical) energy equation as follows. By dotting the momentum equation, (2-11), with **u**, we have

$$\rho\frac{d\mathbf{u}}{dt}\cdot\mathbf{u} = \rho\mathbf{f}\cdot\mathbf{u} + (\nabla\cdot\mathbf{T})\cdot\mathbf{u}\,. \quad (2\text{-}63)$$

The last term can be manipulated using its component form as follows.

$$(\nabla\cdot\mathbf{T})\cdot\mathbf{u} = \mathbf{e}_l\frac{\partial}{\partial x_l}\cdot\mathbf{e}_i T_{ij}\mathbf{e}_j\cdot u_k\mathbf{e}_k = \frac{\partial T_{ij}}{\partial x_i}u_j = \frac{\partial}{\partial x_i}(T_{ij}u_j) - T_{ij}\frac{\partial u_j}{\partial x_i} = \nabla\cdot(\mathbf{T}\cdot\mathbf{u}) - \mathbf{T}:\nabla\mathbf{u}\,,$$

where $\mathbf{T}:\nabla\mathbf{u}$ is the scalar product of two tensors, and can be written further as

$$\mathbf{T}:\nabla\mathbf{u} = \mathbf{T}:(\mathbf{D}+\mathbf{\Omega}) = \mathbf{T}:\mathbf{D} = \mathbf{D}:\mathbf{T}\,,$$

since

$$\mathbf{T}:\mathbf{\Omega} = \mathbf{\Omega}:\mathbf{T} = \Omega_{11}T_{11} + \Omega_{12}T_{21} + \Omega_{13}T_{31} + \Omega_{21}T_{12} + \Omega_{22}T_{22} + \Omega_{23}T_{32} + \Omega_{31}T_{13} + \Omega_{32}T_{23} + \Omega_{33}T_{33}$$

$$= 0 + \Omega_{12}T_{21} - \Omega_{31}T_{13} - \Omega_{12}T_{21} + 0 + \Omega_{23}T_{32} + \Omega_{31}T_{13} - \Omega_{23}T_{32} + 0$$

$$= 0\,,$$

by using the conditions that $\mathbf{\Omega}$ is anti-symmetric and $\mathbf{T}$ is symmetric. Thus (2-63) finally becomes

$$\rho\frac{d}{dt}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) = \rho\mathbf{f}\cdot\mathbf{u} + \nabla\cdot(\mathbf{T}\cdot\mathbf{u}) - \mathbf{D}:\mathbf{T}\,, \quad (2\text{-}64)$$

which is called the kinetic (mechanical) energy equation.

By subtracting (2-64) from (2-62), we have

$$\rho \frac{de}{dt} = -\nabla \cdot \mathbf{q} + \mathbf{D} : \mathbf{T} , \tag{2-65}$$

which is called the internal energy equation. To see further the physical meaning of the term $\mathbf{D} : \mathbf{T}$, we decompose the stress tensor as

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau} , \tag{2-66}$$

then

$$\begin{aligned} \mathbf{D} : \mathbf{T} = \mathbf{D} : (-p\mathbf{I}) + \mathbf{D} : \boldsymbol{\tau} &= -p(D_{11} + D_{22} + D_{33}) + \mathbf{D} : \boldsymbol{\tau} = -p\nabla \cdot \mathbf{u} + \mathbf{D} : \boldsymbol{\tau} \\ &= -p\left(-\frac{1}{\rho}\frac{d\rho}{dt} \right) + \mathbf{D} : \boldsymbol{\tau} , \end{aligned} \tag{2-67}$$

where the continuity equation has been employed. The term $-p\nabla \cdot \mathbf{u}$ represents the rate of work done by compression (i.e., the $-pdv$ work in (2-51b)), which is the reversible part of the rate of work done by the traction force. The rest term, $\mathbf{D} : \boldsymbol{\tau}$, represents the irreversible part of the work, which will be dissipated as heat and is called the dissipation term in the literature. With (2-67), the internal energy equation, (2-65), becomes

$$\rho \frac{de}{dt} = -\nabla \cdot \mathbf{q} - p\left(-\frac{1}{\rho}\frac{d\rho}{dt} \right) + \mathbf{D} : \boldsymbol{\tau} . \tag{2-68}$$

Recall the Gibb's equation, (2-54a), we may rewrite (2-65) as

$$\rho T \frac{ds}{dt} = -\nabla \cdot \mathbf{q} + \mathbf{D} : \boldsymbol{\tau} , \tag{2-69}$$

which is another useful form of the internal energy equation, and is employed frequency in gas dynamics.

(VII) The entropy equation

The entropy equation derived below is a result of the second law of thermodynamics. We shall not employ the entropy equation to solve fluid mechanics problem directly, instead, we shall use it to put restriction on the constitutive equation discussed later in section **(IX)** of this chapter.

For a closed system as that in figure 2-5, the total entropy contained in the material volume is $\oiiint_{V(t)} \rho s dV$, and has the following property,

$$\frac{d}{dt}\oiiint_{V(t)} \rho s dV = \oiint_{S(t)} \frac{-\mathbf{q}\cdot\mathbf{n}}{T} dS + \oiiint_{V(t)} \rho \dot{s}_i dV \,. \tag{2-70}$$

The first term on the right hand side of (2-70) is the reversible part of the change of entropy within the material volume due to the heat addition. The second term on the right hand side of (2-70) represents the irreversible part, which has the characteristics that $\dot{s}_i \geq 0$. By using the Reynolds transport theorem and the divergence theorem, (2-70) becomes

$$\oiiint_{V(t)} \rho \frac{ds}{dt} dV = \oiiint_{V(t)} -\nabla\cdot\left(\frac{\mathbf{q}}{\mathrm{T}}\right) + \oiiint_{V(t)} \rho \dot{s}_i dV \,.$$

Since $V(t)$ is arbitrary, the above equation implies that

$$\rho \frac{ds}{dt} + \nabla\cdot\left(\frac{\mathbf{q}}{T}\right) = \rho \dot{s}_i \geq 0 \tag{2-71}$$

By substituting (2-69) into (2-71), we obtain

$$-\frac{\mathbf{q}\cdot\nabla T}{T^2} + \frac{\mathbf{D}:\boldsymbol{\tau}}{\mathrm{T}} = \rho \dot{s}_i \geq 0 \,. \tag{2-72}$$

Equation (2-72) puts a restriction on the forms of $\mathbf{q}$ and $\boldsymbol{\tau}$ in the constitute equations.

(VIII) Summary of equations

Here we summarize the equations governing the continuum fluid mechanics.

(i) Continuity equation (conservation of mass)

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{u} = 0 \,. \tag{2-4}$$

(ii) Momentum equation (conservation of momentum)

$$\rho \frac{d\mathbf{u}}{dt} = \rho\mathbf{f} - \nabla p + \nabla \cdot \boldsymbol{\tau} \ , \qquad (2\text{-}11a)$$

if the stress tensor decomposition, $\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau}$, is substituted into (2-11).

(iii) Energy equation (conservation of energy)

There are three useful forms of the energy equation as follows, and either one of them may be employed for solving the fluid mechanics problem.

(1) The total energy equation

$$\rho \frac{d}{dt}(e + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}) = -\nabla \cdot \mathbf{q} + \rho\mathbf{f} \cdot \mathbf{u} + \nabla \cdot (\mathbf{T} \cdot \mathbf{u}) \, . \qquad (2\text{-}62)$$

(2) The internal (or thermodynamic, or thermal) energy equation

$$\rho \frac{de}{dt} = -\nabla \cdot \mathbf{q} - p\left(-\frac{1}{\rho}\frac{d\rho}{dt}\right) + \mathbf{D} : \boldsymbol{\tau} \ . \qquad (2\text{-}68)$$

(3) The internal energy equation (using the Gibb's equation)

$$\rho T \frac{ds}{dt} = -\nabla \cdot \mathbf{q} + \mathbf{D} : \boldsymbol{\tau} \ . \qquad (2\text{-}69)$$

(iv) Equations of state

$$p = p(\rho, T) \, , \qquad (2\text{-}48a)$$

and

$$e = e(\rho, T) \, , \qquad (2\text{-}48b)$$

if (2-62) or (2-68) is employed, or

$$s = s(\rho, T) \, , \qquad (2\text{-}48c)$$

if (2-69) is employed.

(v) The entropy equation

$$-\frac{\mathbf{q}\cdot\nabla T}{T^2}+\frac{\mathbf{D}:\boldsymbol{\tau}}{\mathrm{T}}=\rho\dot{s}_i\geq 0\,, \qquad (2\text{-}72)$$

provides a restriction on the forms of **q** and **τ** in the constitute equations. It is not employed directly for solving the problem.

Now we have totally seven scalar equations: one from (2-4), three from (2-11a), one from (2-62), (2-68) or (2-69), one from (2-48a), and one from (2-48b) or (2-48c), but we have sixteen unknowns: ρ , **u** (3 components), *p*, **τ** (6 components), *e* or *s*, **q** (3 components) and T, provided that the body force, **f**, is specified. In order to make the number of unknowns equal the number of equations, we had to postulate certain relationships (9 scalar equations) for **τ** and **q** in terms of the deformation tensors, **D**, and temperature gradient, ∇T, respectively. Such relationships are called the constitutive equations; they vary for different materials (fluids), and are required to satisfy (2-72).

(IX) The constitutive equations

We have different constitute equations for different materials (fluids). Here we consider several kinds of fluids of interest.

(i) Perfect fluid (i.e., non-conductive and inviscid fluid)

The constitute equations are

$$\mathbf{q}=0 \qquad \text{and} \qquad \boldsymbol{\tau}=0\,, \qquad (2\text{-}73)$$

which implies that

$$\dot{s}_i=0 \qquad (2\text{-}74)$$

according to (2-72). All the dissipation mechanisms are neglected, and every process is reversible for the flow of a perfect fluid, i.e., the lower bound (equal sign) of the condition imposed by (2-72) is satisfied. With (2-73), the governing equations become

$$\frac{d\rho}{dt}+\rho\nabla\cdot\mathbf{u}=0\,, \qquad (2\text{-}75a)$$

$$\rho \frac{d\mathbf{u}}{dt} = \rho \mathbf{f} - \nabla p\,, \qquad \text{(Euler equation)} \qquad \text{(2-75b)}$$

$$\rho \frac{de}{dt} = -\,p\left(-\frac{1}{\rho}\frac{d\rho}{dt}\right) \quad \text{or} \quad \rho T \frac{ds}{dt} = 0\,, \qquad \text{(2-75c)}$$

$$p = p(\rho, T)\,, \qquad \text{(2-75d)}$$

and

$$e = e(\rho, T) \quad \text{or} \quad s = s(\rho, T)\,. \qquad \text{(2-75e)}$$

There are seven scalar equations for seven unknowns: ρ , **u** (3 components), *p*, *e* or *s*, and T, for given **f**. The perfect fluid is a good approximation for studying certain high-speed flow of gases, i.e., the gas dynamics problems.

(ii) Ideal fluid (i.e., incompressible and inviscid fluid)

The nomenclature of this fluid is adopted from Currie's book. The incompressible condition implies that

$$\rho = \text{constant}\,, \qquad \text{(2-76)}$$

which may be regarded as an equation of state. The constitutive equation is

$$\boldsymbol{\tau} = 0\,. \qquad \text{(2-77)}$$

With (2-76) and (2-77), the continuity and momentum equations become

$$\nabla \cdot \mathbf{u} = 0\,, \qquad \text{(2-78a)}$$

and

$$\rho \frac{d\mathbf{u}}{dt} = \rho \mathbf{f} - \nabla p\,, \qquad \text{(Euler equation)} \qquad \text{(2-78b)}$$

which are four scalar equations for four unknowns : **u** (3 components) and *p*, with known **f** and ρ. Here the energy equation is not required for solving **u** and *p*, and the constitutive equation for **q** is irrelevant if we do not consider the problem of heat transfer. The assumption of ideal fluid can be applied to study certain hydrodynamics problems.

(iii) Newtonian fluid (compressible and viscous fluid, called the "real" fluid in some texts)

Real fluids are compressible and have certain viscosity in general. The simplest model for the constitutive equation of a real fluid is the Newtonian fluid, which is a special case of the so-called Stokesian fluid. A Stokesian fluid is one for which the component of **τ** can be expressed as

$$\tau_{ij} = \tau_{ij}(D_{kl}, p, T)\,, \qquad \text{(2-79)}$$

where D_{kl} is the component of the deformation tensor. A Newtonian fluid is a special case of a Stokesian fluid in which (1) **τ** is a linear function of the components of **D**, and (2) there are no preferred direction properties (i.e., isotropy).

The most general linear function is

$$\tau_{ij} = \beta_{ijkl} D_{kl}\,, \qquad \text{(2-80)}$$

where β_{ijkl} is a fourth order tensor having 81 components, whose values depend on the two chosen independent thermodynamic variables, say, for example, the pressure, p, and the temperature, T. According to the theory of Cartesian tensor, the most general fourth order isotropic tensor can be written in terms of Kronecker delta as (see H. Jeffreys, "Cartesian tensors," Cambridge Univ. Press, 1984; or Y. C. Fung, "A first course in continuum mechanics," Prentice-Hall, 1969)

$$\beta_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}\,, \qquad \text{(2-81)}$$

where α, β and γ are functions of thermodynamic state, i.e., they depend on the two chosen independent thermodynamic variables. By substituting (2-81) into (2-80), one obtains

$$\begin{aligned}
\tau_{ij} &= (\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk})D_{kl}\,, \\
&= \alpha\delta_{ij}D_{kk} + \beta D_{ij} + \gamma D_{ji} \\
&= \alpha\delta_{ij}D_{kk} + (\beta+\gamma)D_{ij} \qquad \text{(since } D_{ij} = D_{ji}\text{)} \\
&= \lambda\delta_{ij}\nabla\cdot\mathbf{u} + 2\mu D_{ij}\,, \qquad \text{(2-82a)}
\end{aligned}$$

where

$$\lambda \equiv \alpha \qquad \text{and} \qquad \mu = (\beta + \gamma)/2$$

are called the second viscosity and the dynamic viscosity, which are determined experimentally in general. They are functions of temperature and pressure in principle, but depend mainly on temperature in practice. By substituting (2-82a) into the component form of (2-66), we have

$$T_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\nabla \cdot \mathbf{u} + 2\mu D_{ij} \,. \tag{2-82b}$$

Equation (2-82a) or (2-82b) relates the stress tensor to the deformation tensor, which is the constitutive equation for a Newtonian fluid. The vector forms of (2-82a) and (2-82b) are

$$\boldsymbol{\tau} = \lambda \mathbf{I}\nabla \cdot \mathbf{u} + 2\mu \mathbf{D} \,, \tag{2-82c}$$

and

$$\mathbf{T} = -p\mathbf{I} + \lambda \mathbf{I}\nabla \cdot \mathbf{u} + 2\mu \mathbf{D} \,, \tag{2-82d}$$

respectively.

The constitutive equation for the heat flux vector, **q**, is as follows. It is proposed that **q** is linearly proportional to the gradient of the temperature field according to the experimental observation. In component form, we write

$$q_i = K_{ij}\frac{\partial T}{\partial x_j} \,, \tag{2-83}$$

where K_{ij} is a second order tensor. Equation (2-83) can be simplified further by requiring that the conduction of heat in a fluid is isotropic. The most general form of a second order isotropic tensor can be written as (see Jeffrey or Fung's books as mentioned before)

$$K_{ij} = -k\delta_{ij} \,, \tag{2-84}$$

where k is function of thermodynamic state, called the thermal conductivity. By substituting (2-84) into (2-83), we have

$$q_i = -k\frac{\partial T}{\partial x_i}, \tag{2-85a}$$

or in vector form as

$$\mathbf{q} = -k\nabla T\,. \tag{2-85b}$$

Equation (2-85a) or (2-85b) is called the Fourier's law. The minus sign in the equations is used to represent the fact that the heat is transferred from a location at a higher temperature to that at a lower temperature.

By substituting (2-82c) and (2-85b) into (2-72), we found that the second law of thermodynamics implies that

$$\frac{k\nabla T\cdot\nabla T}{T^2}+\frac{\Phi}{T}\geq 0\,. \tag{2-86}$$

Here the energy dissipation, Φ, is defined as

$$\begin{aligned}\Phi \equiv \mathbf{D}:\boldsymbol{\tau} = \boldsymbol{\tau}:\mathbf{D} &= (\lambda\mathbf{I}\nabla\cdot\mathbf{u}+2\mu\mathbf{D}):\mathbf{D}\\ &= \lambda(\nabla\cdot\mathbf{u})\mathbf{I}:\mathbf{D}+2\mu\mathbf{D}:\mathbf{D}\\ &= \lambda(\nabla\cdot\mathbf{u})^{\mathbf{2}}+2\mu\mathbf{D}:\mathbf{D}\\ &= \lambda(D_{11}+D_{22}+D_{33})^2+2\mu(D_{11}{}^2+D_{12}{}^2+D_{13}{}^2\\ &\quad +D_{21}{}^2+D_{22}{}^2+D_{23}{}^2+D_{31}{}^2+D_{32}{}^2+D_{33}{}^2)\\ &= (\lambda+\frac{2}{3}\mu)(D_{11}+D_{22}+D_{33})^2+\frac{2}{3}\mu[(D_{11}-D_{22})^2+(D_{22}-D_{33})^2\\ &\quad +(D_{33}-D_{11})^2]+4\mu(D_{12}{}^2+D_{23}{}^2+D_{31}{}^2)\,,\end{aligned} \tag{2-87}$$

where the symmetric condition for $\mathbf{D}$ has been employed. By substituting (2-87) into (2-86), we found

$$k\geq 0\,,\qquad \mu\geq 0\qquad \text{and}\qquad \kappa\equiv\lambda+\frac{2}{3}\mu\geq 0\,, \tag{2-88}$$

as the absolute temperature, T, is always greater than zero. Here κ is called bulk viscosity, which is sometimes employed instead of the second viscosity, λ. Equation (2-88) put restrictions on the values of k, μ and λ, provided that the second law of thermodynamics is satisfied. These three material constants are determined by experiments in general, and are functions mainly of temperature in practice.

Example 6 : *Interpretation of μ, the viscosity coefficient.*

Consider simple shear flow in Cartesian coordinates *(x, y, z)* :

$$u = u(y)\,, \qquad v = 0\,, \qquad w = 0\,.$$

The flow is incompressible since $\nabla \cdot \mathbf{u} = 0$. The components of the stress tensor

$$\begin{aligned} T_{ij} &= -p\delta_{ij} + \lambda\delta_{ij}\nabla \cdot \mathbf{u} + 2\mu D_{ij} \\ &= -p\delta_{ij} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), \end{aligned}$$

or in tensor form

$$\mathbf{T} = -p\mathbf{I} + \mu\left(\frac{du}{dy}\mathbf{ij} + \frac{du}{dy}\mathbf{ji}\right).$$

The traction (force) on a surface element with normal **j** is

$$\mathbf{t}(\mathbf{j}) = \mathbf{j}\cdot\mathbf{T} = -p\mathbf{j} + \mu\frac{du}{dy}\mathbf{i}\,,$$

which consists a normal part $-p\mathbf{j}$ and a shearing part $\mu\frac{du}{dy}\mathbf{i}$. In general cases, the traction is

$$\begin{aligned} \mathbf{t}(\mathbf{j}) = \mathbf{j}\cdot\mathbf{T} &= T_{yx}\mathbf{i} + T_{yy}\mathbf{j} + T_{yz}\mathbf{k} \\ &= 2\mu D_{xy}\mathbf{i} + (-p + \lambda\nabla\cdot\mathbf{u} + 2\mu D_{yy})\mathbf{j} + 2\mu D_{yz}\mathbf{k}\,. \end{aligned}$$

The shearing forces along the *i* and *k* directions are $2\mu D_{xy}\mathbf{i}$ and $2\mu D_{yz}$ **k**, respectively. The normal force includes the pressure part $-p\mathbf{j}$ and an addition part $(\lambda\nabla\cdot u + 2\mu D_{yy})\mathbf{j}$, which is absent in the above example for simple shear flow.

The interpretation of the second viscosity, λ, or the bulk viscosity, $\kappa(\equiv \lambda + \frac{2}{3}\mu)$, is as follows. The average pressure, $\bar{p}$, can be evaluated as

$$\begin{aligned} -\bar{p} \equiv \frac{1}{3}T_{ii} = \frac{1}{3}\left(T_{11} + T_{22} + T_{33}\right) &= \frac{1}{3}\left[-3p + 3\lambda\nabla\cdot\mathbf{u} + 2\mu(D_{11} + D_{22} + D_{33})\right] \\ &= \frac{1}{3}\left[-3p + (3\lambda + 2\mu)\nabla\cdot\mathbf{u}\right]. \qquad \text{(2-89a)} \end{aligned}$$

Thus the difference between the thermodynamic and the average (mechanical) pressure

$$p - \overline{p} = (\lambda + \frac{2}{3}\mu)\nabla \cdot \mathbf{u}$$

$$= \kappa \nabla \cdot \mathbf{u} \qquad \text{(using definition of } \kappa\text{)}$$

$$= \kappa \left(-\frac{1}{\rho}\frac{d\rho}{dt} \right) \qquad \text{(using continuity)}$$

$$= \kappa \left(\frac{1}{v}\frac{dv}{dt} \right). \qquad \text{(with } v = 1/\rho\text{)} \qquad \text{(2-89b)}$$

The term $(dv/dt)/v$ is the rate of change of specific volume. The thermodynamic pressure equals the average (mechanical) pressure only when the fluid is incompressible. If p, $\overline{p}$ and $(dv/dt)/v$ are measurable, we can use (2-89b) to determine κ. However, it is hard to measure since it requires large value of $\nabla \cdot \mathbf{u}$ so that the measured data are sufficiently greater than the experimental errors.

Stokes proposed that

$$\kappa = 0, \qquad \text{or} \qquad \lambda = -\frac{2}{3}\mu. \qquad \text{(2-90)}$$

Equation (2-90) is called the Stokes relation, which is true for monotonic gasses according to the kinetic theory and experiment. In general, $\kappa \neq 0$, and is of the same order as μ.

With the constitutive equations (2-82d) and (2-85b) for Newtonian fluid, the governing equations become

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0, \qquad \text{(2-91a)}$$

$$\rho \frac{d\mathbf{u}}{dt} = \rho \mathbf{f} - \nabla (p - \lambda \nabla \cdot \mathbf{u}) + \nabla \cdot (2\mu \mathbf{D}), \qquad \text{(2-91b)}$$

$$\rho \frac{de}{dt} = \nabla \cdot (k \nabla T) - p \nabla \cdot \mathbf{u} + \Phi, \qquad \text{(2-91c)}$$

$$T = T(p, \rho), \qquad \text{(2-91d)}$$

and

$$e = e(p, \rho), \tag{2-91e}$$

if the internal energy *e* is chosen for solving the problem (one may choose the entropy, *s*, instead of *e*). The energy dissipation function

$$\begin{aligned}\Phi = \mathbf{D} : \boldsymbol{\tau} &= \lambda(\nabla \cdot \mathbf{u})^2 + 2\mu \mathbf{D} : \mathbf{D} \\ &= \lambda\left(\frac{\partial u_i}{\partial x_i}\right)^2 + 2\mu\left[\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\right]\left[\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\right].\end{aligned} \tag{2-92}$$

Equations (2-91a)-(2-91e) describes a set of seven scalar equations, which are solved for the seven unknown, ρ, **u** (3 components), *p*, *T* and *e*, provided the material parameters, μ, λ and *k*, are specified. These material parameters are expressed as functions of temperature in practice. Note that the two most common fluids, air and water, are well described as Newtonian fluid. Equation (2-91b) is known as the Navier-Stokes equation.

(iv) Incompressible fluid

The definition for incompressible fluid is

$$\rho = \text{constant}, \tag{2-93}$$

which may be regarded as an equation of state. The continuity equation implies $\nabla \cdot \mathbf{u} = 0$. The constitutive equation, (2-82d), then becomes

$$\mathbf{T} = -p\mathbf{I} + 2\mu \mathbf{D}. \tag{2-94}$$

The second viscosity disappears automatically in (2-94) since $\nabla \cdot \mathbf{u} = \mathbf{0}$, which is good news since the second viscosity is difficult to measure. The incompressible fluid described by (2-94) is an incompressible, viscous fluid, which is different from that of the ideal fluid (incompressible, and also inviscid) discussed before. Although all the fluids experience certain compressibility, many liquid and gas flow (at low Mach number) occur in nature and practice can be approximated as incompressible flow. A good discussion for the criteria for incompressibility can be found from Thompson's book (P. A. Thompson, "Compressible-fluid dynamics," McGraw-Hill, 1972, Chapter 3). Also for incompressible fluid, the equation of state, $e = e(\rho, T)$, becomes $e = e(T)$. Thus

$$e = C_v T \,. \tag{2-95}$$

if the definition of the specific heat at constant volume is employed. With (2-93), (2-94), (2-95) and (2-85b), the governing equations become

$$\nabla \cdot \mathbf{u} = 0 \,, \tag{2-96a}$$

$$\rho \frac{d\mathbf{u}}{dt} = \rho \mathbf{f} - \nabla p + \nabla \cdot (2\mu \mathbf{D}) \,, \tag{2-96b}$$

and

$$\rho C_v \frac{dT}{dt} = \nabla \cdot (k \nabla T) + \Phi \,, \tag{2-96c}$$

which describe five scalar equations for five unknowns : $\mathbf{u}$(3 components), p and T, for given μ and k. The viscosity μ and the thermal conductivity k are functions of temperature T in practice. Thus equations (2-96b) and (2-96c) are coupled through the temperature dependence of the viscosity coefficient, $\mu(T)$. If the temperature variation is sufficiently small so that μ and k can be approximated as constant, then we can solve (2-96a) and (2-96b) for $\mathbf{u}$(3 components) and p first, and then calculate T from (2-96c) with $\mathbf{u}$ as known variable. Equation (2-96b) is called the incompressible Navier-Stokes equation.

(X) Boundary and initial conditions

In order to solve the governing equations of the continuum fluid mechanics in the last section, we need to impose appropriate boundary and initial conditions for a given problem.

(i) Boundary conditions

Since there exist spatial derivative terms in the governing equations, it is required to specify certain values of the flow quantities on the boundaries of the fluid domain of interest. There are different kinds of boundary conditions as follows.

(a) At solid boundaries

Experimentally, it is found that the fluid adheres to the boundary surface, i.e.,

$$\mathbf{u} = \mathbf{u}_B , \tag{2-97a}$$

where $\mathbf{u}_B$ is the velocity of the solid boundary. Equation (2-97a) is called the no slip boundary condition, which is valid for fluid with nonzero viscosity. Thus (2-97a) is appropriate for the Navier-Stokes equation, i.e., the governing equations of the Newtonian fluid in (2-91b) and of the incompressible Newtonian fluid in (2-96b). If we put the coordinates on the solid boundary, or the boundary is stationary, $\mathbf{u} = \mathbf{u}_B = 0$, which is a special case of (2-97a).

For inviscid fluid, such as the perfect fluid and the ideal fluid discussed before, (2-97a) is too strong to satisfy since the Euler equation (i.e., (2-75b) and (2-78b)) is of one order less than the Navier-Stokes equation (i.e., (2-91b) or (2-96b)). Therefore, we had to modify (2-97a) as

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{u}_B \cdot \mathbf{n} , \tag{2-97b}$$

where $\mathbf{n}$ is the local unit normal vector of the solid boundary. Equation (2-97b) states that the fluid cannot penetrate the solid boundary, but we do allow the fluid to slip on the solid surface. Equation (2-97b) is thus called the slip boundary condition, or the no cross flow boundary condition. Physically, the fluid adheres to the solid boundary because of its viscous property. If the viscosity is turned off, the fluid can certainly slip on the solid surface.

For the energy equation, it is assumed that there is no temperature jump at solid boundaries, i.e.,

$$T = T_B , \tag{2-98}$$

for both viscous and inviscid fluid, where T_B is the temperature of the solid surface.

There is a little comment on the no slip boundary condition in (2-97a). Recall from chapter 1 that the Knudsen number is a measure of whether we should employ the continuum or the molecular approach to describe the fluid mechanics problem. The continuum assumption is valid when the Knudsen number is sufficiently less than unity. As the Knudsen number increases up to a certain value, say, 0.2, the fluid may slip partially on the solid surface, which accounts for certain molecular effect (see the text in Rarefied gas dynamics or Molecular fluid dynamics).

In some engineering applications such as those in the field of flow control in aerodynamics, the solid boundary may be porous so that the fluid may flow across the boundary. In such cases, the suction/blowing velocity which is normal to the boundary with magnitude v_B is given, and the boundary condition is (consider stationary boundary for example)

$$\mathbf{u}\cdot\mathbf{n} = v_{\mathbf{B}}, \quad \mathbf{u}\cdot\mathbf{n}_{\mathbf{t1}} = \mathbf{0}, \quad \mathbf{u}\cdot\mathbf{n}_{\mathbf{t2}} = \mathbf{0}, \qquad \text{(2-99)}$$

where $\mathbf{n}_{\mathbf{t1}}$ and $\mathbf{n}_{\mathbf{t2}}$ are the two local perpendicular tangential unit vectors on the surface.

(b) Far field conditions

For external flows, the velocity and the temperature far from the body are specified in general, i.e.,

$$\mathbf{u} = \mathbf{u}_\infty \quad \text{and} \quad T = T_\infty . \qquad \text{(2-100)}$$

However, for flow with finite inertia, a wake is formed in the downstream region behind the body, and (2-100) cannot be applied in the wake region. The wake problem will be discussed later in chapter 7.

(c) Inlet and/or outlet conditions

For some internal flows, the velocity and temperature and/or their spatial normal derivatives at the inlet and outlet are specified. Details of these conditions are always discussed in the course of computational fluid dynamics.

(d) At free surface or interface

A free surface is the boundary between the liquid and the air. An interface is the boundary between two different liquids. We may deal with such kind of boundary for many fluid mechanics problems, such as the motion of water waves, which is an important topic in ocean engineering and geophysical fluid dynamics. As the displacement of the free/interface is varying with time during the fluid motion, it is also an unknown, which had to be solved together with the governing equations. Therefore, a double condition (kinematic and dynamic) is imposed on the free/interface. We shall not study this kind of boundary condition in this course because of the tight schedule. Details of the condition can be found from chapter 6 of the book by Currie.

(ii) Initial conditions

If the flow is not steady, then it is necessary to specify the flow quantities having time derivatives at some specified time.

(XI) The vorticity equation

Recall that the vorticity (vector), ω, is a measure of the rotation of an infinitesimal fluid element, having its definition as $\nabla \times \mathbf{u}$. The flow is called irrotational if $\boldsymbol{\omega} = 0$. In many of the flows of interest, a large region of the flow domain is irrotational. Therefore, it may have certain advantage to study the flow in terms of vorticity. First, lets derive an equation governing the evolution of the vorticity field. Consider the Newtonian fluid with conservative body force, i.e., the body force can be represented by a scalar function, ϕ, through $\mathbf{f} = \nabla\phi$. The momentum equation, (2-11a), then becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = \nabla\phi - \frac{\nabla p}{\rho} + \frac{1}{\rho}\nabla\cdot\boldsymbol{\tau} \ . \qquad (2\text{-}101)$$

By taking curl operation of the above equation, and using the vector identity

$$\mathbf{u}\cdot\nabla\mathbf{u} = \nabla\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right) + \boldsymbol{\omega}\times\mathbf{u} \ ,$$

we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\times\nabla\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right) + \nabla\times(\boldsymbol{\omega}\times\mathbf{u}) = \nabla\times\nabla\phi - \nabla\times\left(\frac{\nabla p}{\rho}\right) + \nabla\times\left(\frac{\nabla\cdot\boldsymbol{\tau}}{\rho}\right). \qquad (2\text{-}102)$$

The second term on the left hand side and the first term on the right hand side of (2-102) are identity zero because the curl of the gradient of any scalar function is zero. The third term on the left hand side,

$$\nabla\times(\boldsymbol{\omega}\times\mathbf{u}) = \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\nabla\cdot\mathbf{u} - \mathbf{u}\nabla\cdot\boldsymbol{\omega}$$

$$= \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\nabla\cdot\mathbf{u}\,,$$

since $\nabla\cdot\boldsymbol{\omega} = \nabla\cdot\nabla\times\mathbf{u} = \mathbf{0}$ according to the vector identity. The second term on the right hand side of (2-102),

$$-\nabla\times\left(\frac{\nabla p}{\rho}\right) = -\frac{1}{\rho}\nabla\times\nabla p - \frac{\nabla p\times\nabla\rho}{\rho^2} = -\frac{\nabla p\times\nabla\rho}{\rho^2}$$

using again the fact that the curl of the gradient of a scalar function is zero. Finally, (2-102) becomes

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{\partial\boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} - \boldsymbol{\omega}\nabla\cdot\mathbf{u} - \frac{\nabla p\times\nabla\rho}{\rho^2} + \nabla\times\left(\frac{\nabla\cdot\boldsymbol{\tau}}{\rho}\right), \qquad (2\text{-}103)$$

which is called the Helmoltz equation. The equation says that the rate of change of vorticity following the fluid motion, $d\boldsymbol{\omega}/dt$, is due to the four mechanisms shown by the four terms on the right hand side of (2-103), namely, the vortex stretching term, the compression term, the baroclinic term, and the viscous (or diffusion) term, respectively. These mechanisms are illustrated as follows.

The vortex stretching term $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$ can be expressed as

$$\boldsymbol{\omega}\cdot\nabla\mathbf{u} = \boldsymbol{\omega}\cdot\mathbf{D} + \boldsymbol{\omega}\cdot\boldsymbol{\Omega} = \boldsymbol{\omega}\cdot\mathbf{D} + \frac{1}{2}\boldsymbol{\omega}\times\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\mathbf{D}$$
$$= \mathbf{e}_\alpha\omega_\alpha\frac{\partial u_\alpha}{\partial x_\alpha} + \mathbf{e}_\beta\omega_\beta\frac{\partial u_\beta}{\partial x_\beta} + \mathbf{e}_\gamma\omega_\gamma\frac{\partial u_\gamma}{\partial x_\gamma}\,, \qquad (2\text{-}104a)$$

which implies that the stretching/or shrinking along the principal axes, ($\mathbf{e}_\alpha$, $\mathbf{e}_\beta$, $\mathbf{e}_\gamma$), of the deformation tensor **D**, can increase/or decrease the vorticity. This mechanism is important in studying vortex dynamics and turbulence. For two-dimensional flow, the velocity $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ and the vorticity $\boldsymbol{\omega} = \omega\mathbf{k}$, then $\boldsymbol{\omega}\perp\nabla\mathbf{u}$ or $\boldsymbol{\omega}\cdot\nabla\mathbf{u} = \mathbf{0}$. There is no vortex stretching in two-dimensional flow.

The compression term $-\boldsymbol{\omega}\nabla\cdot\mathbf{u}$ can be written as

$$-\boldsymbol{\omega}\nabla\cdot\mathbf{u} = \frac{\boldsymbol{\omega}}{\rho}\frac{d\rho}{dt}$$

using continuity, which implies that the vorticity can be increased by compression ($d\rho/dt > 0$) or decreased by expansion ($d\rho/dt < 0$).

The baroclinic term $-(\nabla p \times \nabla\rho)/\rho^2$ is zero when p = constant, ρ = constant, and ∇p parallel to $\nabla\rho$. Otherwise, it is not zero and can generate vorticity.

The viscous (or diffusion) term implies that the vorticity can be diffused within the flow field due to the viscous effect of the fluid. Consider the case when both ρ and μ are taken as constants. The viscous term can be written as

$$\nabla \times \left(\frac{\nabla \cdot \boldsymbol{\tau}}{\rho}\right) = \nu\nabla \times \left[\nabla \cdot \left(2\mathbf{D}\right)\right] = \nu\nabla \times \left(\nabla \cdot \nabla\mathbf{u}\right) = \nu\nabla \cdot \nabla\left(\nabla \times \mathbf{u}\right) = \nu\nabla^2\boldsymbol{\omega}, \tag{2-104b}$$

which indeed in the form of a diffusion term, with the kinematic viscosity, ν, as the diffusion coefficient. Here $\nu = \mu/\rho$.

For incompressible fluid, ρ = constant; both the compression and the baroclinic terms are identically zero, and the vorticity equation, (2-103), becomes

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{\partial\boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}, \tag{2-104c}$$

if the viscosity is taken to be constant.

As discussed before, the baroclinic term can be regarded as the source term of vorticity. Vorticity can also be generated when there exists "discontinuity" of velocity, such as the encounter of two streams with different velocities. However, the most common vorticity source is associated with the no slip condition at the solid boundary. For viscous fluid, the velocity $\mathbf{u} = 0$ at solid boundary, but not the vorticity. As the vorticity is generated at the solid wall, it is diffused away into the flow through the diffusion term. The detailed evolution of vorticity then follows (2-103).

(XII) Bernoulli's equation

For inviscid fluid, including both the perfect fluid and ideal fluid discussed before, we can use the so-called Bernoulli's equation (an algebraic equation) to relate the velocity and the pressure field, instead of using the momentum equation (the Euler's equation), which is a partial differential equation. We have different forms of the Bernoulli's equation.

(i) Steady flow of a prefect fluid with conservative body force

The flow is not necessarily barotropic or irrotational, which are special cases of perfect fluid, but must have the condition of conservative body force. The flow is called barotropic if the pressure can be expressed as $p = p(\rho)$, which may be regarded as a special equation of state. Under the conditions of perfect fluid ($\mathbf{q} = 0$, $\boldsymbol{\tau} = 0$), steady flow ($\partial(\cdot)/\partial t = 0$), conservative body force ($\mathbf{f} = \nabla\phi$), the momentum equation becomes

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla\phi - \frac{\nabla p}{\rho}. \tag{2-105}$$

By dotting both sides of (2-105) by **u**, and using the vector identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u}), \tag{2-105a}$$

we have

$$\mathbf{u} \cdot \nabla(\frac{\mathbf{u} \cdot \mathbf{u}}{\mathbf{2}}) - \mathbf{u} \cdot \mathbf{u} \times (\nabla \times \mathbf{u}) = -\frac{\mathbf{u} \cdot \nabla p}{\rho} + \mathbf{u} \cdot \nabla\phi. \tag{2-106}$$

The second term in (2-106) is zero because the vector $\mathbf{u}$ is perpendicular to the vector $\mathbf{u} \times (\nabla \times \mathbf{u})$. Recall the Gibb's equation, we have

$$T\nabla s = \nabla h - \frac{1}{\rho}\nabla p.$$

As s is constant, or $\nabla s = 0$ for steady flow, the above equation implies

$$\frac{\mathbf{u} \cdot \nabla p}{\rho} = \mathbf{u} \cdot \nabla h.$$

Thus (2-106) becomes

$$\mathbf{u} \cdot \nabla\left(\frac{\mathbf{u} \cdot \mathbf{u}}{\mathbf{2}} + h - \phi\right) = 0, \tag{2-107a}$$

or

$$\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{2}}+h-\phi=\text{constant} \qquad \text{(2-107b)}$$

on a given streamline. The constant in (2-107b) is called the Bernoulli's constant. One may have different constants for different streamlines. If the body force is due to gravity, $\phi=-gz$, where g is the gravitational acceleration, and z is the vertical coordinate. In general, the body force can be neglected, and (2-107b) reduces to

$$h_0\equiv h+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}=\text{constant} \qquad \text{(2-107c)}$$

on a given streamline.

Consider the special case for incompressible fluid flows,

$$\frac{\nabla p}{\rho}=\nabla\left(\frac{p}{\rho}\right).$$

Equation (2-106) then becomes (without using the Gibb's equation)

$$\mathbf{u}\cdot\nabla\left(\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{2}}+\frac{p}{\rho}-\phi\right)=0, \qquad \text{(2-108a)}$$

which implies that

$$\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{2}}+\frac{p}{\rho}-\phi=\text{constant} \qquad \text{(2-108b)}$$

on a given streamline. By comparing (2-107b) and (2-108b), and using the definition $h=e+p/\rho$, we obtain

$$e=\text{constant} \qquad \text{(2-108c)}$$

on a given streamline for an incompressible flow.

(ii) Unsteady flow of an irrotational, incompressible, perfect fluid with conservative body force

For irrotational ($\boldsymbol{\omega} \equiv \nabla \times \mathbf{u} = 0$, thus $\mathbf{u} = \nabla \varphi$ according to the vector identity $\nabla \times \nabla \varphi = 0$), incompressible ($\rho = \text{constant}$), perfect fluid ($\mathbf{q} = 0$, $\boldsymbol{\tau} = 0$) with gravity as the conservative body force ($\mathbf{f} = -\nabla gz$), the momentum equation becomes

$$\frac{\partial}{\partial \mathbf{t}}(\nabla \varphi) + \nabla\left(\frac{\mathbf{u} \cdot \mathbf{u}}{\mathbf{2}}\right) = -\nabla\left(\frac{p}{\rho}\right) - \nabla gz\,,$$

with the application of the vector identity, (2-105a). The above equation can be rearrange as

$$\nabla\left(\frac{\partial \varphi}{\partial \mathbf{t}} + \frac{\mathbf{u} \cdot \mathbf{u}}{\mathbf{2}} + \frac{p}{\rho} + gz\right) = 0\,,$$

or

$$\frac{\partial \varphi}{\partial \mathbf{t}} + \frac{\mathbf{u} \cdot \mathbf{u}}{\mathbf{2}} + \frac{p}{\rho} + gz = f(t)\,, \qquad (2\text{-}109)$$

where $f(t)$ is an arbitrary function of time. Equation (2-109) applies for all the points in the flow field, not necessarily for points along a given streamline. If a point in the flow is steady, then $f(t) = \text{constant}$.

(XIII) The Croccos's equation

For steady inviscid flow with conservative body force, the momentum equation becomes

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times \boldsymbol{\omega} = -\frac{\nabla p}{\rho} + \nabla \phi\,. \qquad (2\text{-}110)$$

Here the fluid of interest need not be the perfect fluid, and we may have $\nabla s \neq 0$. For example, we are considering the flow through a shock wave, the fluid on both sides of the shock can be treated as perfect fluid, but the entropy on the downstream side is greater than that on the upstream side. By using the Gibb's equation,

$$T\nabla s = \nabla h - \frac{1}{\rho}\nabla p\,,$$

equation (2-110) becomes

$$\nabla\left(\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{2}}+h-\phi\right)=\mathbf{u}\times\boldsymbol{\omega}+T\nabla s\,, \qquad (2\text{-}111)$$

which is called the Crocco's equation. It states that both the rotational effect (i.e., $\boldsymbol{\omega}\neq 0$) and the non-isentropic effect (i.e., $s\neq \text{constant}$) can make the "Bernoulli constant" (in (2-107b)) not constant along a given streamline.

(XIV) Equations in a non-inertial frame

The governing equations of continuum fluid mechanics that we have discussed before is for the problem in an inertial frame. For some practical problems, it is better or necessary to study the problem in a non-inertial frame. The continuity equation, the energy equation, and the equations of state remain unchanged in a non-inertial frame. All we had to do is to modify the acceleration term of the momentum equation, since the Newton's second law applies only in an inertial frame. Recall from particle dynamics, the acceleration of a particle in an inertial frame, $\left.\frac{d\mathbf{u}}{dt}\right|_I$, can be expressed in terms of the quantities in a non-inertial frame as

$$\left.\frac{d\mathbf{u}}{dt}\right|_I=\frac{d\mathbf{u}_c}{dt}+2\boldsymbol{\Omega}\times\mathbf{u}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})+\frac{d\boldsymbol{\Omega}}{dt}\times\mathbf{r}+\frac{d\mathbf{u}}{dt}\,, \qquad (2\text{-}112)$$

where $\mathbf{r}$ and $\mathbf{u}$ are the position and velocity vector, respectively, in the non-inertial coordinates, $\mathbf{u}_c(t)$ is the translating velocity and $\boldsymbol{\Omega}$ is angular velocity of the non-inertial frame. Here $\boldsymbol{\Omega}$ is rotating about an arbitrary axis passing through the origin of the non-inertial coordinates. As a material fluid point can be identified as a particle, the term $\left.\frac{d\mathbf{u}}{dt}\right|_I$ in (2-112) will be employed to replace the term $\frac{d\mathbf{u}}{dt}$ in the momentum equation (for example, the Euler's equation in (2-75b) and the Navier-Stokes equation in (2-91b)), if the equation is studied in a non-inertial coordinates. The second and the third terms on the right hand side of (2-112) are called the Coriolis acceleration and the centrifugal acceleration, respectively.

Appendix – some useful vector identities (from Currie's book)

In the following formulas, ϕ is any scalar and **a**, **b** and **c** are any vectors.

$$\nabla \times \nabla\phi = 0$$

$$\nabla \cdot (\phi\mathbf{a}) = \phi\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla\phi$$

$$\nabla \times (\phi\mathbf{a}) = \nabla\phi \times \mathbf{a} + \phi(\nabla \times \mathbf{a})$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$(\mathbf{a} \cdot \nabla)\mathbf{a} = \frac{1}{2}\nabla(\mathbf{a} \cdot \mathbf{a}) - \mathbf{a} \times (\nabla \times \mathbf{a})$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$$

Homework

(1) Carry out the detailed derivations of equation (2-104a) and (2-104b).

(2) Write down the equations governing the velocity and pressure in the steady, two-dimensional flow of an inviscid, incompressible fluid in which gravity may be neglected. If the fluid is stratified, the density ρ will depend, in general, on both x and y coordinates. Show that the transformation

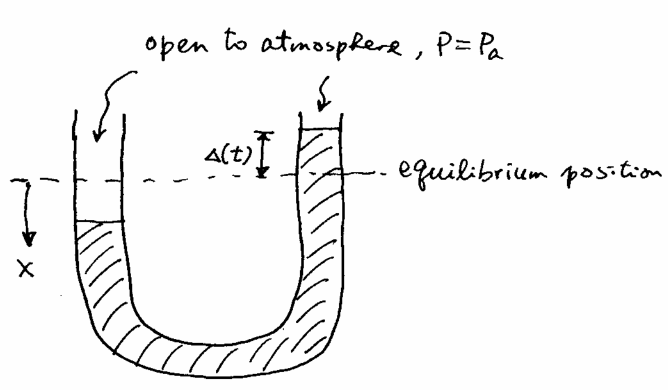
$$u^* = \sqrt{\rho / \rho_0}\,u\,, \quad v^* = \sqrt{\rho / \rho_0}\,v\,,$$

in which ρ_0 is a constant reference density, transforms these governing equations into those of a constant density fluid whose velocity components are u^* and v^*. (from Currie's book, problem 1.1)

(3) Derive the total energy equation when there is a volume source of heat, *h(r, t)*, per unit volume in addition to the heat flux vector. Assume both the Reynolds transport theorem and the Cauchy stress principle hold.

(4) Derive the continuity equation in spherical coordinates in two ways. (i) Starting from first principles using an infinitesimal control volume, which is fixed in space. To do this, set up a small elemental control volume in the spherical coordinate system, and identify the rates of inflow through each face of the control volume; setting the net inflow equal to the rate of mass accumulation in the element. (ii) Starting from the vector form of the continuity equation derived in the course note and using the divergence and/or gradient operators for spherical coordinates.

(5) Assume slug flow (i.e., $\mathbf{u} = \mathbf{u}(\mathbf{x},\mathbf{t})$ alone) in the constant area tube shown below. Also assume that the flow is irrotational, inviscid and incompressible. Use the unsteady Bernoulli's equation to determine the differential equation for $\Delta(t)$, where t is the time. What is the frequency of the oscillation? What is the distribution of pressure in the tube?



CHAPTER 3: EXACT SOLUTIONS OF THE GOVERNING EQUATIONS

The governing equations for continuum fluid mechanics derived in chapter 2 are sets of coupled nonlinear partial differential equations, which cannot be solved analytically in general. Actually, we have only limited number of analytical solutions of the governing equations under restricted conditions. Such analytical solutions are called the exact solutions.

Exact solutions, however simple and restricted, are essential because (1) it can disclose indisputable important features, and provide insight for construction of approximate solution, and (2) it can serve as test cases for plausible approximate and numerical method.

We shall consider here the incompressible viscous flow with constant viscosity since most of the exact solutions belong to this group. The governing equations are the Continuity and the Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0 , \quad \text{(3-1a)}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u} + \mathbf{g} , \quad \text{(3-1b)}$$

where $\nu = \mu / \rho$ is the kinematic viscosity of the fluid, which is taken to be constant in this chapter, and g is the gravitational acceleration. Here the gravitational force is taken to be the body force. Such body force can be removed from the governing equations if the modified pressure (or called the dynamic pressure) is employed, and the derivation is as follows. Consider first the hydrostatics problem. When u = 0, we set $p = p_{HS}$, and (3-1b) becomes

$$0 = -\nabla p_{HS} + \rho \mathbf{g} .$$

The hydrostatic pressure, p_{HS}, can be solved, and the result is

$$p_{HS} = p_0 + \rho \mathbf{g} \cdot \mathbf{r} ,$$

where p_0 is a constant, and r is the position vector. On setting

$$p = p_{HS} + \mathrm{P},$$

where P is called the modified pressure, or the dynamics pressure, equations (3-1a) and (3-1b) become

$$\nabla \cdot \mathbf{u} = 0, \tag{3-2a}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla \mathrm{P}}{\rho} + \nu \nabla^2 \mathbf{u}. \tag{3-2b}$$

The body force is absorbed into the pressure term. The above concept of modified pressure can be employed only under the following conditions: (1) $\rho = \text{constant}$ and the body force f = g, and (2) the boundary conditions involve u alone. This will always be so when the fluid under discussion is gas; but may not be so for liquid. For example, if there exists a free surface or interface, g re-enters the problem through the dynamic boundary condition at the interface (see Chapter 6 of Currie's book). The rest of this chapter is to find solutions of equations (3-2a) and (3-2b) under certain restricted conditions. The terms in (3-2b) are called the unsteady term, the convection term, the pressure term and the viscous (or diffusion) term, respectively.

(I) Flow with parallel streamlines

The major difficulty of solving the problem arises from the nonlinear convection term, $\mathbf{u} \cdot \nabla \mathbf{u}$, which can be written as

$$\mathbf{u} \cdot \nabla \mathbf{u} = q \frac{\partial}{\partial s}(q \mathbf{e}_s), \tag{3-3}$$

in a natural coordinates system shown in Figure 3-1. The natural coordinates system is a local Cartesian coordinates with unit vectors $\mathbf{e}_s$, $\mathbf{e}_{n1}$ and $\mathbf{e}_{n2}$, where $\mathbf{e}_s$ is parallel to the local velocity, and s is the coordinate along the streamline. The local velocity, u, can be expressed as $\mathbf{u} = q\mathbf{e}_s$ (the components in other directions are zero), and equation (3-3) is thus derived. If the magnitude of u, q, is not zero, then $\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{0}$ when both

$$\frac{\partial q}{\partial s}=0 \qquad \text{and} \qquad \frac{\partial \mathbf{e}_s}{\partial s}=0. \tag{3-4}$$

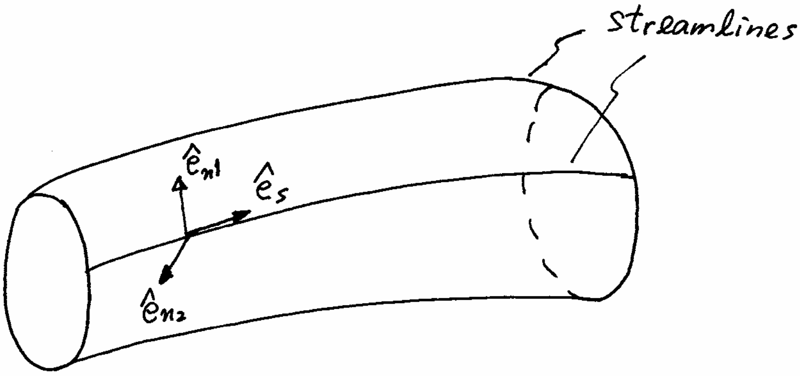


Figure 3-1: The natural coordinates system. The surface of the tube is a stream-surface, which is constructed by streamlines.

With the application of the integral form of the continuity equation of incompressible flow and the definition of streamline, the first equality in (3-4) implies that the stream tube in figure 2-1 does not change its cross section. The second equality in (3-4) implies that the streamline does not change its direction, except at q = 0 (the flow may reverse there). If the streamlines of the flow are parallel, both conditions in (3-4) are satisfied. Thus if the x-axis of the Cartesian coordinates is taken to be parallel to the streamlines, the velocity in (3-2a) and (3-2b) can be written as

$$\mathbf{u}=u(y,z,t)\mathbf{i}\,. \tag{3-5}$$

It follows that the continuity equation, (3-2a), is satisfied automatically by (3-5), and the momentum equation, (3-2b), becomes

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{3-6a}$$

$$0=-\frac{1}{\rho}\frac{\partial P}{\partial y}, \tag{3-6b}$$

and

$$0=-\frac{1}{\rho}\frac{\partial P}{\partial z}. \tag{3-6c}$$

Equations (3-6b) and (3-6c) imply that

$$P = P(x,t),$$

and thus

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = f_1(x,t). \qquad (3\text{-}7a)$$

On the other hand, with the functional form of u in (3-5), equation (3-6a) implies that

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = f_2(y,z,t). \qquad (3\text{-}7b)$$

Thus we have

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = f(t) \qquad (3\text{-}7c)$$

according to (3-7a) and (3-7b), and

$$P = -\rho x f(t) + P_o(t), \qquad (3\text{-}8)$$

where f(t) and $P_o(t)$ are arbitrary functions of time. Finally, the rest unknown, u(y,z,t), of the problem, is governed by

$$\frac{\partial u}{\partial t} = f(t) + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (3\text{-}9)$$

according to (3-6a) and (3-7c). Equation (3-9) is in the form of the heat equation with a time-dependent heat source. The solution of the steady and unsteady problem will be discussed separately. A collection of solutions of (3-9) for different geometries and boundary conditions can be found from the book "Conduction of heat in solids" by H. S. Carslaw and J. C. Jaeger (Second edition, Oxford University Press, 1959).

(i) Steady flow

When the flow is steady, u = u(y, z), f(t) = C = constant, and (3-9) becomes

$$\nabla^2 u = -\frac{C}{\nu}, \qquad (3\text{-}10)$$

where ∇^2 is the Laplacian operator in the plane perpendicular to the flow. For example,

$$\nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \text{and} \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

for Cartesian coordinates and cylindrical coordinates, respectively. Equation (3-10) is a Poisson equation, which is solved subject to Dirichlet boundary conditions. The general procedure is to write

$$u = u_1 + u_2,$$

such that

$$\nabla^2 u_1 = -\frac{C}{\nu} \tag{3-10a}$$

subject to the homogeneous boundary conditions, and

$$\nabla^2 u_2 = 0 \tag{3-10b}$$

subject to the inhomgeneous boundary conditions. The driving mechanism of the problem described by Equation (3-10) is the applied constant pressure gradient, C, which is balanced by the viscous effect.

Consider the steady flow through a long straight pipe of constant arbitrary cross section with contour D(y,z) = 0 as shown in Figure 3-2. The flow is driven by an imposed constant negative pressure gradient along the x-direction. For region sufficiently far from the inlet, the velocity profile can be assumed as parallel to the pipe wall and is independent of x, i.e., the flow is fully developed. It follows that the velocity is of the form $\mathbf{u} = u(y,z)\mathbf{i}$, and u(y,z) is governed by (3-10). The associated boundary condition is the no-slip condition at the wall, i.e., u = 0 on D(x,y) = 0.

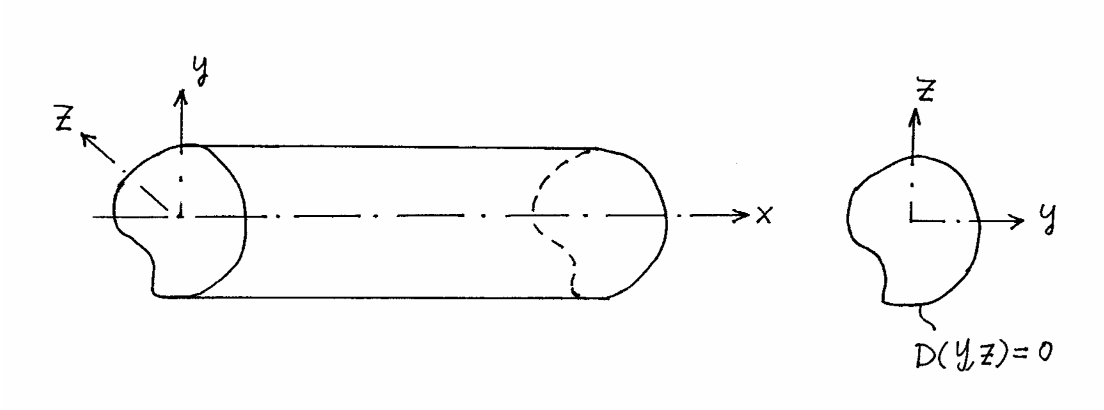


Figure 3-2: A long straight pipe of constant arbitrary cross section with contour D(y,z) = 0.

(a) Hagen-Poiseuille flow

Consider the special case when the cross section is circular, i.e., the steady fully developed flow through a circular straight pipe. This problem was first studied by Hagen (1839) and Poiseuille (1840), and was called the Hagen-Poiseuille flow. As the shape of the contour is circular, it is better to study the problem in polar coordinates, (r, θ). Equation (3-10) then becomes

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=-\frac{C}{\nu}, \qquad (3\text{-}11)$$

which is solved subject to

$$u(a,\theta)=0, \qquad (3\text{-}11a)$$

where a is the radius of the pipe. As the driving mechanism, i.e., the imposed constant pressure gradient, is independent of θ, and both the geometry of the pipe and the boundary condition are also axial symmetric, it is reasonable to expect that the velocity profile is axial symmetric, and takes the form u(r). Thus (3-11) reduces to

$$\frac{d^2 u}{dr^2}+\frac{1}{r}\frac{du}{dr}=\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=-\frac{C}{\nu}, \qquad (3\text{-}12)$$

which has general solution

$$u=-\frac{1}{4}r^2\left(\frac{C}{\nu}\right)+A\ln r+B.$$

The constant of integration, A equals zero since u is bounded at r = 0. The other constant, B, is determined by the no slip condition as described in (3-11a), and $B=(a^2/4)(C/\nu)$. Thus

$$u=\frac{1}{4}a^2\left(\frac{C}{\nu}\right)\left(1-\frac{r^2}{a^2}\right). \qquad (3\text{-}13a)$$

The centerline velocity, u_{CL}, is related to the applied constant pressure gradient through

$$u_{CL} = u(0) = \frac{1}{4}a^2\left(\frac{C}{\nu}\right) = -\frac{1}{4}a^2\left(\frac{1}{\mu}\frac{dP}{dx}\right).$$

Equation (3-13a) may then be written as

$$\frac{u}{u_{CL}} = 1 - \frac{r^2}{a^2}, \tag{3-13b}$$

which shows that the normalized velocity profile is parabolic. The shear stress at the wall,

$$\tau_w \equiv -\tau_{rz} = -\mu\frac{\partial u}{\partial r}\bigg|_{r=a} = \mu\frac{2u_{CL}}{a} = -\frac{a}{2}\frac{dP}{dx}.$$

This is expected since it may be written as

$$\tau_w(2\pi a) = -\pi a^2\frac{dP}{dx},$$

which is a direct consequence of the force balance of the flow in a cylindrical control volume with radius a and an infinitesimal axial length dx. If P, r, u and τ are normalized by $\rho {u_{CL}}^2$, a, u_{CL} and $\rho {u_{CL}}^2$, respectively, we found the normalized pressure gradient

$$\frac{d(P/\rho {u_{CL}}^2)}{d(x/a)} = -\frac{4}{R}, \tag{3-14a}$$

and the shear stress coefficient

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho {u_{CL}}^2} = \frac{4}{R}, \tag{2-14b}$$

where $R = \rho u_{CL} a/\mu$ is the Reynolds number. The volume flow rate, Q, and the average velocity, u_{av}, are defined through

$$Q \equiv u_{av}\pi a^2 \equiv \int_0^a 2\pi r u(r)dr. \tag{3-15}$$

With (3-13b), we found

$$Q = \frac{1}{2}\pi a^2 u_{CL} = -\frac{1}{8}\frac{\pi a^4}{\mu}\frac{dP}{dx} \qquad \text{(3-15a)}$$

and

$$u_{av} = \frac{1}{2}u_{CL}\,. \qquad \text{(3-15b)}$$

If Q and dP/dx are measurable, we can use (3-15a) to determine μ, which is the basis of one of the methods for determining the viscosity of the fluid. Flow in an annular pipe driven by an imposed axial pressure gradient can also be studied in a similar manner except the boundary condition is different.

(b) Flows through pipes with other cross sectional areas

Consider the elliptical cross section with major axes 2a and 2b. The boundary of the pipe in the yz-plane can be described by $y^2/a^2 + z^2/b^2 = 1$. The governing equation according to (3-10) is

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{C}{\nu} = \frac{1}{\mu}\frac{\partial P}{\partial x} = \text{constant}\,, \qquad \text{(3-16a)}$$

which is solved subject to

$$u = 0 \qquad \text{on} \qquad \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1\,. \qquad \text{(3-16b)}$$

According to the boundary condition, (3-16b), one may seek solution of form

$$u(y,z) = A\left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right), \qquad \text{(3-16c)}$$

which satisfies (3-16b) automatically. The constant A is determined by substituting (3-16c) into (3-16a), it follows that

$$A = -\frac{1}{\mu}\frac{\partial P}{\partial x}\left(\frac{1}{1/a^2 + 1/b^2}\right).$$

The solution for steady fully developed flows through pipes with other cross sectional areas, such as triangular pipes and square pipes, can be found in a similar way (see Problem 1 of the Homework).

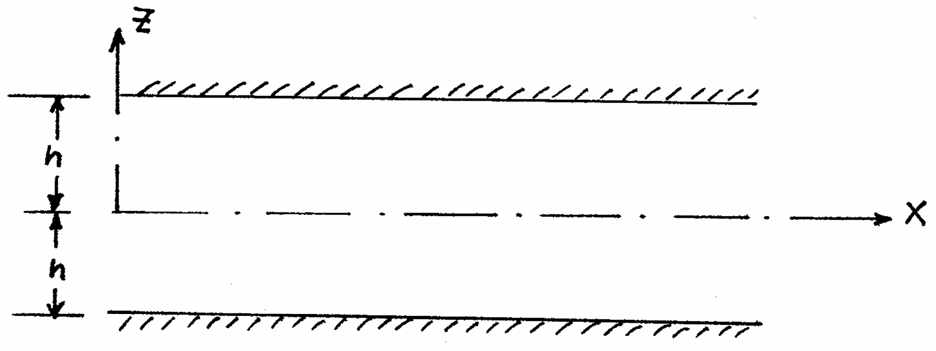


Figure 3-3: Illustration for plane Poiseuille flow. The flow is of infinite extent in the y-direction.

(c) Plane Poiseuille flow

This is the fully-developed flow in a two-dimensional channel with width 2h as shown in figure 3-3. The flow is driven by an applied constant negative pressure gradient along the x-direction. The flow is of infinite extent in the y-direction, and thus the velocity profile is independent of y, i.e., u = u(z) only. The governing equation according to (3-10) becomes

$$\frac{d^2u}{dz^2} = -\frac{C}{\nu} = \frac{1}{\mu}\frac{\partial P}{\partial x} = \text{constant}\,, \qquad (3\text{-}17a)$$

which is solved subject to

$$u = 0 \qquad \text{at} \qquad z = \pm h\,. \qquad (3\text{-}17b)$$

The solution is

$$\frac{u}{u_{CL}} = 1 - \left(\frac{z}{h}\right)^2, \qquad (3\text{-}17c)$$

which is also parabolic as that in the Hagen-Poiseuille flow in example 1. The centerline velocity, u_{CL}, is related to the applied pressure gradient through

$$u_{CL} = -\frac{h^2}{2\mu}\frac{\partial P}{\partial x}\,.$$

However, $u_{av} = (2/3)u_{CL}$, which is different from (3-15b) in Example 1.

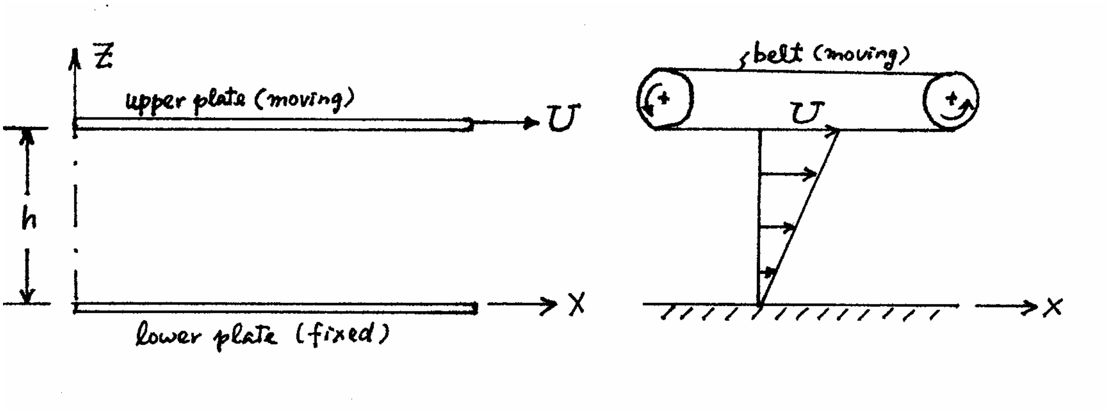


Figure 3-4: Illustration for the plane Couette flow. The figure on the right shows an experimental apparatus for generating a local Couette flow if the moving belt mechanism is immersed in a tank with sufficiently large extent in the x-direction.

(d) Plane Couette flow

The flow in the above three examples are driven by imposed axial pressure gradients. However, flows in channels/pipes can also be driven by the shear associated with the motion of the solid boundary. The simplest example is the so-called Couette flow, which is illustrated in Figure 3-4. Fluid is contained in a two-dimensional channel with width h. There is no imposed pressure contrast, and the flow is driven solely by the motion of the upper plate, which is translating with a constant speed U along its own plane in the x-direction. The governing equation for u(z) according to (3-10) is

$$\frac{d^2u}{dz^2} = 0, \qquad (3\text{-}18a)$$

which is solved subject to

$$u = 0 \qquad \text{at} \qquad y = 0, \qquad (3\text{-}18b)$$

and

$$u = U \qquad \text{at} \qquad y = h. \qquad (3\text{-}18c)$$

The solution is

$$\frac{u}{U} = \frac{z}{h}, \qquad (3\text{-}19d)$$

which is a linear profile. One may combine Example 3 and 4 to study the case when the two-dimensional fluid motion between two parallel plates driven by both the above mechanisms: the imposed pressure gradient and the shear associated with the motion of the upper plate. The resulting flow is called the Poiseuille-Couette flow, which forms the basis of the theory of lubrication.

(ii) Unsteady flow

The flow is unsteady because the driving mechanism is unsteady. There are two main types of unsteady problems, the transient problem and the quasi-steady problem associated with a periodic driving mechanism, which will be discussed later in this section. Similar as before, the driving mechanisms for the present unsteady flows with parallel streamlines include the imposed pressure gradient and the shear associated with the motion of the boundary in its own plane.

(a) Fluid motion above an infinite flat plate driven by the motion of the plate in its own plane

Consider the motion of the fluid above an infinite flat plate, which moves in its own plane as shown in Figure 3-5. The Cartesian coordinates in Figure 3-5 are chosen such that the plate is on the xz-plane and moves along the x-direction with arbitrary time varying speed U(t). The flow is driven solely by the shear associated with the motion of the plate, and there exists no imposed pressure gradient. According to the nature of the driving mechanism and the geometry of the flow domain, we may seek solution of form

$$u = u(y,t) . \qquad (3\text{-}20)$$

The governing equation according to (3-10) becomes

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} , \qquad (3\text{-}21)$$

which is solved subject to the boundary conditions,

$$u = U(t) \qquad \text{at} \qquad y = 0 , \qquad (3\text{-}21a)$$

and

$$u \text{ is bounded} \qquad \text{as} \qquad y \to \infty , \qquad (3\text{-}21b)$$

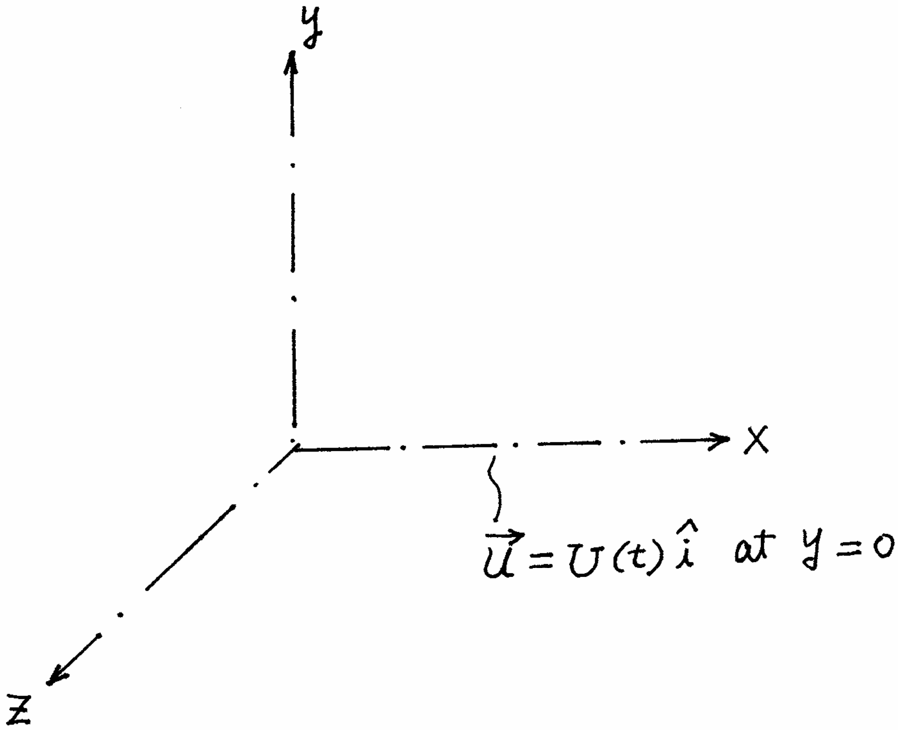


Figure 3-5: Fluid motion in the upper half space driven by the imposed motion of the boundary at y=0.

together with an initial condition

$$u = U_0(y) \qquad \text{at} \qquad t = 0\,. \tag{3-21c}$$

The above equation set can be solved by using the method of Laplace transform. The Laplace transform is defined as

$$\overline{u(y)} = \int_0^\infty e^{-st} u(y,t)\,dt\,,$$

which transforms $u(y,t)$ into $\overline{u(y)}$, with s as a parameter. By multiplying (3-21) by e^{-st}, and integrating both side from t = 0 to ∞, we have

$$-U_0(y) + s\bar{u} = \nu \frac{d^2\bar{u}}{dy^2}\,, \tag{3-22}$$

which is an ordinary differential equation for $\overline{u(y)}$ subject to the boundary conditions

$$\overline{u(0)} = \int_0^\infty e^{st} U(t) dt = \text{known}\,, \tag{3-22a}$$

and

$$\overline{u(y)} \text{ is bounded as } y \to \infty\,. \tag{3-22b}$$

Once $\overline{u(y)}$ is obtained, one may employ the inverse Laplace transform to obtain $u(y,t)$.

(1) Stokes' first problem

The simplest case is

$$U_0(y) = 0 \qquad \text{and} \qquad U(t) = U_C = \text{constant}\,, \tag{3-23}$$

for the initial and boundary conditions, respectively, which is called the Stokes' first problem. Physically, both the fluid and the plate are rest initially, then the infinite plate at y = 0 suddenly starts moving with constant velocity $U_C\mathbf{i}$ at its own plane. The problem is to find the transient response of the fluid motion due to the motion of the plate. Under the condition of (3-23), (3-22) becomes

$$\frac{\partial^2 \overline{u}}{\partial y^2} = \frac{s}{\nu}\overline{u}\,. \tag{3-24}$$

The associated boundary conditions are

$$\overline{u} = \frac{U_C}{s} \qquad \text{at} \qquad y = 0\,, \tag{3-24a}$$

and

$$\overline{u} \to 0 \qquad \text{as} \qquad y \to \infty\,, \tag{3-24b}$$

according to (3-22a) and (3-22b) together with the initial nature of the problem. The general solution of (3-24) is

$$\overline{u} = Ae^{-\sqrt{\frac{s}{\nu}}y} + Be^{\sqrt{\frac{s}{\nu}}y}\,. \tag{3-25}$$

The constant B is zero by applying (3-24b). The constant A equals to U/s according to (3-24a). Thus (3-25) becomes

$$\bar{u} = \frac{U_C}{s} e^{-\sqrt{\frac{s}{\nu}} y} . \tag{3-25a}$$

By looking up in the table of Laplace inverse transform, we found

$$\frac{u(y,t)}{U_C} = 1 - erf\left(\frac{y}{2\sqrt{\nu t}} \right), \tag{3-26}$$

where

$$erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-\eta^2} d\eta \tag{3-27}$$

is the error function, which increases rapidly to unity as x increases. By looking up at the mathematical handbook, we found $erf(2) = 0.995$, and practically, $erf(3) = 1$. The resulting normalized velocity profile according to (3-26) is sketched in Figure 3-6. The thickness of the fluid layer, y_L, dragged by the plate is (define $y = y_L$ when $u = 0.005U_C$; u is essentially zero outside the fluid layer)

$$\frac{y_L}{2\sqrt{\nu t}} \cong 2,$$

or

$$y_L \cong 4\sqrt{\nu t} . \tag{3-28a}$$

The length scale of the fluid layer is $\sqrt{\nu t}$, which increases with both the kinematic viscosity (ν) and the time (t), and is independent of the coordinate (x). The shear stress at the wall can be calculated as

$$\tau_w \equiv \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \mu U_C \left. \frac{\partial}{\partial y} \left(-\frac{2}{\sqrt{\pi}} \int_0^{\frac{y}{2\sqrt{\nu t}}} e^{-\eta^2} d\eta \right) \right|_{y=0} = \mu U_C \left(-\frac{2}{\sqrt{\pi}} \frac{1}{2\sqrt{\nu t}} \right) = -\frac{\mu U_C}{\sqrt{\pi \nu t}}, \tag{3-28b}$$

where the Leibniz rule has been employed for the derivation.

Remark: The Leibniz rule is as follows.

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x,t) dt = \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x} dt + \frac{db}{dx} f(x,b) - \frac{da}{dx} f(x,a)$$

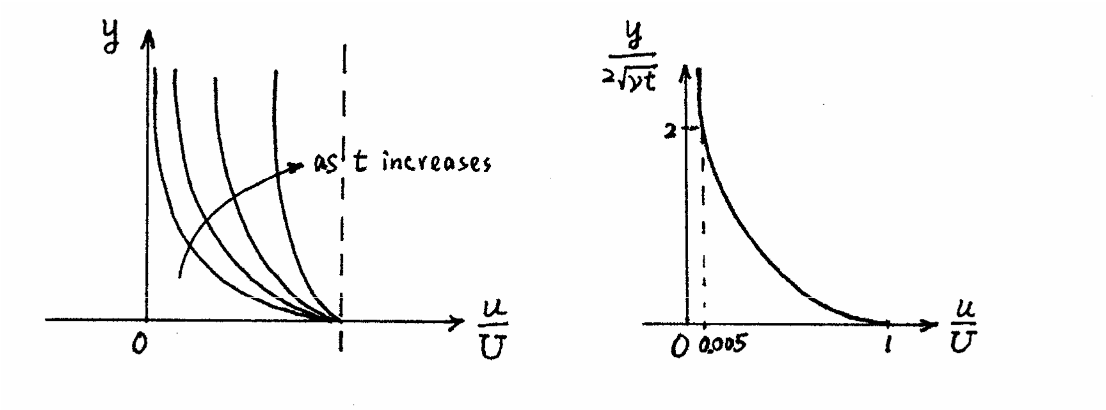


Figure 3-6: Sketch of the velocity profiles for the Stokes' first problem. The profiles at different times collapse into one if they are plotted in terms of the dimensionless variable, $y/(2\sqrt{\nu t})$, as shown on the right figure.

(2) Method of similarity transformation

An alternative method, called the method of similarity transformation, will be introduced here for solving the above Stokes' first problem. Although the method of similarity transformation finds no obvious advantage for solving the present linear problem, it may be employed to solve nonlinear problem under restricted conditions, and will be employed later in this chapter.

The Stokes' first problem is governed by

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{3-29}$$

with boundary conditions,

$$u = U \qquad \text{at} \qquad y = 0, \tag{3-29a}$$

and

$$u \to 0 \qquad \text{as} \qquad y \to \infty, \tag{3-29b}$$

together with an initial condition

$$u = 0 \qquad \text{at} \qquad t = 0. \tag{3-29c}$$

Before we start to solve the problem, lets estimate the terms in the governing equation, (3-29). If we take the velocity scale of the problem to be U_C, the time scale to be t since we are studying the transient response of the fluid motion, and set the length scale as y_s, we may estimate

$$\frac{\partial u}{\partial t} \sim \frac{U_C}{t} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \sim \frac{1}{y_S}\frac{U_C}{y_S} \sim \frac{U_C}{{y_S}^2},$$

and thus (3-29) implies

$$\frac{U_C}{t} \sim \nu \frac{U_C}{{y_s}^2}.$$

It follows that

$$y_s \sim \sqrt{\nu t}\,, \tag{3-29d}$$

which is actually the length scale discussed before in (3-28a). The achievement of the above crude analysis (called the scaling analysis) is remarkable since we have obtained the correct length scale of the problem without actually solving the differential equation. The scaling analysis is quite useful in the research work.

The nature of the length scale in (3-29) suggests that we may try to seek solution of dimensionless form as

$$\frac{u}{U_C} = F(\xi)\,, \qquad \text{with} \qquad \xi = \frac{y}{\sqrt{\nu t}}. \tag{3-30}$$

By using the chain rule for differentiation, we have

$$\frac{\partial u}{\partial t} = U_C \frac{dF}{d\xi}\frac{\partial \xi}{\partial t} = U_C \frac{dF}{d\xi}\left(-\frac{1}{2}\frac{y}{\sqrt{\nu}}t^{-\frac{3}{2}}\right) = -U_C \frac{1}{2}\frac{\xi}{t}\frac{dF}{d\xi},$$

$$\frac{\partial^2 u}{\partial y^2} = U_C \frac{\partial}{\partial y}\left(\frac{dF}{d\xi}\frac{\partial \xi}{\partial t}\right) = U_C \frac{\partial}{\partial y}\left(\frac{dF}{d\xi}\frac{1}{\sqrt{\nu t}}\right) = U_C \frac{d}{d\xi}\left(\frac{dF}{d\xi}\frac{1}{\sqrt{\nu t}}\right)\frac{\partial \xi}{\partial t} = U_C \frac{1}{\nu t}\frac{d^2 F}{d\xi^2},$$

and thus the partial differential equation, (3-29), becomes

$$-\frac{\xi}{2}\frac{dF}{d\xi} = \frac{d^2 F}{d\xi^2}, \tag{3-31}$$

which is an ordinary differential equation. Furthermore, the boundary condition, (3-29a), becomes

$$F(0) = 1, \qquad (3\text{-}31a)$$

and both the boundary condition, (3-39b), and the initial condition, (3-29c), imply

$$F(\infty) = 0. \qquad (3\text{-}31b)$$

After integration, the general solution of (3-31) is

$$F(\xi) = \int_0^{\xi} e^C e^{-\frac{1}{4}\zeta^2} d\zeta + D,$$

where C and D are the integration constants, and ζ is the dummy variable for integration. By applying the boundary conditions (3-31a) and (3-31b), we found

$$D = 1 \qquad \text{and} \qquad e^C = -\frac{1}{\sqrt{\pi}},$$

respectively. Thus the solution is

$$\frac{u(y,t)}{U_C} = F(\xi) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{\xi}{2}} e^{-\eta^2} d\eta = 1 - erf(\xi/2) = 1 - erf\left(\frac{y}{2\sqrt{\nu t}}\right), \qquad (3\text{-}32)$$

which is the same as that in (3-26) as expected.

In summary, the transformation described by (3-30) transforms the partial differential equation (3-29) with three conditions (3-29a-c) into an ordinary differential equation (3-31) with two conditions (3-31a-b). The type of the transformation described by (3-30) is called the similarity transformation, ξ is called the similarity variable, and the solution of the ordinary differential equation (3-31) subject to (3-31a) and (3-31b) is called the similarity solution. Although the solution procedure of an ordinary differential equation in general is much simpler than that of a partial differential equation, there are strict limitations on the method of similarity transformation. Mathematically, it is required that: (1) the partial differential equation transforms into an ordinary differential equation, and (2) the boundary and initial conditions transform into appropriate boundary conditions, including the situation that two conditions can collapse into one condition after the similarity transformation. Physically, it is required that there exists no imposed length (or time) scale in the flow.

(3) Solution for general plate motion

If both the fluid and the plate are rest initially, and the plate starts to move with an arbitrary velocity U(t)i when $t \geq 0$ (see Figure 3-7), we may use the method of superposition (with the solution of the Stokes' first problem as the basic solution; denoted the basic solution here in dimensionless form by $u_b(y,t)$) to build up the solution since the governing equation is linear. The result is

$$u(y,t) = \int_0^t u_b(y,t-t')\frac{dU(t')}{dt'}dt' + U(0)u_b(y,t), \qquad (3\text{-}33)$$

by using the Duhamel's theorem (see Carslaw & Jaeger, 1959).

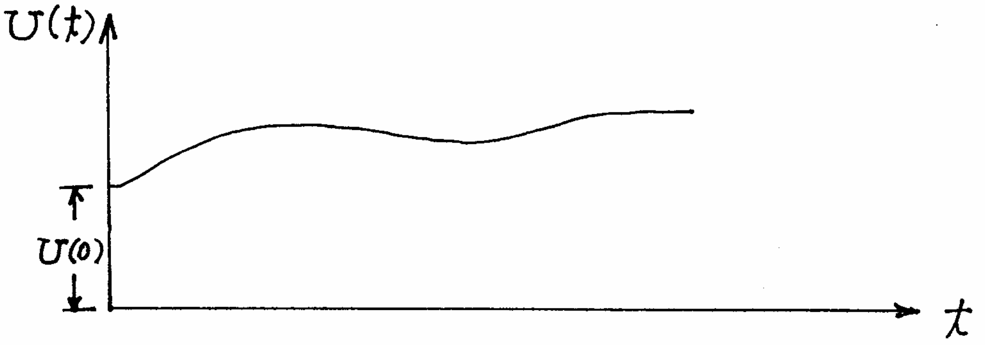


Figure 3-7: A sketch for the time varying speed of the plate.

Of course, this problem can also be solved by using the method of Laplace transform as discussed in (3-22). However, it is wonder if we can solve this problem using the method of similarity transformation, and the rest of this section is to study its possibility. As in (3-30), let

$$u(y,t) = U(t)f(\eta), \quad \text{with} \quad \eta = \frac{y}{\sqrt{\nu t}}, \qquad (3\text{-}34)$$

The governing equation becomes

$$\frac{d^2 f}{d\eta^2} + \frac{1}{2}\eta\frac{df}{d\eta} - 4\left(\frac{t}{U}\frac{dU}{dt}\right)f = 0. \qquad (3\text{-}35)$$

The boundary condition at y = 0 reduces to

$$f(0) = 1, \qquad (3\text{-}35a)$$

and both the boundary condition as $y \to \infty$ and the initial condition at $t = 0$ collapse into

$$f(\infty) = 0. \qquad (3\text{-}35b)$$

The coefficient of the last term in (3-35) is a function of time in general, and thus (3-35) is reduced to an ordinary differential equation only when

$$\frac{t}{U}\frac{dU}{dt} = M = \text{constant}, \tag{3-36a}$$

or

$$U = U_m \left(\frac{t}{t_m}\right)^M , \tag{3-36b}$$

where U_m and t_m are constants. Therefore, we have similarity solution for the case of time varying plate motion only when (3-36a) or (3-36b) is satisfied. In particular, if M = 1, (3-35) reduces to

$$\frac{d^2 f}{d\eta^2} + \frac{1}{2}\eta \frac{df}{d\eta} - 4f = 0 , \tag{3-36}$$

and the solution of (3-36) subject to (3-35a) and (3-35b) is

$$f(\eta) = (1+\eta^2)(1 - erf(\eta)) - \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2} . \tag{3-36a}$$

Solution can be obtained similarly for cases with other values of M.

(4) Stokes' second problem

Consider the case when the flow is driven by the harmonic oscillation of the plate. The velocity u(y,t) of the problem is governed by

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad , \tag{3-37}$$

with boundary conditions,

$$u(0,t) = U_0 \cos \omega t , \tag{3-37a}$$

$$u(\infty,t) = 0 , \tag{3-37b}$$

and initial condition,

$$u(y,0) = 0 . \tag{3-37c}$$

Here the amplitude and frequency of the oscillation, U_0 and ω, are taken to be constants. The solution of (3-37) subject to (3-37a-c) can be found by using the method of Laplace transform as studied before in

(3-22), and includes both a transient response part (which decays to zero at large time) and a long-term response part (which also oscillates harmonically but with phase shift). The Stokes' second problem is to study the long-term response of the fluid motion, i.e., the so-called quasi-steady solution.

In order to solve the quasi-steady solution of (3-37) subject to (3-37a-c), lets consider a complementary problem for $u_i(y,t)$ as follows.

$$\frac{\partial u_i}{\partial t} = \nu \frac{\partial^2 u_i}{\partial y^2} \quad , \tag{3-38}$$

with the boundary conditions

$$u_i(0,t) = U_0 \sin \omega t \, , \tag{3-38a}$$

$$u_i(\infty,t) = 0 \, , \tag{3-38b}$$

and initial condition

$$u_i(y,0) = 0 \, . \tag{3-38c}$$

Next, we consider a complex velocity defined by

$$\bar{u}(y,t) = u(y,t) + iu_i(y,t) \, , \tag{3-39}$$

where $i = \sqrt{-1}$. By adding (3-37) to i times (3-38), we obtain

$$\frac{\partial \bar{u}}{\partial t} = \nu \frac{\partial^2 \bar{u}}{\partial y^2} \, . \tag{3-40}$$

The associated boundary conditions are

$$\bar{u}(0,t) = U_0 e^{i\omega t} \, , \tag{3-40a}$$

$$\bar{u}(\infty,t) = 0 \, . \tag{3-40b}$$

The initial condition is

$$\bar{u}(y,0) = 0 \, , \tag{3-40c}$$

but is irrelevant to the solution since we are seeking the long-term response. Once we have solved $\bar{u}(y,t)$, we may obtained $u(y,t)$ by taking the real part of $\bar{u}(y,t)$. The solution procedures are as follows. First, the boundary condition, (3-40a), suggests that the solution takes the form,

$$\frac{\bar{u}(y,t)}{U_0} = g(y)e^{i\omega t}, \tag{3-41}$$

which may be regarded as a kind of separation of variables. By substituting (3-41) into (3-40), (3-40a) and (3-40b), we found

$$i\omega g = \nu \frac{d^2 g}{dy^2}, \tag{3-42}$$

with

$$g(0) = 1, \tag{3-42a}$$

and

$$g(\infty) = 0. \tag{3-42b}$$

The general solution of (3-42) is

$$g(y) = Ae^{\sqrt{\frac{i\omega}{\nu}}y} + Be^{-\sqrt{\frac{i\omega}{\nu}}y}.$$

With $\sqrt{i} = (1+i)/\sqrt{2}$, the boundary condition (2-42b) implies $A = 0$, and (2-42a) implies that $B = 1$. Thus

$$g(y) = e^{-\sqrt{\frac{\omega}{2\nu}}y} e^{-i\sqrt{\frac{\omega}{2\nu}}y},$$

and

$$u(y,t) = U_0 Re\left\{ e^{-\sqrt{\frac{\omega}{2\nu}}y} e^{-i\sqrt{\frac{\omega}{2\nu}}y} e^{i\omega t} \right\} = U_0 e^{-\sqrt{\frac{\omega}{2\nu}}y} \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y \right). \tag{3-43}$$

The amplitude of the fluid oscillation, $U_0 e^{-\sqrt{\frac{\omega}{2\nu}}y}$, decays exponentially as y increases. The effect of the boundary (i.e., the plate) oscillation is confined to a layer of finite thickness, δ. If the layer thickness is defined as

$$y = \delta \qquad \text{when} \qquad u = 0.01U_0,$$

then

$$\delta \cong 4.6\sqrt{2\nu/\omega}.$$

There also exists a phase difference, $\sqrt{\frac{\omega}{2\nu}}y$, between the plate and fluid oscillation.

By using the so-called unsteady Galilean transformation (see next section and Section 10.7 of Panton's book), the result in this sub-section can be applied to study the problem for "an oscillating stream over a stationary plate," and hence we can estimate "the thickness of the boundary layer for wave motion over a flat plate" (see Panton's book, p.269-272). Furthermore, the present result can also be applied to study the damping behavior of certain lateral vibrating structures in micro-electrical-mechanical-system (MEMS). For example, see Cho, Pisano and Howe, "Viscous damping model for laterally oscillating microstructures," Journal of Microelectromechanical systems, Vol.3, No.2, pp.81-87, 1994.

Finally, it is worth to point out that the method of similarity transformation cannot be applied in the present problem since there exists an imposed time scale, ω^{-1}. Such time scale corresponds to an imposed length scale $\sqrt{\nu/\omega}$, which is different from the length scale associated with viscous diffusion, $\sqrt{\nu t}$.

(5) Unsteady Galilean transformation

When a rigid body translates with a constant speed $\mathbf{U}_\infty$ in an unbounded fluid at rest, it is convenient to solve the problem in a coordinate system fixed at the body (Figure 3-8(b)) since the boundary condition can be applied easily. For example, if the body is a sphere with radius a, we may set $\mathbf{u}=0$ at $r=a$, where r is the radius component of the spherical coordinates fixed at the center of the sphere. However, if the problem is solved in a coordinates fixed at the surrounding stagnant fluid, we have $\mathbf{u}=\mathbf{U}_\infty$ on the body surface, where the body surface is described by a spherical surface which translates with time, and is thus a time varying function. The governing equations are the same for both the coordinates (x,y,z) in Figure 3-8(a) and the coordinates $(\hat{x},\hat{y},\hat{z})$ in Figure 3-8(b), which are fixed in space and at the body, and are called the fixed and translating coordinates, respectively.

Now, consider the case when the body is translating with a time varying speed $\mathbf{V}(t)$, instead of a constant speed, along a straight line as shown in Figure 3-8(c). We may also choose a translating coordinates system, $(\hat{x},\hat{y},\hat{z})$, which is fixed at the body, for solving the problem. It is interest to see if the governing equations are the same for different coordinates in the present unsteady translating problem. In the inertia coordinates (x,y,z) in Figure 3-8(c), the governing equations for the incompressible flow, written in index form, are

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \text{(3-44a)}$$

and

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}. \qquad \text{(3-44b)}$$

The associated boundary conditions are

$$u_i = V_i(t) \quad \text{on solid boundary,} \qquad \text{(3-44c)}$$

and

$$u_i = 0 \quad \text{far away from the body (i.e., at infinity).} \qquad \text{(2-44d)}$$

Here x_i are the components of x (x,y,z), u_i are the components of the fluid velocity u(x_i,t), and $V_i(t)$ are the components of V(t). The relations

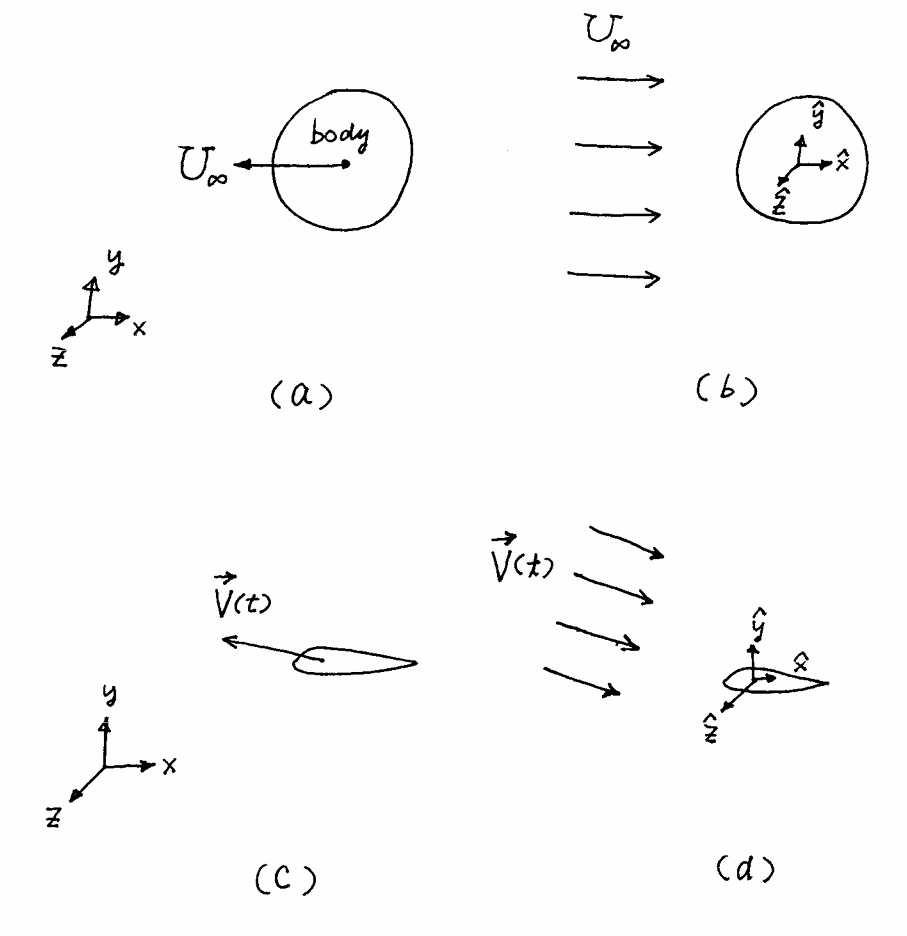


Figure 3-8 : Illustration for the steady (a, b) and the unsteady (c, d) Galilean transformations.

between the non-inertial translating coordinates $(\hat{x},\hat{y},\hat{z})$ in Figure 3-8(d) and the fixed coordinates (x,y,z) in Figure 3-8(c) are

$$\hat{x}_i = x_i - \int_0^t V_i(t')dt', \tag{3-45a}$$

and

$$\hat{u}_i = u_i - V_i(t)\,. \tag{3-45b}$$

Here $\hat{x}_i$ are the components of $\hat{\mathbf{x}}$ $(\hat{x},\hat{y},\hat{z})$. Equations (3-45a) and (3-45b) are called the unsteady Galilean transformation. By using (3-45a) and (3-45b) together with the chain rule, we can derive the governing equations in the translating coordinates $(\hat{x},\hat{y},\hat{z})$ from (3-44a-d) as

$$\frac{\partial \hat{u}_i}{\partial \hat{x}_i} = 0 \tag{3-46a}$$

and

$$\frac{\partial \hat{u}_i}{\partial t} + \hat{u}_j \frac{\partial \hat{u}_i}{\partial \hat{x}_j} = -\frac{\partial \hat{P}}{\partial \hat{x}_i} + \nu \frac{\partial^2 \hat{u}_i}{\partial \hat{x}_j \partial \hat{x}_j}, \tag{3-46b}$$

with boundary conditions

$u_i = 0$ on solid boundary, (3-46c)

and

$u_i = -V_i(t)$ far away from the body (i.e., at infinity). (3-46d)

where

$$\hat{P} = P + \rho \hat{x}_i \frac{dV_i}{dt}\,. \tag{3-47}$$

Thus the governing equations have the same form in both coordinates, but the pressure fields are different as described by (3-47). When $V_i(t) = U_\infty = \text{constant}$, (3-47) reduces to $\hat{P} = P$ as in the steady Galilean transformation described in Figures 3-8(a) and (b). An example is to employ the solution of the Stokes second problem to evaluate the damping of the water wave (see Problem 5 of the homework).

(b) Fluid motion inside pipes and channels

We have studied two types of unsteady problem in a semi-infinite domain in the last section, the transient problem via the Stokes' first problem, and the quasi-steady problem via the Stokes' second problem.

We may also have these two types of fluid motion inside pipes and channels.

(1) The transient problem

Consider the transient response of the set up of the Couette flow in Figure 3-9(a), i.e., the planar flow between two infinite rigid plates with the lower plate moved suddenly and the upper plate held fixed or vice versa. The governing equation is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{3-48}$$

with boundary conditions

$$u(0,t) = U_c = \text{constant}, \quad u(d,t) = 0 \quad \text{for} \quad t \geq 0, \tag{3-48a}$$

and initial condition

$$u(y,0) = 0 \quad \text{for} \quad 0 \leq y \leq d. \tag{3-48b}$$

The method of similarity transformation cannot be applied here since there exists an imposed length scale, d. Of course, the problem can be studied via the Laplace transform. However, we solve the problem here using the method of separation of variables. In order to apply the method of separation of variables for the present problem with non-homogeneous boundary condition, we first change the dependent variable into $w(y,t)$ by setting

$$w(y,t) = U_c\left(1 - \frac{y}{d}\right) - u(y,t). \tag{3-49}$$

Equations (3-48) and (3-48a,b) then become

$$\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial y^2}, \tag{3-49a}$$

with boundary conditions

$$w(0,t) = 0, \quad w(d,t) = 0 \quad \text{for} \quad t \geq 0, \tag{3-49b}$$

and initial condition

$$w(y,0) = U_c\left(1 - \frac{y}{d}\right) \quad \text{for} \quad 0 \leq y \leq d. \tag{3-49c}$$

By using the method of separation of variables, we found

$$w(y,t)=\sum_{n=1}^{\infty}A_n\exp\left(-n^2\pi^2\frac{\nu t}{d^2}\right)\sin\frac{n\pi y}{d}.$$

The coefficient, A_n, is determined by satisfying the initial condition with the application of the orthogonal property, and

$$A_n=\frac{2}{d}\int_0^d U_c\left(1-\frac{y}{d}\right)\sin\left(\frac{n\pi y}{d}\right)dy=\frac{2U_c}{n\pi}.$$

Thus

$$u(y,t)=U_c\left(1-\frac{y}{d}\right)-\frac{2U_c}{\pi}\sum_{n=1}^{\infty}\exp\left(-n^2\pi^2\frac{\nu t}{d^2}\right)\sin\frac{n\pi y}{d}, \qquad (3\text{-}49)$$

which is sketched in Figure 3-9(b). The first term on the right hand side of (3-49) is the steady part, which is the solution of the steady Couette flow studied before; the second term on the right hand side of (3-49) is a transient term, which decays to zero as $t\to\infty$.

The transient responses to the starting of the plane and circular Poiseuille flow with the applied pressure gradient as the driving mechanism can be studied in a similar way as above (see Problem 7 of the homework).

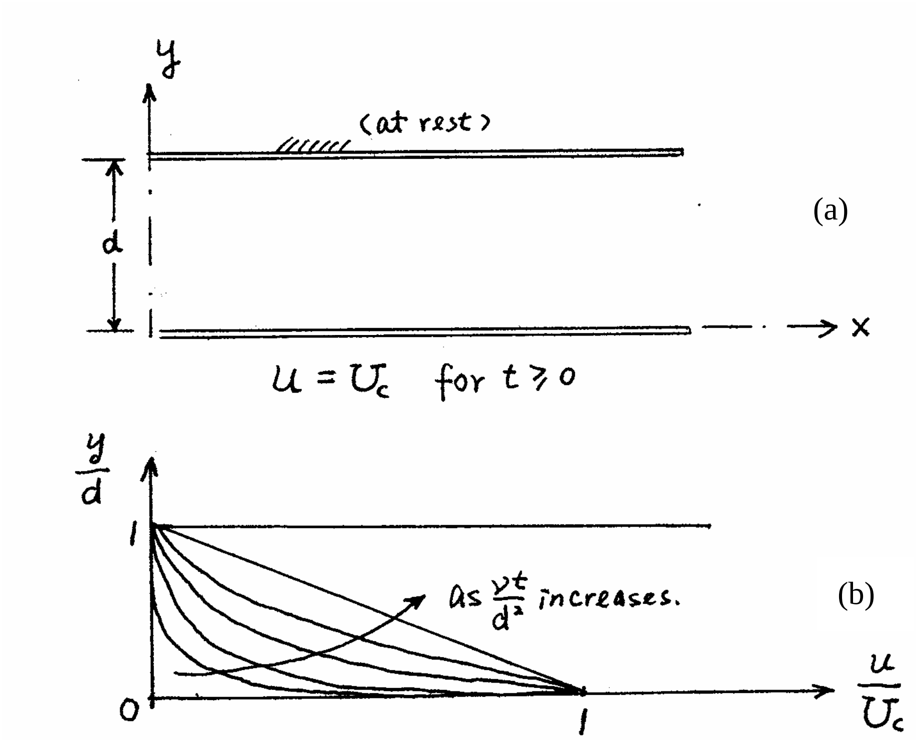


Figure 3-9: (a) Configuration of the transient of Couette flow. (b) Qualitative result of the velocity profiles according to (3-49).

(2) The quasi-steady problem

Here we consider the pulsating flow between two parallel plates (see Figure 3-10) driven by an imposed axial oscillating pressure gradient. A characteristic length scale, 2a (width of the channel), exists, and thus we are not expecting to have a similarity solution for this problem. However, we may apply the method using complex variables as that in the Stokes' second problem discussed above if our goal is to obtain the long-term behavior. The governing equation for the velocity component, u(y,t), along the x-direction in Figure 3-10 is

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}, \tag{3-50}$$

with the pressure gradient

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = K \cos \omega t = K \,\mathrm{Re}\left(e^{i\omega t}\right), \tag{3-50a}$$

and boundary conditions

$$u(a,t) = u(-a,t) = 0 \,. \tag{3-50b}$$

Here the amplitude and frequency, K and ω, of the driving pressure gradient are taken to be constants. Similar as before in the Stokes' second problem, let

$$u(y,t) = \mathrm{Re}\left(w(y)e^{i\omega t}\right), \tag{3-51}$$

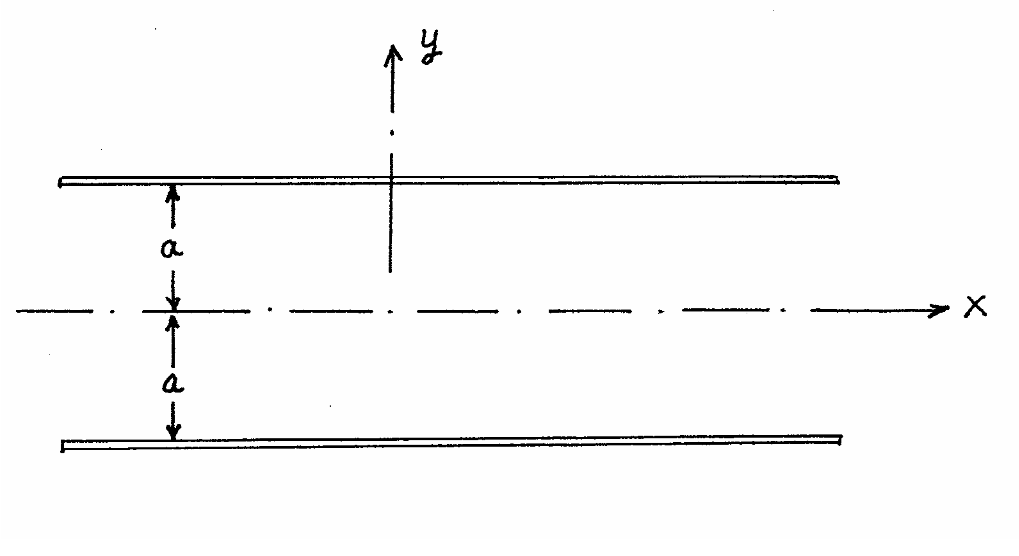


Figure 3-10: Configuration of the flow between two parallel plates driven by an imposed oscillating pressure gradient.

equations (3-50) and (3-50b) then imply

$$\frac{d^2 w}{dy^2} - \frac{i\omega}{\nu} w = \frac{K}{\nu}, \tag{3-52}$$

and

$$w(a) = w(-a) = 0. \tag{3-52a}$$

The general solution of (3-52) is (with $\sqrt{i} = (1+i)/\sqrt{2}$)

$$w(y) = i\frac{K}{\omega} + A\cosh\left[(1+i)\sqrt{\frac{\omega}{2\nu}}y\right] + B\sinh\left[(1+i)\sqrt{\frac{\omega}{2\nu}}y\right].$$

The constants of integration, A and B, are obtained by using (3-52a), and the results are

$$A = \frac{iK}{\omega\cosh\left[(1+i)\sqrt{\frac{\omega}{2\nu}}a\right]}, \qquad B = 0.$$

Thus

$$w(y) = i\frac{K}{\omega}\left\{1 - \frac{\cosh\left[(1+i)\sqrt{\frac{\omega}{2\nu}}y\right]}{\cosh\left[(1+i)\sqrt{\frac{\omega}{2\nu}}a\right]}\right\}, \tag{3-53}$$

and

$$u(y,t) = \mathrm{Re}\left(w(y)e^{i\omega t}\right) = \mathrm{Re}\left\{\frac{iK}{\omega}\left[1 - \frac{\cosh\left(\frac{1+i}{\sqrt{2}}N\frac{y}{a}\right)}{\cosh\left(\frac{1+i}{\sqrt{2}}N\right)}\right]e^{i\omega t}\right\}, \tag{3-54}$$

where $N \equiv \omega a^2/\nu$ is a dimensionless parameter governing the flow.

For small N,

$$\cosh\frac{1+i}{\sqrt{2}}N\frac{y}{a} \approx 1 + \frac{1}{2}\left(\frac{1+i}{\sqrt{2}}N\frac{y}{a}\right)^2 + \ldots\ldots \approx 1 + \frac{1}{2}iN^2\left(\frac{y}{a}\right)^2 + \ldots\ldots,$$

thus

$$u(y,t) \approx -\frac{1}{2}\frac{a^2}{\nu}\left(K\cos\omega t\right)\left(1-\frac{y^2}{a^2}\right), \qquad \text{(3-54a)}$$

which states that the quasi velocity has a parabolic profile. Equation (3-54a) is the same as the steady solution in (3-17c) if the steady pressure gradient is replaced by the periodic pressure gradient in (3-50a) under this low frequency limit (i.e., for $N << 1$).

For large N,

$$\cosh\left(\frac{1+i}{\sqrt{2}}N\frac{y}{a}\right) = \frac{1}{2}\left(e^{\frac{1+i}{\sqrt{2}}N\frac{y}{a}} + e^{-\frac{1+i}{\sqrt{2}}N\frac{y}{a}}\right) \approx \frac{1}{2}e^{\frac{1+i}{\sqrt{2}}N\frac{y}{a}},$$

and thus

$$u(y,t) \approx \mathrm{Re}\left\{\frac{iK}{\omega}\left[1-e^{\frac{1+i}{\sqrt{2}}N\left(\frac{y}{a}-1\right)}\right]e^{i\omega t}\right\}. \qquad \text{(3-54b)}$$

For region not close to the wall (called the core region), (3-54b) further reduces to (note that y/a < 1)

$$u(y,t) \approx \mathrm{Re}\left\{\frac{iK}{\omega}e^{i\omega t}\right\} = \frac{-K\sin\omega t}{\omega} \qquad \text{(3-54c)}$$

under the condition $N \to \infty$, which is also the solution of (3-50) with the viscous term set to zero. Thus the flow in the core region is essentially inviscid. For region next to the wall (called the boundary layer region), we may re-scale the problem by setting

$$\eta = N\frac{a-y}{a},$$

such that $\eta = O(1)$ within the boundary layer region. Equation (3-54b) reduces to

$$u(y,t) = \mathrm{Re}\left\{\frac{iK}{\omega}\left(1-e^{-\frac{1+i}{\sqrt{2}}\eta}\right)e^{i\omega t}\right\} \approx -\frac{K\sin\omega t}{\omega} + \frac{K}{\omega}e^{-\frac{\eta}{\sqrt{2}}}\sin\left(\omega t - \frac{1}{\sqrt{2}}\eta\right). \qquad \text{(3-54d)}$$

The first term on the right hand side of (3-54d) is the same as the flow in the core region in (3-54c), the second term accounts for the viscous effect, which is required for $u(y,t)$ to satisfy the no-slip boundary condition at the wall.

Other examples for quasi-steady flows include the pulsating flow in a circular straight pipe driven by an imposed periodic pressure gradient, the oscillating plane Couette flow driven by the harmonic oscillation of the upper and/or lower plate(s), and the pulsating flow in a circular straight pipe driven by the axial oscillation of the pipe wall.

(II) Flow with circular streamlines

We have found that the nonlinear convection term in the Navier-Stokes equation is identically zero for cases with parallel streamlines. When the streamlines are curved, the convection term is not zero in general. However, for cases with circular streamlines, the equation governing the azimuthal velocity component is linear although the convection term is nonzero. Consider the planar flow for example. Let's choose a polar coordinates system (r,θ) with velocity components (u_r,u_θ). The governing equations are the continuity equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}=0\,, \tag{3-55a}$$

the r-component Navier-Stokes equation,

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{{u_\theta}^2}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}$$
$$+\nu\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right), \tag{3-55b}$$

and the θ-component Navier-Stokes equation,

$$\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}=-\frac{1}{\rho r}\frac{\partial P}{\partial \theta}$$
$$+\nu\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right), \tag{3-55c}$$

For flows with closed circular streamlines, we may seek solution with

$$u_r = 0\,. \tag{3-56a}$$

It follows from the continuity equation, (3-55a), that

$$u_\theta = V(r,t)\,. \tag{3-56b}$$

The radial component Navier-Stokes equation, (3-55b), then reduces to

$$-\frac{V^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}\,, \tag{3-57a}$$

which shows a balance between the centrifugal force and the force associated with the radial pressure gradient. In particular, the convection term reduces to $-V^2/r$, which is not zero in general. Also the θ-component Navier-Stokes equation, (3-55c), reduces to

$$\frac{\partial V}{\partial t} = -\frac{1}{\rho r}\frac{\partial P}{\partial \theta} + \nu\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2}\right). \tag{3-57b}$$

With the functional dependence of V in (3-56b), (3-57a) implies that

$$P = f(r,t) + g(\theta,t)\,, \tag{3-58a}$$

or

$$\frac{\partial P}{\partial \theta} = \frac{\partial g}{\partial \theta} = g_1(\theta,t)\,, \tag{3-58b}$$

where $f(r,t)$ and $g(\theta,t)$ are arbitrary functions. With the functional relationships in (3-56b) and (3-58b), (3-57b) implies that

$$\frac{\partial P}{\partial \theta} = C_1(t) \tag{3-58c}$$

alone, where $C_1(t)$ is an arbitrary function. After integration, we have

$$P = C_1(t)\theta + C_2(r,t) \tag{3-58d}$$

according to (3-58c), where $C_2(r,t)$ is an arbitrary function. As $P(r,\theta,t) = P(r,\theta + 2\pi,t)$, it is required that $C_1(t) = 0$. Thus

$$P = C_2(r,t)\,, \tag{3-58e}$$

or

$$\frac{\partial P}{\partial \theta} = 0\,. \tag{3-58f}$$

With (3-58f), (3-57b) reduces to

$$\frac{\partial V}{\partial t} = \nu\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2}\right) = \nu\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\frac{V}{r}\right)\right), \tag{3-59}$$

which is a linear equation governing $V(r,t)$. After we have obtained $V(r,t)$, we may calculate the pressure field by integrating (3-57a).

(i) Steady flow

For steady flow, (3-59) reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\frac{V}{r}\right)\right) = 0\,, \tag{3-60}$$

which has general solution

$$V = \frac{b}{r} + ar\,, \tag{3-61}$$

where a and b are constants of integration. We have the following three situations according to different boundary conditions.

(1) Flow inside an infinitely long circular cylinder with radius R_2 which is rotating about its own axis with constant angular speed Ω_2

We have $b = 0$ for finite solution at $r = 0$ and $a = \Omega_2$ according to the no-slip condition. Thus

$$V = r\Omega_2\,, \tag{3-61a}$$

which is the same as a solid body rotation.

(2) Flow outside an infinite long circular cylinder with radius R_1 which is rotating about its own axis with constant angular speed Ω_1

We have $a = 0$ for finite solution as $r \to \infty$ and $b = R_1^2 \Omega_1$ according to the no-slip condition. Thus

$$V = \frac{R_1^2 \Omega_1}{r}, \tag{3-61b}$$

which describes the velocity field of a vortex.

(3) Flow between two infinite concentric long circular cylinders with inner radius R_1 and outer radius R_2 rotating at constant angular speed Ω_1 and Ω_2, respectively.

The constants of integration, a and b, are determined according to the no-slip conditions

$$V = R_1 \Omega_1 \qquad \text{at} \qquad r = R_1,$$

and

$$V = R_2 \Omega_2 \qquad \text{at} \qquad r = R_2.$$

The result is

$$V = \left(\frac{\Omega_1 - \Omega_2}{1/R_1^2 - 1/R_2^2} \right) \frac{1}{r} + \frac{\Omega_1 R_1^2 - \Omega_2 R_2^2}{R_1^2 - R_2^2} r, \tag{3-61c}$$

for $R_1 \le r \le R_2$. As $R_2 \to \infty$ and with $\Omega_2 = 0$, (3-61c) reduces to (3-61b). As $R_1 \to 0$, (3-61c) reduces to (3-61a). With $\Omega_1 = 0$ and $R_1 \to R_2$ (i.e., $R_2 - R_1 << R_1 \approx R_2$), (3-61c) reduces to

$$V \approx R_2 \Omega_2 \left(\frac{r - R_1}{R_2 - R_1} \right), \tag{3-62}$$

which is the same as the linear velocity profile of the plane Couette flow in (3-19d) if $R_2 \Omega_2$, $r - R_1$ and $R_2 - R_1$ are identified as U, z and h, respectively in (3-19d). Thus we may generate a Couette flow (called the circular Couette flow) in the gap between two concentric circular cylinders if the outer cylinder is rotating with constant speed and the inner cylinder is held fixed, provided that the gap width is sufficiently less than the radii of the cylinders. Such apparatus (concentric cylinders) can generate "better" Couette flow than the apparatus shown in figure 3-4 since there exists no end effect of the apparatus.

For $\Omega_2 = 0$, (3-61c) reduces to

$$V = \left(\frac{\Omega_1}{1/R_1^2 - 1/R_2^2} \right) \frac{1}{r} + \frac{\Omega_1 R_1^2}{R_1^2 - R_2^2} r . \qquad \text{(3-63a)}$$

The shear stress at the inner cylinder is

$$\tau_{r\theta} = \mu \left(r \frac{\partial}{\partial r} \left(\frac{V}{r} \right) \right)_{r=R_1} = -2\mu \frac{R_2^2}{R_2^2 - R_1^2} \Omega_1 . \qquad \text{(3-63b)}$$

The shear force per unit length of the cylinder is

$$F = 2\pi R_1 \tau_{r\theta} = -4\pi\mu \frac{\Omega_1 R_1^2 R_2^2}{R_2^2 - R_1^2} . \qquad \text{(3-63c)}$$

For a given apparatus, i.e., Ω_1, R_1 and R_2 are known, if F is measurable, we can use (3-63c) to determine the viscosity of the fluid, μ. Finally, the rate of work done by the inner cylinder per unit length is

$$W = -F(\Omega_1 R_1) = 4\pi\mu \frac{\Omega_1^2 R_1^2 R_2^2}{R_2^2 - R_1^2} . \qquad \text{(3-64d)}$$

On the other hand, the dissipation function

$$\Phi = \mu \left[r \frac{\partial}{\partial r} \left(\frac{V}{r} \right) \right]^2 = 4\mu \frac{R_2^4 R_1^4 \Omega_1^2}{\left(R_2^2 - R_1^2 \right)^2} \frac{1}{r^4} ,$$

and the total energy dissipation per unit axial length of the cylinder is

$$\int_{R_1}^{R_2} 2\pi r \Phi dr = 4\mu \frac{R_2^4 R_1^4 \Omega_1^2}{\left(R_2^2 - R_1^2 \right)^2} \frac{1}{r^4} , \qquad \text{(3-64e)}$$

which equals the rate of work done in (3-64d) for this steady problem as expected.

(ii) Unsteady flow

The general unsteady flow is governed by (3-59) with appropriate boundary and initial conditions. We may also have both the transient and quasi-steady problems as discussed before. The decay of an ideal line vortex (called the Oseen vortex) is an example for the transient problem, which can be found from Chapter 11 of the book by Panton.

(III) Nonlinear problems

When the streamlines are not straight and parallel as those in (I) and not circular and closed as those in (II), the convection term is not zero, and the governing equations for the flow are nonlinear. There are only limited numbers of exact solutions of this kind, and some of them are discussed as follows.

(i) Radial flow: Steady flow between two non-parallel plane walls

Consider the steady planar converging (flow toward the origin, r = 0) or diverging (flow outward from the origin) flow between two non-parallel planes as shown in Figure 3-11 with the volume flow rate, Q, fixed. Q > 0 for diverging flow and Q < 0 for converging flow. The flow is driven by the applied pressure contrasts at both ends, or may be regarded that the flow is induced by an imposed line source or sink at the

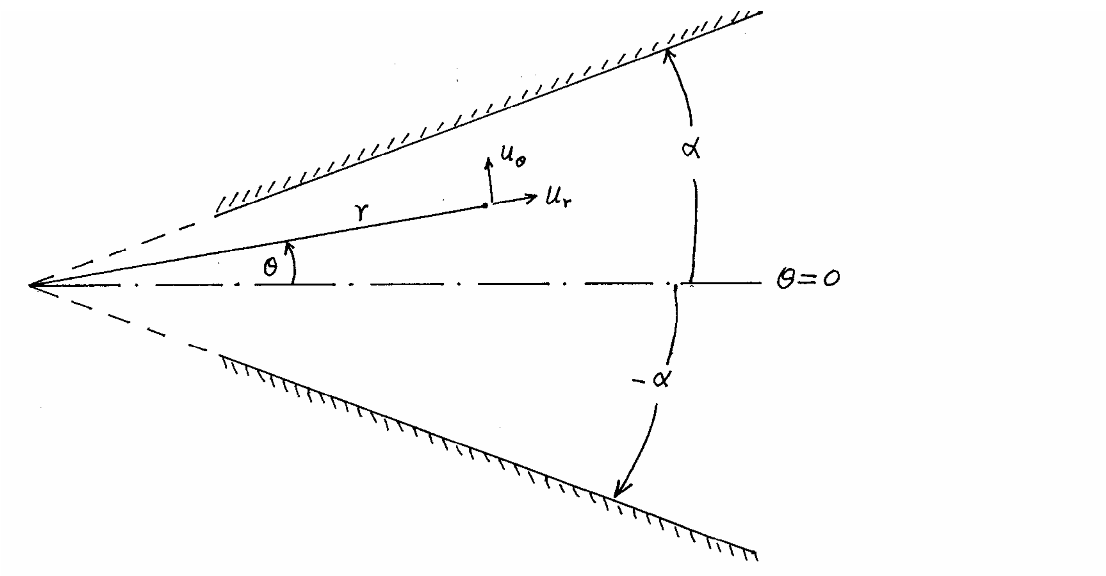


Figure 3-11: Configuration of the flow between two non-parallel planes.

origin of Figure 3-11. The present problem was first studied by Jeffrey (1915) and independently by Hamel (1917), and is thus called the Jeffrey-Hamel flow. If the fluid is inviscid and incompressible, and the flow is further irrotational, the flow may be regarded as a piece of sink (or source) flow (will be discussed later in Chapter 6), and is purely radial. The velocity field is

$$u_\theta = 0 \qquad \text{and} \qquad u_r = Q/(2\alpha r)\,, \tag{3-65}$$

with 2α the angle between the two planes.

For the present viscous flow, it was found that the flow is still purely radial, i.e., $u_\theta = 0$ and the streamlines are lines of constant θ in Figure 3-11. However, the radial velocity component for viscous flow, $u_r = u_r(r,\theta)$, instead of $u_r = u_r(r)$ alone in (3-65) for inviscid flow. For purely radial flow,

$$u_\theta = 0\,, \tag{3-66a}$$

the continuity equation in the polar coordinates in Figure (3-11),

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0\,,$$

reduces to $r u_r = f(\theta)$ for arbitrary function $f(\theta)$, or

$$u_r = \frac{f(\theta)}{r}\,, \tag{3-66b}$$

which represents a source/sink flow (same r-dependence as in (3-55)). With (3-66a) and (3-66b), the radial and the circumferential components of the Navier-Stokes equation reduce to

$$u_r \frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}\right), \tag{3-67a}$$

and

$$0 = -\frac{1}{\rho r}\frac{\partial P}{\partial \theta} + \nu\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\,, \tag{3-67b}$$

respectively. After integration, (3-67b) implies

$$\frac{P}{\rho} = \frac{2\nu}{r^2} f(\theta) + g(r)\,, \tag{3-68}$$

where $g(r)$ is an arbitrary function. With (3-66b) and (3-68), (3-67a) reduces to

$$-\frac{f^2}{r^3}=\frac{4\nu f}{r^3}-\frac{dg}{dr}+\nu\left(\frac{2f}{r^3}-\frac{f}{r^3}-\frac{f}{r^3}+\frac{1}{r^3}\frac{d^2f}{d\theta^2}\right),$$

or

$$f^2+4\nu f+\nu\frac{d^2f}{d\theta^2}=r^3\frac{dg}{dr}. \tag{3-69}$$

The left hand side of (3-69) is function of θ alone, while the right hand side is only function of r. Thus (3-69) implies that

$$r^3\frac{dg}{dr}=C, \tag{3-69a}$$

and

$$f^2+4\nu f+\nu\frac{d^2f}{d\theta^2}=C, \tag{3-69b}$$

where C is a constant to be determined, which is related either to the specified constant flow rate, Q, or the given pressures at both ends. The solution of (3-69a) is

$$g(r)=-\frac{1}{2}\frac{C}{r^2}+\text{constant}. \tag{3-70}$$

To solve (3-69b), lets define a dimensionless variable, F(θ), through

$$F(\theta)\equiv\frac{f(\theta)}{\nu}=\frac{ru_r}{\nu}, \tag{3-71}$$

which may be regarded as a local Reynolds number. With (3-71), (3-69b) becomes

$$\frac{d^2F}{d\theta^2}+F^2+4F=\frac{C}{\nu^2}\equiv -A, \tag{3-72}$$

which is solved subject to the no-slip conditions

$$F(\alpha)=F(-\alpha)=0, \tag{3-72a}$$

and the constraint

$$\int_{-\alpha}^{\alpha} u_r(r,\theta) r d\theta = Q = \text{ given constant,}$$

which can be expressed through (3-71) in dimensionless form as

$$\int_{-\alpha}^{\alpha} F(\theta) d\theta = \frac{Q}{\nu}. \qquad \text{(3-72b)}$$

By inspecting the problem governed by equations (3-71), (3-72a) and (3-72b), we found that the velocity profile, $F(\theta)$, is symmetric about the axis, $\theta = 0$. Thus we may replace (3-72a) by

$$\left.\frac{dF}{d\theta}\right|_{\theta=0} = 0 \qquad \text{and} \qquad F(-\alpha) = 0. \qquad \text{(3-72c)}$$

Equation (3-72) subject to (3-72c) and (3-72b) can be solved numerically, and the results are shown in Figure 3-12. The results are expressed in dimensionless form with the radial velocity component, u_r, normalized by its maximum value, which occurs at $\theta = 0$. The angle θ is normalized by α. As $\alpha = 0$ for the situation of parallel plates, i.e., the plane Poiseuille flow, the horizontal axis for the curve with $\alpha R = 0$ in Figure 3-12 is equivalent to z/h in Figure 3-3. The profile for $\alpha R = 0$ is parabolic as shown in (3-17c). Here $R = (u_r r/\nu)|_{\theta=0} = F(0)$. The curves for

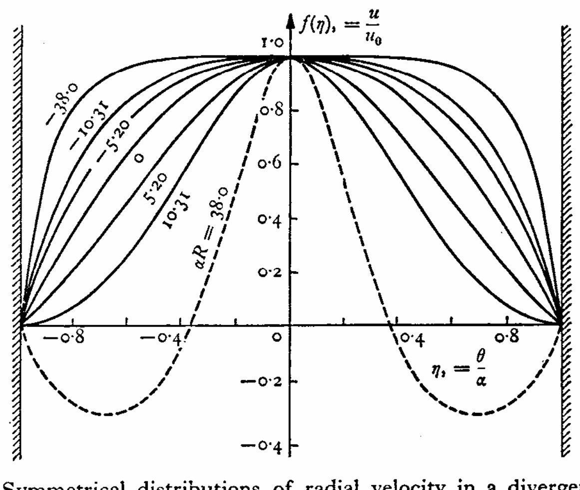


Figure 5.6.4. Symmetrical distributions of radial velocity in a divergent channel for which $\alpha = \pi/36$ at various values of $\alpha R (= \alpha^2 r u_0/\nu)$. Positive values of αR represent outflow at the centre of the channel.

Figure 3-12: Velocity profiles for the flow between two non-parallel planes. This figure is adopted from Batchelor's book (1967). u in the figure is the radial velocity component, u_r, in the text. u_0 is the radial velocity along the centerline, i.e., at $\theta = 0$.

$\alpha R < 0$ correspond to converging flows, which becomes flatter and flatter as $|\alpha R|$ increases. The curves for $\alpha R > 0$ correspond to diverging flows. Numerical result shows that the shear stress at the wall vanishes (i.e., $dF/d\theta|_{\theta=\pm\alpha} = 0$) when $\alpha R = 10.31$. Reverse flows occur next to the walls, and the profile (see case for $\alpha R = 38.0$) shows a maximum at $\theta = 0$ with two relative minimums (negative).

The problem can also be studied analytically, and an implicit solution is obtained as follows. By multiplying (3-72) by an integrating factor, $dF/d\theta$, we obtain

$$\frac{1}{2}\frac{d}{d\theta}\left[\left(\frac{dF}{d\theta}\right)^2\right] + \frac{1}{3}\frac{d}{d\theta}\left(F^3\right) + 2\frac{d}{d\theta}\left(F^2\right) = -A\frac{dF}{d\theta}.$$

After integration, we have

$$\frac{1}{2}\left(\frac{dF}{d\theta}\right)^2 + \frac{1}{3}F^3 + 2F^2 + AF - B = 0. \qquad (3\text{-}73)$$

Here B is the constant of integration, which can be related to the normalized centerline velocity, $F_0 \equiv F(\theta = 0)$, as

$$B = \frac{1}{3}F_0^{\,3} + 2F_0^{\,2} + AF_0,$$

or the dimensionless shear stress at wall, $F_w' \equiv dF/d\theta|_{\theta=\pm\alpha}$, as

$$B = \frac{F_w'^2}{2},$$

through the application of the boundary conditions depicted in (3-72c).

The solution of (3-73) subject to (3-72c) can be solved, and expressed in implicit form as

$$\theta = \pm\int_0^F \frac{d\tilde{F}}{\sqrt{2B - 2A\tilde{F} - 4\tilde{F}^2 - 2\tilde{F}^3/3}} - \alpha \qquad (3\text{-}74)$$

for $2B - 2A\tilde{F} - 4\tilde{F}^2 - 2\tilde{F}^3/3 \neq 0$ (i.e., $dF/d\theta \neq 0$), where the "$\pm$" sign depends on direction of the flow near the wall $\theta = -\alpha$, we choose the plus sign for $dF/d\theta > 0$ and the minus sign for $dF/d\theta < 0$. Equation (3-74) can be expressed as elliptic integral (see L. Rosenhead, "Laminar boundary layers," Dover (1988), p.149).

The qualitative behavior for the profile of the diverging flow shown in Figure 3-12 can be studied as follows. Equation (3-73) can be written as

$$\frac{1}{2}\left(\frac{dF}{d\theta}\right)^2 + V = 0, \tag{3-75}$$

where

$$V = \frac{1}{3}F^3 + 2F^2 + AF - B \tag{3-75a}$$

is a cubic function of F, and can be rewritten in the following form as

$$V = -\frac{1}{3}(a-F)(F-b)(F-c)\,. \tag{3-75b}$$

The constants a, b and c can be related to A and B by comparing (3-75a) with (3-75b). Equation (3-75) is the same as the equation of motion for a point mass under the influence of a force potential, $V(F)$. The first and the second terms in (3-75) represent the kinetic energy and the potential energy, respectively. According to (3-75) and (3-72c), we observe:

(1) At the wall, $F = 0$ and hence

$$-V(0) = \frac{1}{3}abc > 0 \tag{3-76}$$

for any real value of F_w'.

(2) As $|F| \to \infty$, $V \to F^3/3$.

(3) For real value of $dF/d\theta$, we must have $V(F) \le 0$. This implies that the solution is in the lower half of the FV-plane.

(4) The relative maximum or minimum value of the profile for $F(\theta)$ occurs at $dF/d\theta = 0$ or $V = V(F) = 0$. According to (3-75b), there are three roots of $V = V(F) = 0$. There are two situations for these three roots: one real root and a pair of complex root (as in the case for $\alpha R = 5.2$ in Figure 3-12), and three real roots (as in the case for $\alpha R = 38.0$ in Figure 3-12). For the former case, lets denote the real root by a. For the later case, we arrange the roots in the order a > b > c.

(5) For diverging flow, $Q > 0$. a should be positive, and equals to the maximum velocity which occurs at $\theta = 0$.

With the above five items, the locus of the variations of V with F for cases with one and three real roots are plotted qualitatively in Figure 3-13(a) and 3-13(b), respectively. Also sketched in the figure are the corresponding velocity profiles across the channel.

For case with one real root, the curve intersects with the horizontal axis at only one point (at F = a), which is denoted by point “M” in the figure. The intersection of the curve with the vertical axis (denoted by point “W”) corresponds to place where the velocity equals to zero (F = 0), which occurs at the wall. The portion of the curve between “W” and “M” are the locus of V versus F for the flow within the plates. As we move from point “W” to “M” and then back to point “W” along the curve in the left figure of Figure 13(a), the velocity changes from zero at the lower wall to the maximum value at the centerline and then back to zero at the upper wall as shown in the right figure of Figure 13(a).

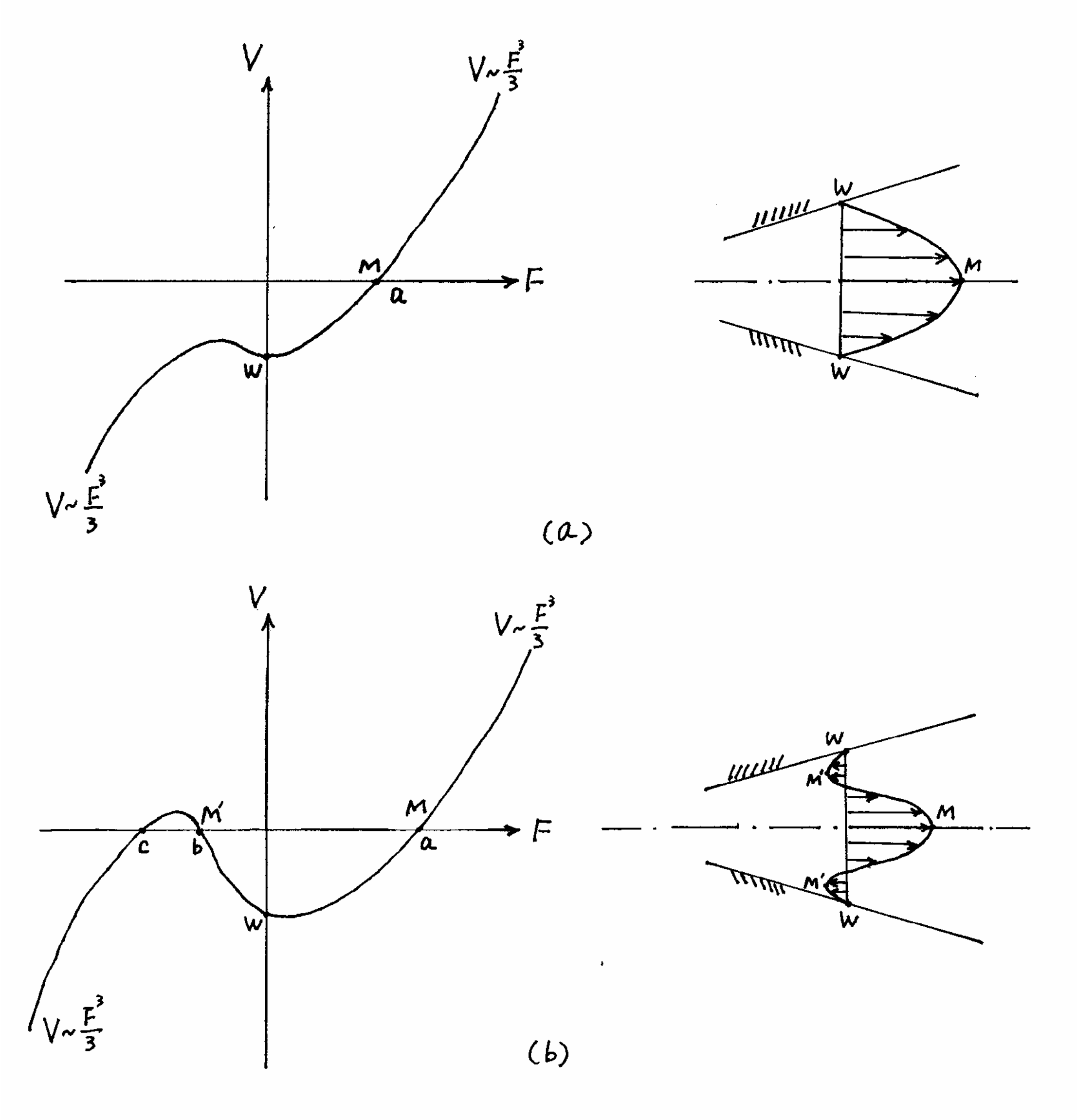


Figure 3-13: Qualitative sketches for the locus of V(F) and the corresponding velocity across the channel. (a) Case with one real root. (b) Case with three real roots.

For case with three real roots, the curve intersects the horizontal axis at three points, F = a, b, and c, as shown on the left figure of Figure 13(b). The point at F = a is denoted by M, and that at F = b by M'. The velocity at the intersection between the curve and the vertical axis is zero (F = 0) is denoted by "W" similar as before. The portion of the curve between point M' and M refers to the flow within the channel, which contains two parts, the negative part (F < 0, reverse flow) from M' to W, and the positive part (F > 0, positive flow) form W to M. The point W corresponds to the state with F = 0, the point M corresponds to state with maximum positive velocity, and the point M' corresponds to the state with maximum negative velocity. The correspondence between the characteristics of the velocity profile (see Figure 3-13(b), right) and the locus on the V-F curve (Figure 3-13(b), left) is as follows. On the V-F curve, we start from W (the velocity at the lower wall), move leftward to M' (the flow reaches maximum reverse velocity), then turn back toward the right, passing through W (the velocity is zero), and reach M (maximum velocity at the centerline), which constitute the first half of the "journey" within the lower half of the channel. The second half of the "journey" corresponds to the upper half of the channel, we start from M, move leftward to W (zero velocity within the flow), then M' (maximum reverse velocity), and then turn rightward, and finally stop at W (the upper wall).

(ii) Curved streamlines: stagnation flow

This is an example for flow with general curved streamlines. The problem was first studied by Hiemenz (1911). Imagine that a uniform flow is approaching an infinite rigid wall, which is located perpendicular to the flow, as shown in Figure 3-14(c). The fluid cannot flow through the wall, and thus turn its direction. The resulting flow field is called the stagnation flow. For two-dimensional flow in a plane, there exists a dividing streamline, i.e., the y-axis in Figure 3-14(c). The flow is symmetric about the dividing streamline. The velocity at the point denoted by "S" is zero, and thus we call the point "S" the stagnation point. The fluid turns a right angle and flows along the wall away from the stagnation point. For axial symmetric case, the y-axis in Figure 3-14(c) is the symmetric axis. The fluid turns as it approaches the wall, and flows away from the stagnation point along radial directions. In practice, the flow around a blunted nosed body as those in Figure 3-14(a) (for planar flow) or 3-14(b) (for axial symmetric flow) can be approximated locally as a stagnation flow. Here we shall consider both cases simultaneously. Consider the coordinates (x,y) in Figure 3-14(c). It may represent the Cartesian coordinates in planar problem, or the cylindrical coordinates in

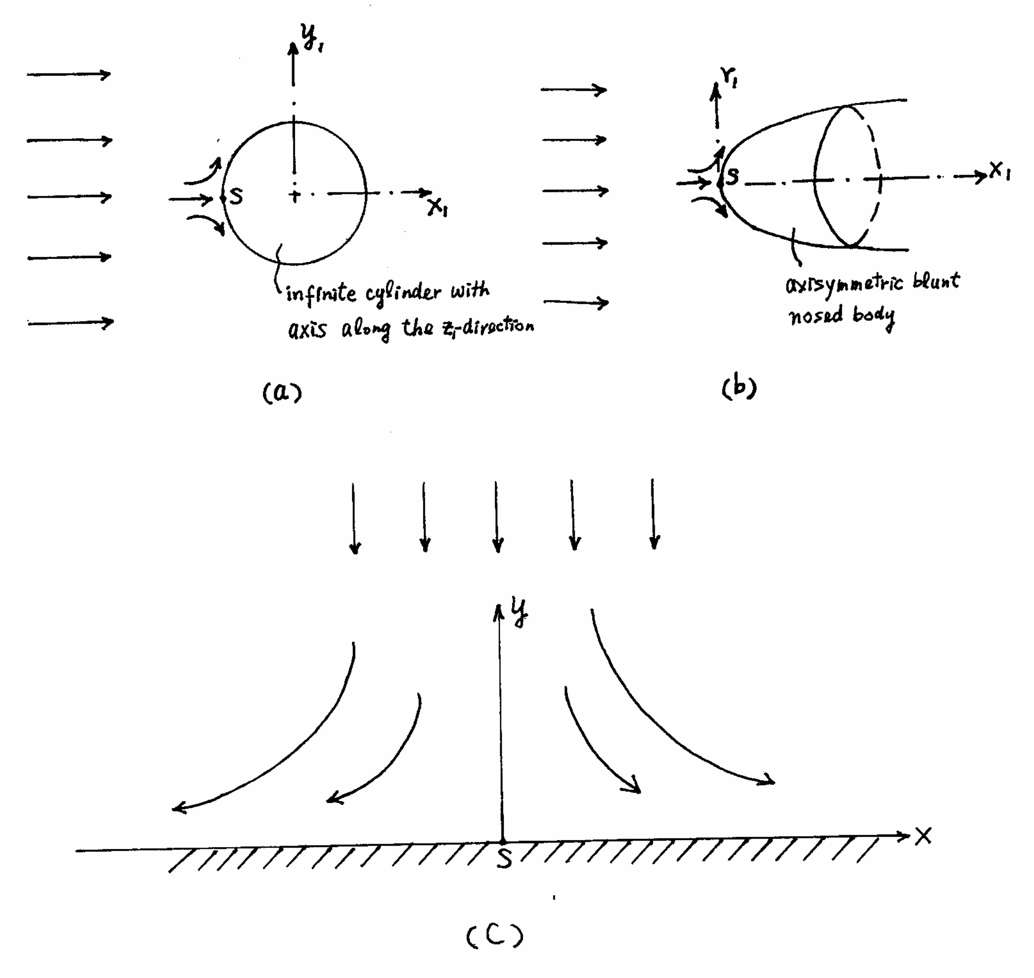


Figure 3-14: For uniform flow over (a) a long circular cylinder, and (b) an axial symmetric blunt nosed body, the flows near the forward stagnation point (denoted by S in the figure) can be approximated as a 2-dimensional (planar) and axial symmetric stagnation flow, respectively. The configuration of the stagnation flow is sketched in (c). x and y are the axes of a Cartesian coordinates for the planar flow, but are the radial and axial axes of a cylindrical coordinates for the axial symmetric flow. The origin of the coordinates is set at the stagnation point.

the axial symmetric problem. Let (u,v) be the velocity components in the (x,y) directions. The continuity and the Navier-Stokes equations can be expressed as

$$\frac{\partial}{\partial x}\left(x^{n-2}u\right)+\frac{\partial}{\partial y}\left(x^{n-2}v\right)=0\,, \tag{3-77a}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{u^{n-2}}{x^{n-2}}\right)+\frac{\partial^2 u}{\partial y^2}\right), \tag{3-77b}$$

and

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left(\frac{1}{x^{n-2}}\frac{\partial}{\partial x}\left(x^{n-2}\frac{\partial v}{\partial x}\right)+\frac{\partial^2 v}{\partial y^2}\right). \tag{3-77c}$$

The flow is planar when $n=2$, but is axial symmetric when $n=3$. Equation (3-77a) is satisfied automatically by defining a stream function $\psi(x,y)$ such that

$$u=\frac{1}{x^{n-2}}\frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v=-\frac{1}{x^{n-2}}\frac{\partial \psi}{\partial x}. \tag{3-78}$$

For frictionless stagnation flow (will be discussed later in chapter 6), the stream function $\psi \sim xy$ for planar flow, and $\psi \sim x^2 y/2$ for axial symmetric flow, or may be written as

$$\psi \equiv \psi_I = \frac{B}{n-1}x^{n-1}y\,, \tag{3-79a}$$

where B is a constant. The corresponding inviscid velocity field is

$$u_I = \frac{B}{n-1}x \qquad \text{and} \qquad v_I = -By\,. \tag{3-79b}$$

We have $u_I = 0$ when $x=0$ and $v_I = 0$ when $y=0$. The point $x=0$ and $y=0$ is a stagnation point. However, the no-slip boundary condition at wall ($y=0$) cannot be satisfied.

For (real) viscous fluid, we modify (3-79a) as

$$\psi = \frac{B}{n-1}x^{n-1}f(y)\,, \tag{3-80a}$$

in order to satisfy the no-slip condition at $y=0$. Note that the x-dependence relationship, $\psi \sim x^{n-1}$, in (3-80a) is still the same as that in (3-79a). The velocity components corresponding to (3-80a) are

$$u = \frac{B}{n-1} x f'(y) \quad \text{and} \quad v = -Bf(y), \tag{3-80b}$$

where $f'(y) = df/dy$. By substituting (3-80b) into (3-77c) we found

$$-\frac{1}{\rho}\frac{\partial P}{\partial y} = B^2 ff' + \nu Bf'', \tag{3-81a}$$

which is function of y alone. Thus

$$\frac{\partial^2 P}{\partial x \partial y} = 0. \tag{3-81b}$$

With (3-81b), we may differentiate (3-77b) by eliminating P, and obtain

$$f^{iv} + \frac{B}{\nu}\left(ff''\right)' - \frac{B}{\nu}\frac{1}{(n-1)}\left(f'^2\right)' = 0. \tag{3-82}$$

The no-slip condition at $y = 0$ implies that

$$f'(0) = 0 \quad \text{and} \quad f(0) = 0. \tag{3-82a}$$

For region sufficiently away from the wall, the viscous effect is negligible, and the flow is expected to match with the inviscid result. Thus we require

$$f'(\infty) = 1 \quad \text{and} \quad f(\infty) = y, \tag{3-82b}$$

by comparing (3-80b) with (3-79b). By integration, we obtain

$$f''' + \frac{B}{\nu}\left(ff''\right) - \frac{B}{\nu}\frac{1}{(n-1)}\left(f'^2\right) = C \tag{3-83a}$$

from (3-82). The constant of integration C is determined from (2-82b) as

$$C = -\frac{B}{\nu}\frac{1}{(n-1)}.$$

Here we have employed $f''(\infty) = f'''(\infty) = 0$, which implies that the flow matches smoothly with the inviscid flow as $y \to \infty$. Thus (3-83a) becomes

$$f'''+\frac{B}{\nu}\left(ff''\right)+\frac{B}{\nu}\frac{1}{(n-1)}\left(1-f'^2\right)=0\,. \qquad \text{(3-83b)}$$

Note that B has the dimension as “1/time” and f has dimension as “length”. By setting

$$\eta = y\sqrt{B/\nu} \qquad \text{and} \qquad F(\eta)=\sqrt{B/\nu}\,f(\eta)\,, \qquad \text{(3-84)}$$

equation (3-83b) becomes

$$F'''+FF''+\frac{1}{n-1}\left(1-F'^2\right)=0\,, \qquad \text{(3-85)}$$

which is in dimensionless form, and is solved subject to

$$F(0)=F'(0)=F'(\infty)-1=0 \qquad \text{(3-85a)}$$

according to (3-82a) and (3-82b). A numerical program for the planar case (n=2) is shown in Figure 3-15, and the numerical results are shown in Figure 3-16.

From Figure 3-16, the x-component velocity approaches a constant, the y-component velocity approaches a linear relationship, and the shear stress approaches zero as the dimensionless distance from the wall, η, increases. This is expected since the flow approaches the inviscid flow sufficiently far from the wall. The viscous effect is confined in a region next to the wall. If the thickness of the viscous region, δ, is defined through

$$y=\delta \qquad \text{when} \qquad F'(\eta)=0.99\,, \qquad \text{(3-86)}$$

we found

$$\delta\sqrt{B/\nu}=2.4 \qquad \text{for the planar stagnation flow} \qquad \text{(3-86a)}$$

according to the numerical result in Figure 3-16. It is important to note that δ is a constant and is independent of x! The thickness of the viscous region does not grow with x because the approaching flow confines the vorticity generated associated with the no-slip condition next to the wall. Similarly, we can solve the axial symmetric problem numerically, and find

$$\delta\sqrt{B/\nu} = 2.75 \quad \text{for the axial symmetric stagnation flow.} \quad (3\text{-}86b)$$

The shear stress at the wall is calculated as

$$\tau_w = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_{y=0} = \mu\left(\frac{Bx}{n-1}F''(0)\sqrt{\frac{B}{\nu}} + 0\right) = \mu\frac{Bx}{n-1}F''(0)\sqrt{\frac{B}{\nu}}, \quad (3\text{-}87)$$

which increases linearly with x. There is no contribution from the term $\mu\partial v/\partial y|_{y=0}$. For the planar case, we have $n = 2$, and $F''(0) = 1.23$ according to the numerical result. The shear stress at wall is usually written in dimensionless form as

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho u_I^2} = \frac{2F''(0)\sqrt{n-1}}{\sqrt{\mathbf{R}}}, \quad (3\text{-}87a)$$

where C_f is called the skin friction coefficient, and $\mathbf{R} = u_I x/\nu$ is the Reynolds number. Finally, with the velocity field obtained, we can evaluate the pressure field by integrating (3-81a).

(iii) Other nonlinear exact solutions

There are also some other nonlinear problems having exact solutions. For examples, the flow induced by a rotating disk (the "von Karman's viscous pump", Problem 4 of the Homework), the steady jet from a point source of momentum (called the Squire or Landau's jet, see Batchelor's book), and some other interesting problem in Yih's book (pp.332-334).

(IV) Concluding Remarks

For steady incompressible flow with constant viscosity, the Reynolds number is the only dimensionless parameter governing the flow. Mathematically, exact solution is valid for all Reynolds number. Unfortunately, this is not true! As claimed by L.D. Landau and E. M. Lifshitz (1959), "Yet not every solution of the equation of motion, even if it is exact, can actually occur in nature. The flows that occur in nature must not only obey the equations of fluid dynamics, but also be stable." For example, we may observe parabolic velocity profile in a circular pipe under the steady, fully developed condition only when the Reynolds number (based on the mean velocity and pipe diameter) is less than a critical value, say, less than 2100. When the Reynolds number exceeds the critical value, the flow becomes unstable, and we observe turbulent flow when the Reynolds number is sufficiently large. The shape of the mean turbulent profile is more flat (may be approximated as a 1/7-power profile) in comparing with the parabolic profile. The book by Drazin & Reid is a good introduction to the study of the stability of fluid motion (P. G. Drazin and W. H. Reid, "Hydrodynamic stability," Cambridge University Press, 1981).

A Fortran program for solving (2-D stagnation flow)

$$F''' + FF'' - F'^2 + 1 = 0, \quad F(0) = F'(0) = F'(\infty) - 1 = 0$$

```
      implicit real*8(a-h,o-z)
      dimension x(10001), y1(10001), y2(10001), y3(10001)
      write(*,9)
    9 format(' input h, y30, n, m')
      read (*,*) h, y30, n, m
      x(1) = 0.d0
      y1(1) = 0.d0
      y2(1) = 0.d0
      y3(1) = y30
      do 100 i=1,n-1
      y1i=y1(i)
      y2i=y2(i)
      y3i=y3(i)
      a1=h*ff1(y2i)
      b1=h*ff2(y3i)
      c1=h*ff3(y1i,y2i,y3i)
      a2=h*ff1(y2i+.5d0*b1)
      b2=h*ff2(y3i+.5d0*c1)
      c2=h*ff3(y1i+.5d0*a1,y2i+.5d0*b1,y3i+.5d0*c1)
      a3=h*ff1(y2i+.5d0*b2)
      b3=h*ff2(y3i+.5d0*c2)
      c3=h*ff3(y1i+.5d0*a2,y2i+.5d0*b2,y3i+.5d0*c2)
      a4=h*ff1(y2i+b3)
      b4=h*ff2(y3i+c3)
      c4=h*ff3(y1i+a3,y2i+b3,y3i+c3)
      x(i+1)=x(i)+h
      y1(i+1)=y1(i)+(a1+2.d0*a2+2.d0*a3+a4)/6.d0
      y2(i+1)=y2(i)+(b1+2.d0*b2+2.d0*b3+b4)/6.d0
      y3(i+1)=y3(i)+(c1+2.d0*c2+2.d0*c3+c4)/6.d0
      write(*,120) x(i),y2(i)
  100 continue
      open(unit=3,file='output.dat')
      do 104 i=1,n,m
  104 write(3,120) x(i),y1(i),y2(i),y3(i)
      close(3)
  120 format(4(1x,e12.5))
      stop
      end
c     ****************
      function ff1(y2)
c     ****************
      implicit real*8(a-h,o-z)
      ff1=y2
      return
      end
c     ****************
      function ff2(y3)
c     ****************
      implicit real*8(a-h,o-z)
      ff2=y3
      return
      end
c     **********************
      function ff3(y1,y2,y3)
c     **********************
      implicit real*8(a-h,o-z)
      ff3=-y1*y3+y2*y2-1.d0
      return
      end
```

$y1 \leftrightarrow F$, $y2 \leftrightarrow F'$, $y3 \leftrightarrow F''$

$h \leftrightarrow$ step size in x ($h = 0.001$ is ok)

$y30 \leftrightarrow$ guessed $F''(0)$ (the correct guess is 1.23259)

$n \leftrightarrow$ number of steps in x ($n = 5001$ is ok)

$m \leftrightarrow$ output your results every m steps.

$ff1 \leftrightarrow y1' = y2$

$ff2 \leftrightarrow y2' = y3$

$ff3 \leftrightarrow y3' = -ff'' + f'^2 - 1 = -(y1)(y3) + (y2)^2 - 1$

Figure 3-15: The Fortran program for solving the planar stagnation flow using shooting method (with fourth order Runge-Kutta method).

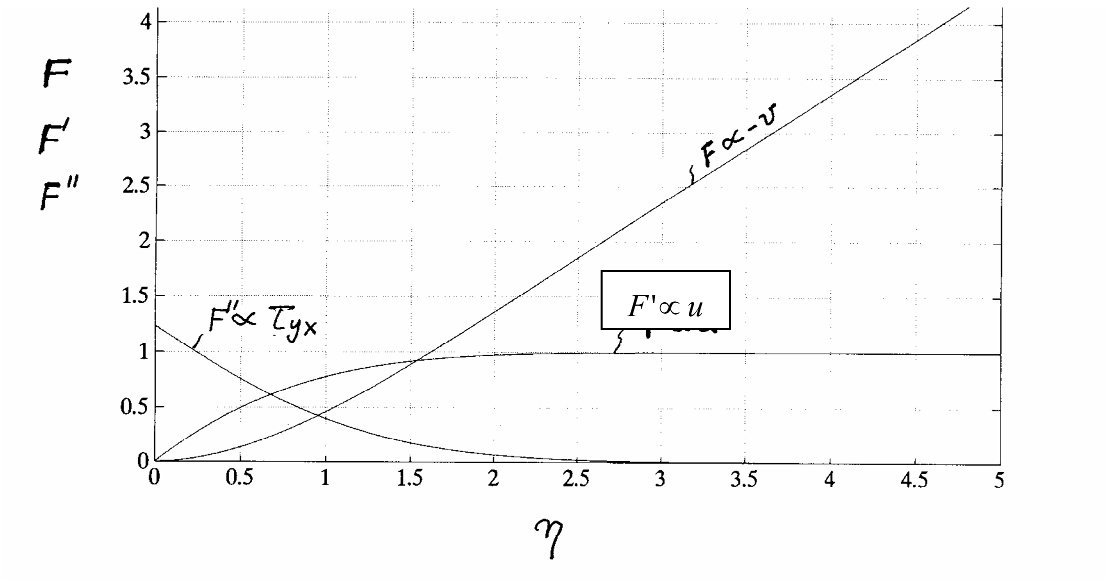


Figure 3-16: Numerical results for the planar stagnation flow.

Homework

(1) The cross section of a tube in an equilateral triangle with sides of length l and a horizontal base. Flow in the tube is produced by an imposed pressure gradient dp/dx. Verify that the velocity profile is given by

$$u(y,z) = \frac{1}{2\sqrt{3}\mu l}\left(-\frac{dp}{dx}\right)\left(z - \frac{\sqrt{3}}{2}l\right)\left(3y^2 - z^2\right),$$

where the coordinate origin is at the apex of the triangle with z bisecting the angle and positive downward, and y is horizontal. Check that the flow rate is

$$Q = \frac{\sqrt{3}}{320}\frac{l^4}{\mu}\left(-\frac{dp}{dx}\right).$$

This problem is adopted from Panton's book, Problem 11.1.

(2) Consider the annulus formed between a rod of radius r_0 and a tube of radius r_1. Find the steady fully developed velocity profile driven by both the following mechanisms simultaneously. (i) The rod is translating in the axial direction at a constant speed V_0 and rotating about its own axis with a constant angular speed Ω. (ii) A constant pressure gradient dp/dx is imposed along the axial direction (from Panton's book, Problem 7.4).

(3) Extend the plane Poiseuille flow analysis to describe stratified flow. One fluid, filling half the channel, has density ρ_1 and viscosity μ_1; the other fluid has ρ_2 and μ_2. Check your result by setting $\rho_1 = \rho_2$ and $\mu_1 = \mu_2$. What happens if the upper fluid is much less dense and less viscous than the lower fluid, as would be the case with air flowing over water? (Hint: let the velocity and the shear stress be continuous at the interface)

(4) Consider the flow around an infinite long circular cylinder with radius a. The flow is driven by the rotation the cylinder about its own axis with angular velocity $\omega = 0$ when $t < 0$ and $\omega = \Omega = \text{constant}$ when $t \geq 0$. Here t is the time.

(a) Starting from the incompressible Navier-Stokes equation in cylindrical coordinates, show that the velocity in θ-component satisfy the "diffusion equation" under certain assumptions. State your assumptions and the appropriate boundary and initial conditions.

(b) Show that the problem cannot be solved by similarity transformation.

(c) Solve the problem by using the Laplace transform or other appropriate method.

(5) Consider an infinite stream oscillating according to $u(t) = u_0 \sin \Omega t$. What pressure gradient would cause this oscillation? A solid wall is inserted into the flow so that it is parallel to the motion. What is the shear stress on the wall? What is the phase of the shear stress with respect of the velocity $u(y \to \infty, t)$? What is the y location, as a function of time, where the particle acceleration is a maximum (of either sign)? How much of the acceleration is due to pressure and how much to viscosity? (Panton's book, problem 11.4)

(6) The flat bottom surface under a liquid of depth h is moved in its own plane according to $u(t) = u_0 \sin \Omega t$. Find the velocity and vorticity profiles in the liquid. (Hint: What boundary condition would you apply on the upper free surface? This problem is from Panton's book, problem 11.10)

(7) Solve the transient starting Poiseuille flow in a circular pipe. Plot the velocity profiles for different values of $t\nu / a^2$ together with the steady solution. Here t is the time, ν is the kinematic viscosity of the fluid, and a is the radius of the pipe.

(8) Consider the so-called von Karman's viscous pump problem as follows. Incompressible Newtonian fluid with constant viscosity is filled in the semi-infinite domain above the rigid plane z = 0. Consider the steady flow which results if the infinite plane z = 0 rotates at constant angular velocity $\omega \hat{k}$ about the z-axis. The viscous drag of the rotating surface would set up a swirling flow toward the plane. Let (u_r, u_θ, u_z) be the velocity components in the cylindrical coordinates (r, θ, z) and p be the dynamic pressure. Assume that the flow is independent of θ.

(a) Write down the governing equations for u_r, u_θ, u_z and p, i.e., the continuity and the incompressible Navier-Stokes equation in cylindrical coordinates with the flow properties independent of θ.

State the appropriate boundary conditions.

(b) Karman proposed (1921) that the appropriate solution to be

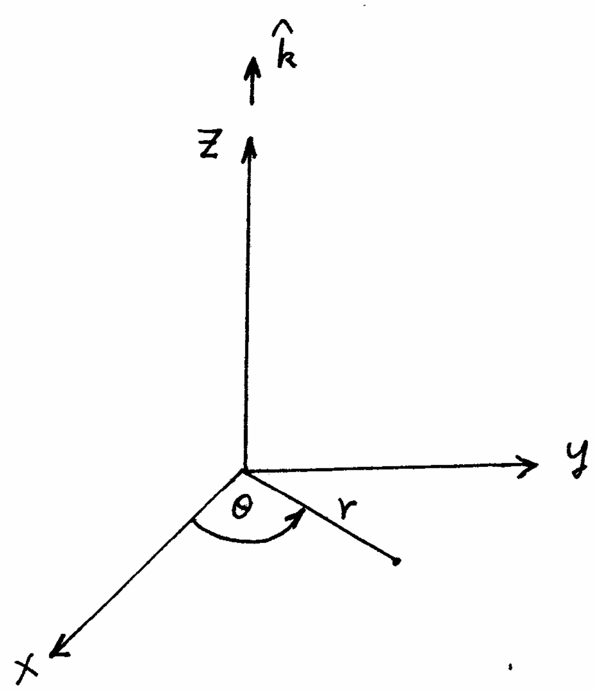
$u_r / r, u_\theta / r$, p = function of z alone,

and hence wrote

$$u_r = r\omega F(z^*), \quad u_\theta = r\omega G(z^*), \quad u_z = \sqrt{\omega\nu}H(z^*), \quad p = \rho\omega\nu P(z^*), \quad (1)$$

where $z^* = z\sqrt{\omega/\nu}$. Derive the ordinary differential equations governing the dimensionless function F, G, H and P by substituting equation (1) into the governing equations in (a), and state the appropriate boundary conditions for the ordinary differential equations.

(d) Express the shear stress on the rotating surface in terms of the function G at z*=0, and estimate the torque exerted by the fluid on the plane from $r = 0$ to $r = r_0$, where r_0 is a finite value.



CHAPTER 4: THE FLOW PHYSICS FROM SMALL TO LARGE REYNOLDS NUMBER

As discussed in chapter 3, we have only limited numbers of exact solution to the governing equations of the fluid motion, and the equations cannot be solved "exactly" in general. In order to solve the governing equations, we may either employ the numerical methods (which is the topic of Computational Fluid Dynamics), or simplify them into simpler equations under certain conditions so that we may carry out further analysis. The results of the simplified equations are approximate but not "exact" solutions to the governing equations, i.e., the continuity and Navier-Stokes equations. In the rest of this course, we shall discuss two types of approximations under condition of either very small or very large Reynolds number.

We shall consider here only the steady incompressible flow with constant viscosity. The governing equations are

$$\nabla \cdot \mathbf{u} = 0 \qquad (4\text{-}1a)$$

and

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \mu \nabla^2 \mathbf{u}\,, \qquad (4\text{-}1b)$$

where P is the modified pressure. The terms in (4-1b) represent the inertia, the pressure and the viscous forces, respectively. At solid surface, the no-slip condition should be satisfied for a given flow.

The problem can be normalized by choosing suitable velocity scale, U_s, length scale, L_s, and pressure scale, P_s. We may choose U_s and L_s as the characteristic speed and length, respectively, of the problem of interest. The choice of the pressure scale depends on different situations as follows.

(I) Cases for small inertia (low Reynolds number flows)

When the inertia force is small, we require that the pressure and the viscous terms in (4-1b) be of the same order, i.e., $-\nabla P \sim \mu \nabla^2 \mathbf{u}$, or

$$\frac{P_s}{L_s} \sim \mu \frac{U_s}{L_s^{\ 2}} .$$

Thus

$$P_s \sim \frac{\mu U_s}{L_s} .$$

On setting

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{U_s}, \qquad \hat{\mathbf{x}} = \frac{\mathbf{x}}{L_s}, \qquad \hat{\nabla} = L_s \nabla \qquad \text{and} \qquad \hat{P} = \frac{P}{\mu U_s / L_s}, \qquad (4\text{-}2)$$

equations (4-1a) and (4-1b) can be written in dimensionless form as

$$\nabla \cdot \hat{\mathbf{u}} = 0 \qquad (4\text{-}3a)$$

and

$$\mathbf{R}\hat{\mathbf{u}} \cdot \hat{\nabla}\hat{\mathbf{u}} = -\hat{\nabla}\hat{P} + \hat{\nabla}^2 \hat{\mathbf{u}} , \qquad (4\text{-}3b)$$

where

$$\mathbf{R} = \frac{\rho U_s L_s}{\mu} \qquad (4\text{-}4)$$

is the Reynolds number, which represents the ratio of the inertia to the viscous force since

$$\frac{\rho \mathbf{u} \cdot \nabla \mathbf{u}}{\mu \nabla^2 \mathbf{u}} \sim \frac{\rho U_s \frac{U_s}{L_s}}{\mu \frac{U_s}{L_s^{\ 2}}} \sim \frac{\rho U_s L_s}{\mu} = \mathbf{R} .$$

As $\mathbf{R} \to 0$, the inertia force is negligible in comparing with the viscous force, and (4-3a) and (4-3b) reduce to

$$\nabla \cdot \hat{\mathbf{u}} = 0 \qquad (4\text{-}5a)$$

and

$$\mathbf{0} = -\hat{\nabla}\hat{P} + \hat{\nabla}^2 \hat{\mathbf{u}} . \qquad (4\text{-}5b)$$

The equivalent dimensional equations of (4-5a) and (4-5b) are

$$\nabla \cdot \mathbf{u} = 0 \qquad (4\text{-}6a)$$

and

$$\mathbf{0} = -\nabla P + \mu \nabla^2 \mathbf{u} . \qquad (4\text{-}6b)$$

Equations (4-6a) and (4-6b) are the simplified governing equations for low Reynolds number flows (i.e., for $\mathbf{R} << 1$), and (4-6b) is called the Stokes equation. The low Reynolds number flows are sometimes called the Stokes flows (because (4-4b) is called the Stokes equation), or the creeping flows (because the flow velocity is usually small for this type of flow). Strictly speaking, there are three situations for the Reynolds number to be small: when the characteristic velocity is small, when the characteristic length is small, and when the viscosity is large. Recall that the inertia force is identically zero for flows with parallel streamlines as discussed in chapter 3, such parallel flows together with the low Reynolds number flows described by (4-6a) and (4-6b) are sometimes called the inertial-free flows (because the inertia term is neglected) or inertialess flows (in Liggett's book). The governing equations become linear when the inertia term is neglected, which implies that we can carry out further theoretical analysis under $\mathbf{R} << 1$. More details of the low Reynolds number flows will be discussed in chapter 5.

(II) Cases for large inertia (high Reynolds number flows)

By balancing the inertia with the pressure term, i.e., $\rho \mathbf{u} \cdot \nabla \mathbf{u} \sim -\nabla P$, we have

$$\rho U_s \frac{U_s}{L_s} \sim \frac{P_s}{L_s},$$

or

$$P_s \sim \rho U_s^{\ 2}.$$

On setting

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{U_S}, \qquad \hat{\mathbf{x}} = \frac{\mathbf{x}}{L_s}, \qquad \hat{\nabla} = L_s \nabla \qquad \text{and} \qquad \hat{P} = \frac{P}{\mu U_s^{\ 2}}, \qquad (4\text{-}7)$$

equations (4-1a) and (4-1b) can be written in dimensionless form as

$$\nabla \cdot \hat{\mathbf{u}} = 0 \qquad (4\text{-}8a)$$

and

$$\hat{\mathbf{u}} \cdot \hat{\nabla} \hat{\mathbf{u}} = -\hat{\nabla} \hat{P} + \frac{1}{\mathbf{R}} \hat{\nabla}^2 \hat{\mathbf{u}}. \qquad (4\text{-}8b)$$

As $R \to \infty$, equations (4-8a) and (4-8b) reduce to

$$\nabla \cdot \hat{\mathbf{u}} = 0 \qquad (4\text{-}9a)$$

and

$$\hat{\mathbf{u}} \cdot \hat{\nabla} \hat{\mathbf{u}} = -\frac{1}{\rho} \hat{\nabla} \hat{P}. \qquad (4\text{-}9b)$$

The equivalent dimensional form of (4-9a) and (4-9b) are

$$\nabla \cdot \mathbf{u} = 0 \tag{4-10a}$$

and

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P \,, \tag{4-10b}$$

which are exactly the equations governing the inviscid flow. Equation (4-10b) is the steady Euler equation, which is of one order less than (4-1b). Thus the no-slip boundary condition at wall should be replaced by the associated slip boundary condition.

In order to satisfy the no-slip condition (the real situation) for flow with large Reynolds number, we must retain at least some part of the viscous term in the equation of motion. Therefore, it is proposed that there exists a thin region next to the solid boundary with thickness δ (see Figure 4-1), where the viscous force is not negligible and of the same order as the inertia force. Such thin region is called the boundary layer, and was first proposed by Prandtl at 1904. Define a local Cartesian coordinates (*x,y,z*) within the boundary layer. Let *x*, *y*, and *z* be the streamwise, the cross-streamwise, and the spanwise direction, respectively, with L_x, δ and L_z the representative length scales. We have

$$\delta << L_x \sim L_z \sim L_s \quad \text{for flow inside the boundary layer.} \tag{4-11}$$

The viscous term for the streamwise momentum equation within the boundary layer locally can then be evaluated as

$$\mu \nabla^2 u = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \sim \mu \left(\frac{U_s}{L_s^{\,2}} + \frac{U_s}{\delta^2} + \frac{U_s}{L_s^{\,2}} \right) \sim \mu \frac{U_s}{\delta^2} \,,$$

which is much greater than that outside the boundary layer, $\mu U_s / L_s^{\,2}$, according to (4-11). By balancing the viscous term with the inertia term, i.e., $\rho \mathbf{u} \cdot \nabla \mathbf{u} \sim \mu \nabla^2 \mathbf{u}$, inside the boundary layer, we have

$$\rho U_s \frac{U_s}{L_s} \sim \mu \frac{U_s}{\delta^2} \,,$$

or

$$\frac{\delta}{L_s} \sim \sqrt{\frac{\mu}{\rho U_s L_s}} = \frac{1}{\sqrt{\mathbf{R}}} \,, \tag{4-12}$$

which is much less than unity as $\mathbf{R} \to \infty$, and is consistent with (4-11).

Thus there are two regions having different characteristics for the flow field (see figure 4-1(b)) at large Reynolds number. Most of the region is essentially inviscid, and is called the inviscid main flow. The viscous effect is confined in a thin region next to the wall, called the boundary layer. The thickness of the boundary layer is much less than its streamwise extent, which is of the same order as the length scale of the inviscid main flow. Mathematically, the boundary layer is inserted for satisfying the no-slip boundary condition. Vorticity is generated at the wall associated with the no-slip condition, and swept downstream by the flow. The diffusion of the vorticity by the viscous effect across the stream is relatively small in comparing with the streamwise convective effect due to the imposed flow at large Reynolds number, and thus the boundary layer is kept thin although its thickness is evolving slowly downstream. The boundary layer at the upper surface and that at the lower surface of the body merge, and form a wake region behind the body. The wake region contains mainly the relatively high vorticity fluid generated inside the boundary layers upstream, and is fundamentally different from that in the inviscid main flow. In fact, due to the drag of the body, the velocity inside the wake is less than that in the inviscid main flow, say, U_∞, in Figure 4-1, even at far downstream location from the body. This can be understood by carrying out a momentum analysis of the flow inside a large control volume containing the body, as will be discussed later in Chapter 7. For large Reynolds number, the wake is a slender region, with its cross streamwise extent much less than the streamwise extent, and can also be analyzed using the boundary layer assumption in (4-11). Thus we shall group the boundary layer next to the body surface and the wake together, and called them simply the boundary layer region in the text. The boundary layer region has two main characteristics: (1) it is a slender region, and thus the spatial variation of the flow properties across the flow is much greater than that along the flow, and (2) viscous effect is important in the region. When the Reynolds number is large enough, the flow inside the boundary layer and the wake are turbulent, and the thickness of the boundary layer region becomes larger in comparing with that in laminar case. However, the main characteristics remain unchanged. If the flow inside the boundary layer experiences a sufficiently severe adverse pressure gradient, it may separate from the body surface, and the boundary layer grows rapidly. The boundary layer assumption in (4-11) is not valid locally where the boundary layer separates. Also the boundary layer assumption fails in the wake region near the body (called the near wake). Numerical solution of the continuity and Navier-Stokes equations is required for understanding the local detailed features of such cases.

However, the structure of the inviscid main flow can still be analyzed using the continuity and Euler equations, but the shape of the body for inviscid calculation should be modified.

Therefore, in case with high Reynolds number, we may replace “the original problem governed by the continuity and Navier-Stokes equations subject to the no-slip boundary condition at solid surface as in Figure 4-1(a)” by “the problem with two regions governed separately by simpler equations as in Figure 4-1(b)”. The whole flow domain is separated into two parts in Figure 4-1(b), the inviscid main flow and the thin boundary layer region, which includes the boundary layer next to the solid surface and the wake region in the downstream of the body. The

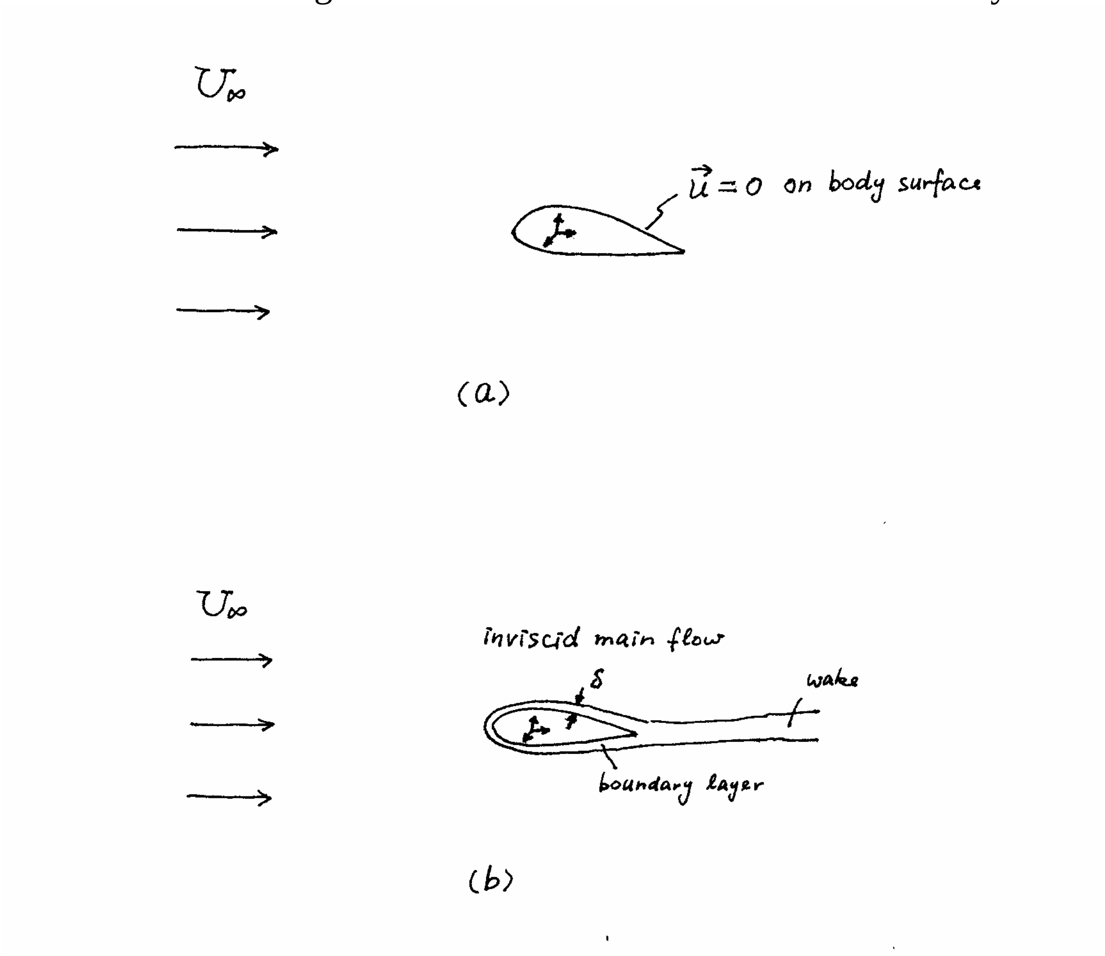


Figure 4-1: (a) The flow governed by the continuity and Navier-Stokes equations subject to the no-slip boundary condition at solid wall is replaced by (b) the inviscid main flow plus the thin boundary layer region when the Reynolds number is sufficiently large. The viscous effect is confined inside the boundary layer region, which consists the boundary layer and the wake region downstream of the body.

inviscid main flow includes most of the flow region of the original problem, and is governed by the continuity and the Euler equations (with slip boundary condition). The flow in the boundary layer region is governed by the so-called boundary layer equations (will be derived later), which include the continuity and a "simplified" form of the Navier-Stokes equation. For the boundary layer next to the solid surface, the boundary layer equations are solved subject to the no-slip condition at the solid boundary and the matching condition at the outer edge of the boundary layer. The matching condition enforces that the flow fields obtained in both the inviscid main flow region and the boundary layer region obtained from different sets of governing equations to be matched smoothly in a common region. The wake region sufficiently far downstream from the body (i.e., the far wake) can also be approximated as a thin boundary layer (without solid boundary), which is also governed by the boundary layer equations.

As an illustration for deriving the boundary layer equations, consider the two-dimensional planar flow over a wedge at large Reynolds number. An incompressible uniform stream is approaching the wedge along its axis with speed, U_∞. If the fluid is inviscid, a velocity field can be obtained using the potential flow theory (will be discussed later in Chapter 6), and the flow is slipping along the wedge surface with speed $U_s(x)$, which is a function of x and is not equal to the approaching uniform speed U_∞. In order to satisfy the no-slip boundary condition at the wedge surface, a thin boundary layer is inserted between the inviscid main flow and the solid boundary as illustrated in Figure 4-2. A Cartesian coordinates (x, y) is set up in the figure, with (u, v) the corresponding velocity components. The governing equations written in component forms are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \text{(4-13a)}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad \text{(4-13b)}$$

and

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \qquad \text{(4-13c)}$$

with P the reduced pressure and ν the kinematic viscosity of the fluid. The relative importance of various terms in the above equations can be studied using the scaling analysis as follows. Let L_x and δ be the length scale for the x and y directions, respectively. The scale for u can be chosen as U_s, a representative speed of $U_s(x)$, and the scale for P is thus $\rho U_s^{\,2}$. The scale for v, V_s, can be determined through the continuity equation, (4-13a), i.e.,

$$\frac{U_s}{L_x} \sim \frac{V_s}{\delta}.$$

Thus

$$V_s \sim \frac{\delta}{L_x} U_s. \tag{4-14a}$$

We have

$$V_s << U_s \quad \text{as} \quad \delta << L_x \tag{4-14b}$$

according to the boundary layer assumption, (4-11). With (4-14a), the scales for various terms in (4-13c) are

$$U_s \frac{U_s \delta / L_x}{L_x} + \left(U_s \delta / L_x\right) \frac{U_s \delta / L_x}{\delta} \sim \frac{1}{\rho} \frac{\rho U_s^{\,2}}{\delta} + \nu \frac{U_s \delta / L_x}{L_x^{\,2}} + \nu \frac{U_s \delta / L_x}{\delta^2},$$

respectively, or their relative importance can be expressed as

$$\frac{\delta^2}{L_x^{\,2}} + \frac{\delta^2}{L_x^{\,2}} \sim 1 + \frac{1}{\mathbf{R}} \frac{\delta^2}{L_x^{\,2}} + \frac{1}{\mathbf{R}},$$

where **R** is the Reynolds number. All the terms are negligible in comparing with the pressure term in (4-13c) under the conditions $\delta << L_x$ and $\mathbf{R} \to \infty$, and thus (4-13c) reduces to

$$\frac{\partial P}{\partial y} = 0. \tag{4-15}$$

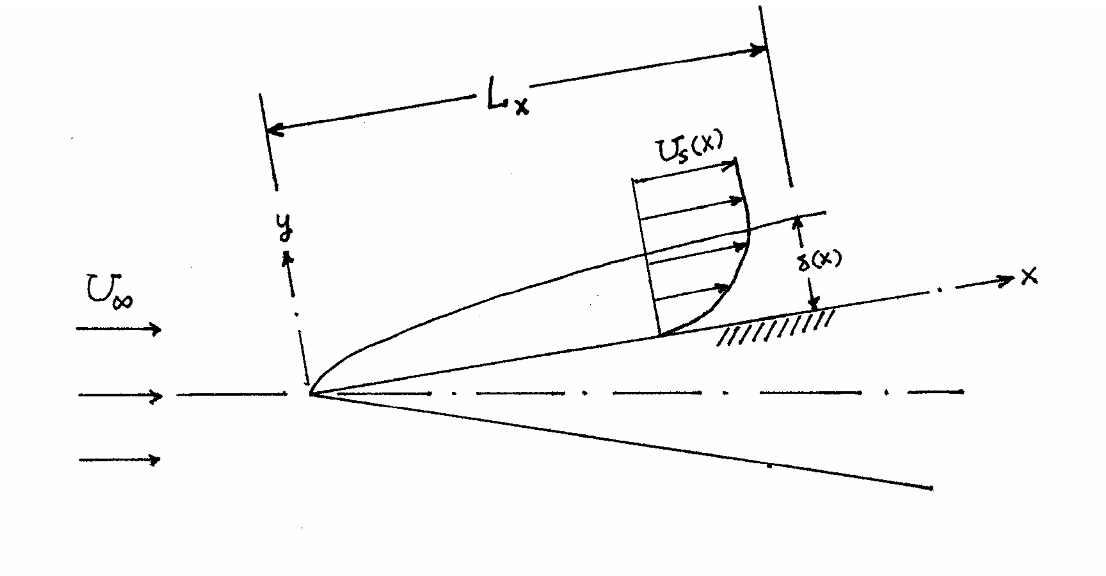


Figure 4-2: Sketch of the boundary layer for flow over a two-dimensional wedge.

Equation (4-15) implies that the pressure variation across the boundary layer is negligible, or that the pressure gradient along the *x*-direction inside the boundary layer, $\partial P/\partial x$, can be evaluated using the flow outside the boundary layer, i.e.,

$$\frac{\partial P}{\partial x} = \frac{dP_s}{dx}, \tag{4-15a}$$

where $P_s = P_s(x)$ is the pressure along the wedge surface according to the solution of inviscid main flow. With $v = 0$, the x-momentum equation along the wedge surface for inviscid flow according to (4-9b) is

$$\rho U_s \frac{dU_s}{dx} = -\frac{dP_s}{dx}. \tag{4-15b}$$

Thus the pressure gradient inside the boundary layer can be expressed as

$$-\frac{\partial P}{\partial x} = \rho U_s \frac{dU_s}{dx}. \tag{4-15c}$$

The relative importance of the terms in (4-13b) are evaluated in a similar way as

$$U_s\frac{U_s}{L_x}+U_s\frac{\delta}{L_x}\frac{U_s}{\delta}\sim\frac{1}{\rho}\frac{\rho U_s^{\,2}}{L_x}+\nu\frac{U_s}{L_x^{\,2}}+\nu\frac{U_s}{\delta^2},$$

or

$$1+1\sim 1+\frac{1}{\mathbf{R}}+\frac{1}{\mathbf{R}}\frac{L_x^{\,2}}{\delta^2}.$$

The streamwise diffusion term, $\nu\partial^2 u/\partial x^2$, in (4-13b) is negligible under $\mathbf{R}\to\infty$. The relative importance of the cross-streamwise diffusion term, $\nu\partial^2 u/\partial y^2$, in comparing with the inertia (and pressure) term depends on $\left(L_x^{\,2}/\delta^2\right)/\mathbf{R}$, which is a product of a very large value $\left(L_x^{\,2}/\delta^2\right)$ and a small value $1/\mathbf{R}$. There are three possibilities:

(i) If $(\delta/L_x)^2\to\mathbf{0}$ slower than $1/\mathbf{R}\to\mathbf{0}$ as $\mathbf{R}\to\infty$, the streamwise diffusion term is negligible and (4-13b) reduces to the x-component of the Euler equation. The no-slip condition cannot be satisfied, and this is not a correct possibility.

(ii) If $(\delta/L_x)^2\to\mathbf{0}$ faster than $1/\mathbf{R}\to\mathbf{0}$ as $\mathbf{R}\to\infty$, the streamwise diffusion term dominates and (4-13b) reduces to $\partial^2 u/\partial y^2=0$. Its solution subject to the no-slip condition is a linear profile, $u=cy$, which cannot be matched smoothly (the velocity is continuous but not its derivates) with the "external" inviscid main flow. Therefore, this is also not a correct possibility.

(iii) If $(\delta/L_x)^2\to\mathbf{0}$ is of the same order as $1/\mathbf{R}\to\mathbf{0}$ as $\mathbf{R}\to\infty$, the streamwise diffusion term is of the same order as the inertia and pressure term, and (4-13b) reduces to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}. \qquad \text{(4-16)}$$

It is important to note that (4-16) is a "parabolic" type partial differential equation, which is simpler than the "elliptic" partial differential equation, (4-13b).

Therefore, with the application of (4-15a-c), equations (4-13a) and (4-13c) reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad \text{(4-17a)}$$

and

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_s\frac{dU_s}{dx}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{4-17b}$$

which is called the boundary layer equations. These equations are solved subject to

$$u=0 \quad \text{at} \quad y=0, \tag{4-17c}$$

$$u=U_s(x) \quad \text{as} \quad y\to\infty, \tag{4-17d}$$

and a suitable condition at a specified x.

The boundary layer equations in (4-17a) and (4-17b) are for planar steady boundary layer flow over a "flat" surface. In general, the body surface is curved. In such a case, we define a curvilinear coordinate system, (x,y), as shown in Figure 4-3, where x is measured along the surface from a fixed point, and y is measured along the local normal direction with y=0 on the surface. Under the boundary layer assumptions, we obtain the same equations as those in (4-17a) and (4-17b) for the boundary layer flow over a curved surface as that in Figure 4-3, provided that the boundary layer thickness is much less than the local radius of curvature of the body.

For unsteady flow, the surface velocity of the inviscid flow is function of both x and t, i.e., $U_s(x,t)$. Similar analyses as those in steady flow above can be applied, and the boundary layer equations for planar unsteady incompressible flow over a curved surface in Figure 4-3 are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{4-18a}$$

and

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U_s}{\partial t}+U_s\frac{\partial U_s}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}. \tag{4-18b}$$

For those who are interest in the problems of unsteady boundary layers, please read the books by Schlichting (Chapter 15) and Rosenhead (Chapter 7). In a similar way, the boundary layer theory above can also be extended to boundary layer flows over axial-symmetric and general three-dimensional bodies (see Schlichting, Chapter 11 and Rosenhead, Chapter 8).

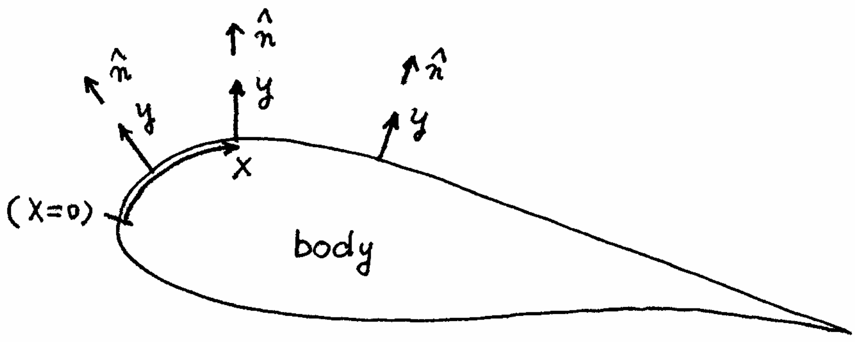


Figure 4-3: The curvilinear coordinates system for planar boundary layer flow over a curved surface.

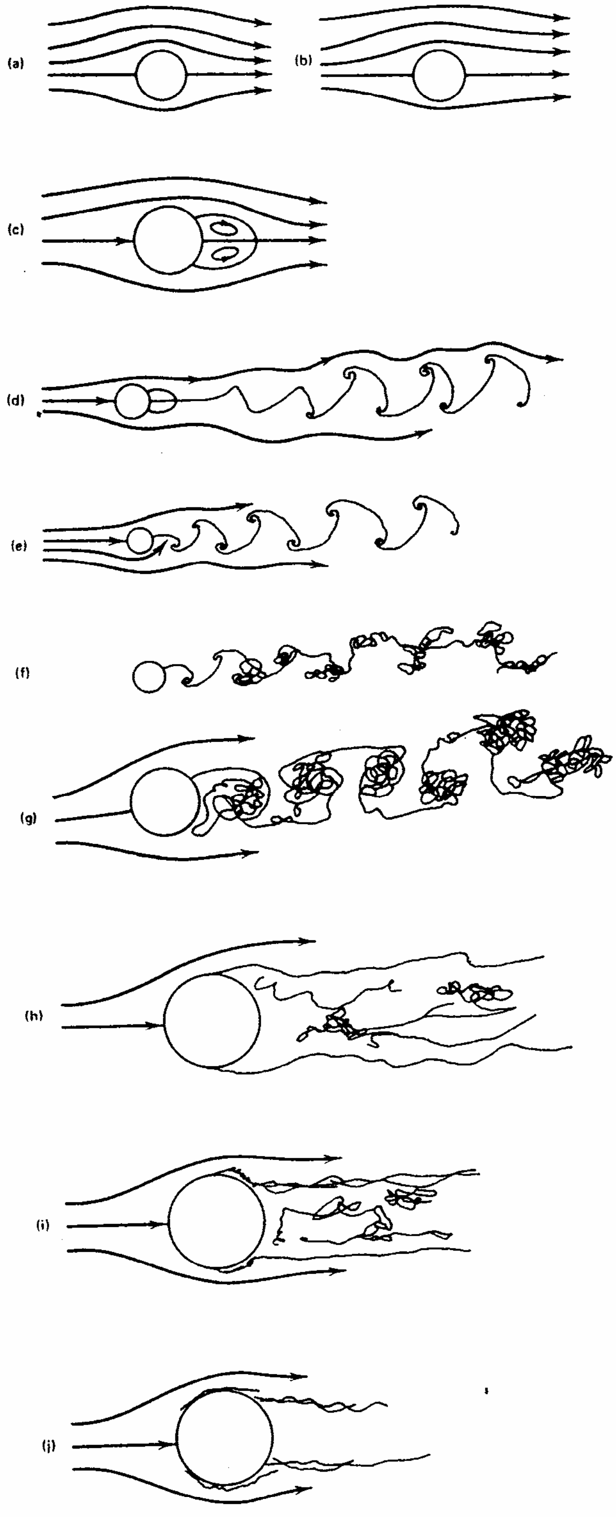
For unsteady flow, the Reynolds number is not the only governing dimensionless parameter of the problem. There exists an additional parameter, called the Strouhal number, which expresses the ratio of the unsteady term to the convection term. In fact, the numbers of the governing dimensionless parameters increase as the problem becomes more and more complicated. White (1991) had summarized more than ten dimensionless parameters, which are important for vary problems in Chapter 2 of his book.

(III) Cases for intermediate Reynolds numbers

When the Reynolds number, **R**, is of intermediate value, say, for $\mathbf{R} = O(1) - O(100)$, the inertia, the pressure and the viscous terms in (4-1b) are of the same order, and the governing equations cannot be simplified. Note that the range for $\mathbf{R} = O(1) - O(100)$ corresponds to $\delta / L_s = O(1) - O(10)$ according to (4-12). For moderate Reynolds number, the problem can be studied via numerical method (see Chapter 15 of Panton's book).

(IV) An example for illustrating the flow physics from small to large Reynolds number

The change of the flow from small to large Reynolds number is complicated and interesting. Here we conclude this chapter by considering the problem of steady planar flow over a circular cylinder. Detailed calculation procedures and results can be found from Chapter 15 of Panton’s book. Essential flow physics can be summarized and sketched in Figure 4-4 below. It is interesting to note that the flow is unsteady at intermediate Reynolds number as the Karman vortex street sets on although the approaching flow is constant and the cylinder is fixed.



Flow regimes for a cylinder: (a) Re = 0, symmetrical; (b) 0 < Re < 4; (c) 4 < Re < 40, attached vortices; (d) 40 < Re < 60–100, Kàrmàn vortex street; (e) 60–100 < Re < 200, alternate shedding; (f) 200 < Re < 400, vortices unstable to spanwise bending; (g) 400 < Re, vortices turbulent at birth; (h) Re < 3×10^5, laminar boundary layer separates at 80°; (i) 3×10^5 < Re < 3×10^6, separated region becomes turbulent, reattaches, and separates again at 120°; (j) 3×10^6 < Re, turbulent boundary layer begins on front and separates on back.

Figure 4-4: Sketch of the essential flow physics for uniform flow over a circular cylinder (from Panton's book).

CHAPTER 5: LOW REYNOLDS NUMBER FLOWS

As discussed in chapter 4, we may have low Reynolds number flow when the characteristic velocity is small, when the characteristic length is small, and/or when the viscosity is large. For steady flow, the governing equations for low Reynolds number flows are the continuity and the Stokes equations (4-6(a) and 4-6(b)),

$$\nabla \cdot \mathbf{u} = 0 \tag{5-1a}$$

and

$$\mathbf{0} = -\nabla P + \mu \nabla^2 \mathbf{u} \, . \tag{5-1b}$$

(I) Simplifications of the governing equations

Equations may be simplified further in the following ways, and their applications depend on different situations.

(i) Equation with pressure as dependent variable

By Taking divergence of the momentum equation, (5-1b), we found

$$0 = -\nabla^2 P + \mu \nabla^2 (\nabla \cdot \mathbf{u}) \, .$$

The last term in the above equation is zero by using the (5-1a). Thus we have

$$\nabla^2 P = 0 \, , \tag{5-2}$$

which is solved by imposing suitable pressure conditions. This formulation is employed frequency for lubrication problem.

(ii) Equation with velocity as dependent variable

By taking Laplacian operation of (5-1b), we found

$$0 = -\nabla\left(\nabla^2 P\right) + \mu \nabla^2 (\nabla^2 \mathbf{u}) \, .$$

With (5-2), the above equation becomes

$$\nabla^4 \mathbf{u} = \nabla^2(\nabla^2 \mathbf{u}) = 0 , \qquad (5\text{-}3)$$

which is a biharmonic equation for the vector quantity, **u**. We may employ two of the component equations of (5-3) to determine two velocity components, and the rest velocity component is determined by the continuity equation, (5-1a).

(iii) Vorticity and stream function as dependent variables

By taking curl of (5-1b), and using the definition of vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} , \qquad (5\text{-}4a)$$

we found

$$0 = \nabla \times \nabla P + \mu \nabla^2 \boldsymbol{\omega} .$$

The first term on the right hand side is identically zero according to the vector identity. Thus

$$\nabla^2 \boldsymbol{\omega} = \mathbf{0} . \qquad (5\text{-}4b)$$

We have seven equations, (5-4a), (5-4b) and (5-1a), for six unknowns, three components for **u** and three components for ω. Thus only two of the component equations of (5-4b) will be employed. This formulation is always employed for external flow.

For two-dimensional flow, we may define a stream function, ψ, through

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x} , \qquad (5\text{-}5a)$$

such that the continuity equation, (5-1a), is satisfied automatically. Here (x,y) is the Cartesian coordinates with (u,v) the corresponding velocity components. The vorticity according to its definition in (5-4a) is

$$\boldsymbol{\omega} = \hat{\mathbf{k}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\hat{\mathbf{k}}\nabla^2\psi \,, \qquad (5\text{-}5b)$$

where $\hat{\mathbf{k}}$ is the unit vector perpendicular to the (x,y) plane. It follows from (5-4b) that

$$\nabla^2\nabla^2\psi = 0 \,. \qquad (5\text{-}5c)$$

Similar results may be obtained for axial-symmetric flow (see later in section III).

We shall discuss two problems in the next two sections, the lubrication problem (an internal flow) in section II and the flow past a sphere in section III (an external flow).

(II) Lubrication problem

We shall consider a simple case, the plane slider, first, and then extend to the general lubrication theory. The basis of the lubrication theory is to induce fluid motion between two surfaces, usually via the relative motion of the surfaces, so that a large pressure can be built up between such two surfaces in comparing with that in the surroundings, and thus the two surfaces can be kept separated.

(i) Plane slider

Although we can derive the followings using the governing equations of the low Reynolds number flow, i.e., (5-1a) and (5-1b), we shall derive the theory from the continuity and Navier-Stokes equations directly. The sketch of the problem is shown in Figure 5-1, which occurs in the (x,y) plane with velocity components (u,v). A stationary block with certain weight and/or even under loading is immersed in the fluid above a moving blade with uniform velocity $U\mathbf{i}$. The gap between the bottom surface of the block and the blade has width $d(x)$ and length L. The key assumptions are that (1) the gap width is much less than the gap length and (2) the angle between the lower surface of the block and the blade is much less than unity. i.e.,

$$d(x) << L \qquad \text{and} \qquad \alpha \approx \tan\alpha = \frac{d_1 - d_2}{L} << 1. \qquad (5\text{-}6)$$

Thus we are studying the flow in a two-dimensional long channel with slowly varying cross sectional area, with fluid motion driven by the uniform motion of the lower surface in its own plane. The continuity and the Navier-Stokes equations, written in components form, are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (5\text{-}7a)$$

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad (5\text{-}7b)$$

and

$$\rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \qquad (5\text{-}7c)$$

First, we estimate the relative importance of the terms in the above equations. The scales for *x*, *y* and *u* are chosen as L, $d_1 - d_2$ (or αL) and

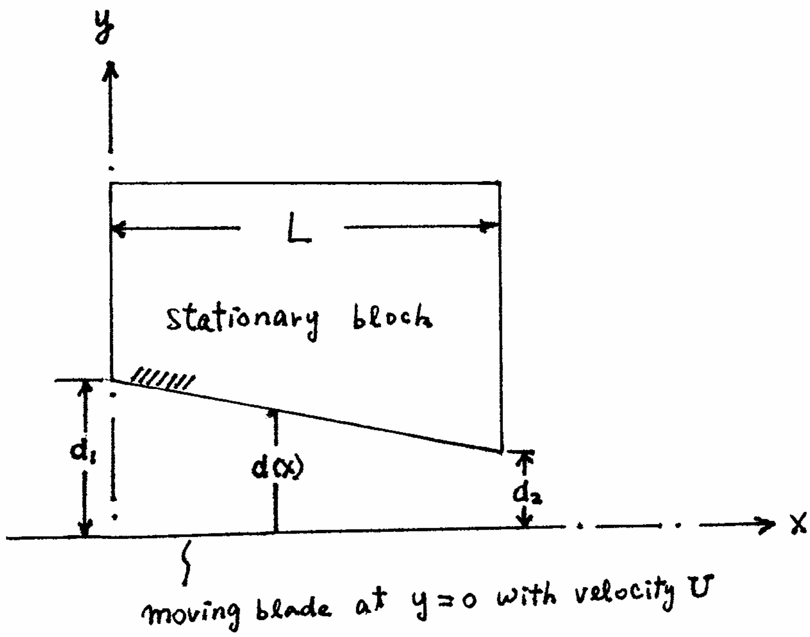


Figure 5-1: Sketch of the flow for the problem of a plane slider.

U, respectively. The velocity scale for v is thus αU according to (5-7a). Let P_s be the pressure scale. The scales of the terms in (5-7b) are

$$\rho U\frac{U}{L}+\rho\alpha U\frac{U}{\alpha L}\sim\frac{P_s}{L}+\mu\frac{U}{L^2}+\mu\frac{U}{\alpha^2L^2},$$

or

$$\frac{\rho UL}{\mu}\alpha^2+\frac{\rho UL}{\mu}\alpha^2\sim\frac{P_s}{\mu\dfrac{U}{\alpha^2L}}+\alpha^2+1. \qquad (5\text{-}8)$$

The first two terms in (5-8), i.e., the convection terms in the Navier-Stokes equation are negligible for $\alpha<<1$ except when the Reynolds number, $\mathbf{R}\equiv\rho U(\alpha L)/\mu\geq O(\alpha^{-1})$, which is a large value; for example, if $\alpha=0.1^o=0.001745$radian, $\mathbf{R}\cong 573$. The second term on the right hand side, i.e., the diffusion term along the x-direction is also negligible for $\alpha<<1$. The pressure term is passive, which is a direct consequence of the fluid motion in the channel driven by the moving blade. It cannot be neglected otherwise there is no mechanism for balancing the y-diffusion term. Therefore, we set $P_s=\mu U/(\alpha^2L)$, and (5-7b) reduces to

$$0=-\frac{\partial P}{\partial x}+\mu\frac{\partial^2u}{\partial y^2}, \qquad (5\text{-}9a)$$

which is the same governing equation as that in plane Poiseuille flow in Chapter 3. Here we have employed only the assumption $\alpha<<1$ with **R** of order less than α^{-1} for deriving (5-7b). The scales for (5-7c) are

$$\rho U\frac{\alpha U}{L}+\rho\alpha U\frac{\alpha U}{\alpha L}\sim\frac{\mu\dfrac{U}{\alpha^2L}}{\alpha L}+\mu\frac{\alpha U}{L^2}+\mu\frac{\alpha U}{\alpha^2L^2},$$

or

$$\frac{\rho UL}{\mu}\alpha^4+\frac{\rho UL}{\mu}\alpha^4\sim 1+\alpha^4+\alpha^2.$$

All the terms are negligible in the above equation under $\alpha<<1$ with **R** less than $O(\alpha^{-3})$ except the pressure term. Thus (5-7c) reduces to

$$\frac{\partial P}{\partial y}=0, \qquad (5\text{-}9b)$$

which states that the pressure is essential constant across the channel, or

$$\frac{\partial P}{\partial x} = \frac{dP}{dx} = \text{function of } x \text{ alone.} \qquad (5\text{-}9c)$$

Therefore, in view of (5-9a) and (5-9c) together with the boundary conditions, the flow in the two-dimensional channel with a slowly varying cross sectional area in Figure 5-1 is locally a Poiseuille flow (with a local width $d(x)$ and a local pressure gradient $\partial P/\partial x$) superimposed by a Couette flow.

The solution of (5-9a) subject to the no-slip boundary condition is

$$u(x,y) = -\frac{1}{\mu}\frac{dP}{dx}\frac{y}{2}(d-y) + U\frac{d-y}{y}. \qquad (5\text{-}10)$$

The first and the second term in (5-10) represent a local plane Poiseuille flow and a Couette flow, respectively. Here dP/dx is function of x instead of a constant in the "exact" plane Poiseuille flow in Chapter 3. We can calculate v from (5-1a) by using (5-10), which is different from zero as that in Chapter 3. Thus the streamlines are not parallel here although v is indeed a small value in comparing with u.

The volume flow rate, Q, is evaluated as

$$Q = \int_0^{d(x)} u(x,y)dy = -\frac{dP}{dx}\frac{d^3}{12\mu} + \frac{Ud}{2}, \qquad (5\text{-}11)$$

which is a specified constant in the present problem. From (5-11), we can express the pressure gradient as

$$\frac{dP}{dx} = 6\mu\left(\frac{U}{d^2} - \frac{2Q}{d^3}\right). \qquad (5\text{-}11a)$$

With $d = d_1 - \alpha x$, (5-11a) can be integrated easily, and the result is

$$P - P_1 = \frac{6\mu}{\alpha}\left[U\left(\frac{1}{d} - \frac{1}{d_1}\right) - Q\left(\frac{1}{d^2} - \frac{1}{d_1^2}\right)\right], \qquad (5\text{-}12)$$

where the initial condition $P = P_1$ at $x = 0$ has been employed. If there is no pressure contrast (the situation as that shown in figure 5-1) at both ends of the channel, i.e., $P = P_2 = P_1$ at $x = L$, we can relate the volume flow rate to the blade speed as

$$Q = U\frac{d_1 d_2}{d_1 + d_2}. \tag{5-13a}$$

Then (5-12) may be written as

$$P - P_1 = \frac{6\mu U}{\alpha}\frac{(d_1 - d)(d - d_2)}{d^2(d_1 + d_2)}, \tag{5-13b}$$

which is indeed greater than zero. A qualitative picture for the pressure distribution in (5-13b) is sketched in Figure 5-2. The flow induced by the moving blade can indeed generate a pressure field, which is greater than its surrounding pressure, P_1. With (5-13b), we can calculate the hydrodynamic force on the block as

$$N = \int_0^L (P - P_1)dx = \frac{6\mu U}{\alpha^2}\left[\ln\frac{d_1}{d_2} - 2\frac{d_1 - d_2}{d_1 + d_2}\right], \tag{5-14a}$$

which is greater than zero. The tangential force on the lower surface, i.e., the moving blade, is evaluated by using (5-10) as

$$T = -\int_0^L \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} dy = \frac{2\mu U}{\alpha}\left[3\frac{d_1 - d_2}{d_1 + d_2} - 2\ln\frac{d_1}{d_2}\right]. \tag{5-14b}$$

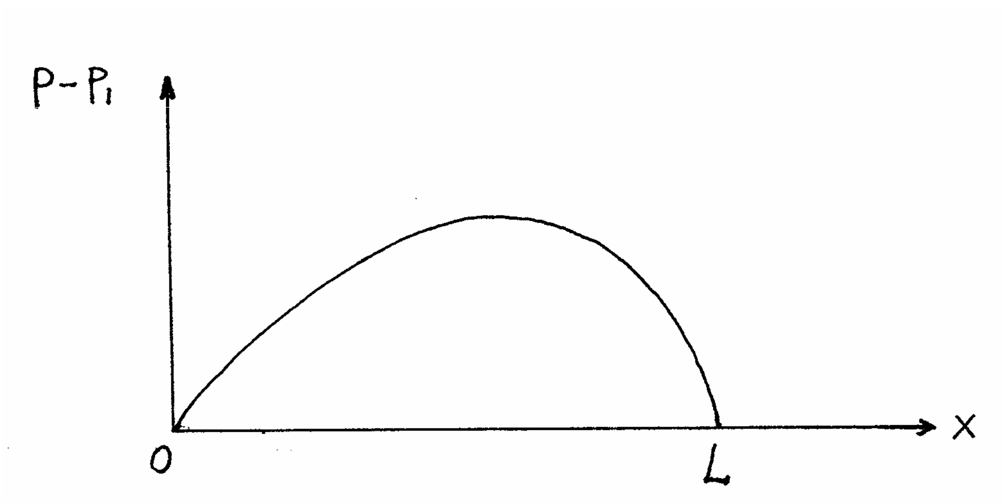


Figure 5-2: A qualitative sketch for equation (5-13b).

The coefficient of friction,

$$\frac{T}{N} = \alpha f(d_1, d_2) = \alpha \left[\frac{\dfrac{d_1 - d_2}{d_1 + d_2} - \dfrac{2}{3}\ln\dfrac{d_1}{d_2}}{\ln\dfrac{d_1}{d_2} - 2\dfrac{d_1 - d_2}{d_1 + d_2}} \right]. \tag{5-15}$$

For practical situation, $(d_2 - d_2)$, d_1 and d_2 are of the same order, or $f(d_{1,} d_2) = O(1)$, then $T/N = O(\alpha)$, which implies that we may apply a small input force (T) to generate a large output force (N).

(ii) Reynolds lubrication equation

Now, we generate the above result to a general case for flow inside a channel as shown in Figure 5-3. The width of the channel, $d(x, y, t)$, is a slowly varying function of the spatial coordinates x and y, and also a time varying function. The flow is generated by the motion of the lower surface, which is moving with local velocity $U\mathbf{i} + V\mathbf{j}$ at the origin of the coordinates (x, y, z) shown in the figure. The upper surface is allowed to move along the z-direction with respect to the lower surface. With the scaling analysis similar as above, we found equations similar to those in (5-9a) and (5-9b) provided that the time scale is chosen to be *L/U* for evaluating the unsteady term of the Navier-Stokes equation. Thus at a fixed x, the local flow is a combination of the Poiseuille flow and Couette flow, i.e.,

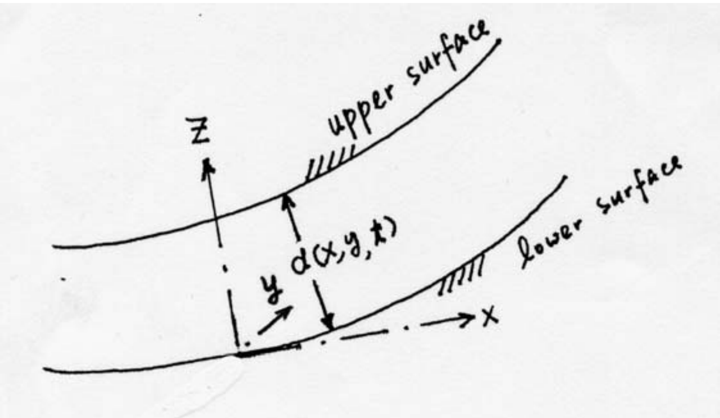


Figure 5-3: Sketch for the flow inside a long channel with slowly varying width. At a given x, the lower surface is moving with velocity $U\mathbf{i} + V\mathbf{j}$ and the upper surface with $W\mathbf{k}$, where $W = \partial d / \partial t$.

$$u = -\frac{1}{2\mu}\frac{\partial P}{\partial x}z(d-z) + U\frac{d-z}{d}, \qquad (5\text{-}16a)$$

and

$$v = -\frac{1}{2\mu}\frac{\partial P}{\partial y}z(d-z) + V\frac{d-z}{d}. \qquad (5\text{-}16b)$$

The instantaneous flow rate in the x and y direction are calculated as

$$Q_x = \int_0^d u(x, y, z, t)dz = -\frac{\partial P}{\partial x}\frac{d^3}{12\mu} + \frac{Ud}{2}, \qquad (5\text{-}17a)$$

and

$$Q_y = \int_0^d v(x, y, z, t)dz = -\frac{\partial P}{\partial y}\frac{d^3}{12\mu} + \frac{Vd}{2}, \qquad (5\text{-}17b)$$

respectively. By integrating the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

across the channel, and using the Leibniz' rule, we found

$$\frac{\partial Q_x}{\partial x} - \frac{\partial d}{\partial x}u\Big|_{y=d} + \frac{\partial Q_y}{\partial y} - \frac{\partial d}{\partial y}v\Big|_{y=d} + w\Big|_{y=d} - w\Big|_{y=0} = 0.$$

The second, the fourth and the sixth term of the above equation are zero, and the fifth term equals to $\partial d/\partial t$ according to the boundary conditions. Thus

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \frac{\partial d}{\partial t} = 0.$$

With (5-17a) and (5-17b), the above equation finally becomes

$$\frac{\partial}{\partial x}\left(d^3\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(d^3\frac{\partial P}{\partial y}\right) = 6\mu U\frac{\partial d}{\partial x} + 6\mu V\frac{\partial d}{\partial y} + 12\mu\frac{\partial d}{\partial t}, \qquad (5\text{-}18)$$

which is a linear partial differential equation governing the pressure, $P(x, y, t)$, within the channel, provided that $d(x, y, t)$ is a given function. Equation (5-18) is called the Reynolds equation (incompressible version), which forms the basis of lubrication theory. If $d(x, y, t)$ is given, equation (5-18) can be solved rather "easily", say, using numerical method. For practical situation, $d(x, y, t)$ is unknown in advance and is governed by the equation of motion of the body associated with the upper surface, say, the Newton's second law. Thus the fluid mechanics problem (equation (5-18)) is coupled with the dynamic problem, and should be solved simultaneously for the unknowns, $P(x, y, t)$ and $d(x, y, t)$. An example is the slider bearing inside the disk drives as shown in Figure 5-4. The slider bearing is sometimes called the air bearing, which consists of the slider where the sensor (the "magnetic head") installed, the rotating disk, and the thin air film between the slider and the disk. The length and the width of the slider are of order 1mm, and the average thickness of the air film is reduced continuously from $20\mu m$ at 1960's to $0.33\mu m$ at 1981, $0.05\mu m$ at 1995, and $0.02\mu m$ at 2000. This tendency is associated with the fact that the storage of the disk drive is inversely proportional to the spacing of the slider and the disk, i.e., the thickness of the air film. The fluid motion within the air film can be analyzed using an extended compressible version of the above Reynolds lubrication equation, (5-18), with certain modification accounting for the rarefied gas effect. Note that (5-18) is derived under the incompressible condition, the compressible version of (5-18) is derived using the compressible continuity and Navier-Stokes equation together with suitable equations of states, and is a non-linear partial differential equation. The shape of the slider is crucial for the design of an ultra thin air film (see E. Cha and D. B. Bogy, Trans. ASME, Journal of Tribology, V.117, pp.36-46, 1995).

For fluid motion driven by the relative normal motion of the surfaces, for examples, the upper surface approaches the lower surface and squeezes the fluid out of the gap between the surfaces (see W. S. Griffin, H. H. Richardson and S. Yamanami, Trans. ASME, J. Basic Engineering, V.88, pp.451-456, 1966), the upper surface oscillates harmonically and rapidly with respect to the lower surface (see J. J. Blech, Trans. ASME, J. Lubrication Technology, V.105, pp.615-619, 1983), the compressibility of the fluid cannot be neglected. Such problem is called the "squeeze film problem", and a nonlinear partial differential equation for pressure is obtained by combining the continuity equation, the Navier-Stokes equation and suitable equation of state. Such equation can be obtained from the compressible version of the Reynolds equation by setting $U=V=0$. The squeeze film problem has been employed recently in studying certain microelectromechanical systems (see M, Andrews, I. Harris and G. Turner, Sensors and Actuators A, V.36, pp.79-87, 1993).

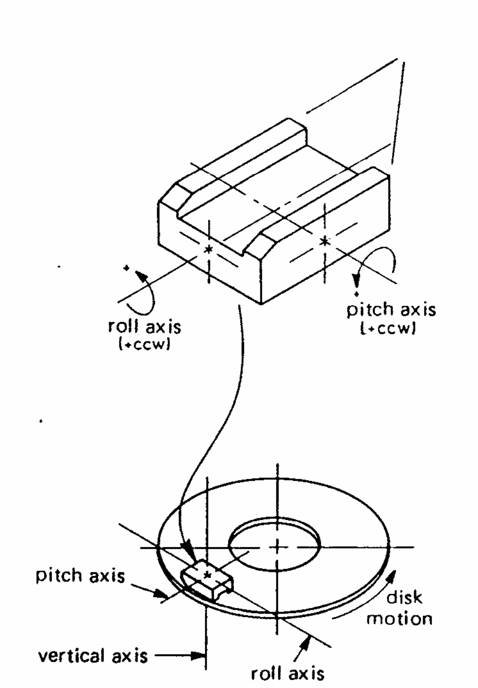


Figure 5-4: Sketch of the flying of a slider over a rotating disk in a disk drive. The figure is adopted from Bolasna et al. (IBM Disk Storage Technology, Feb. 1980).

(III) Uniform flow past a sphere at small Reynolds number

Here we consider an external flow as sketched in Figure 5-5, which was studied first by Stokes (1851). A uniform stream, $U\hat{\mathbf{e}}_z$, is approaching a sphere with radius a. A spherical coordinates system (r,θ,ϕ) is set up with its origin at the center of the sphere, and $(\hat{\mathbf{e}}_r,\hat{\mathbf{e}}_\theta,\hat{\mathbf{e}}_\phi)$ the corresponding unit vectors. The corresponding velocity components in this spherical coordinates are denoted by (u_r,u_θ,u_ϕ). For low Reynolds number flows, the governing equations together with the flow geometry imply that the flow is symmetric with respect to the z-axis as indicated in Figure 5-5, i.e., all the flow variables are independent of ϕ, and $u_\phi = 0$; and the velocity and pressure fields are the same for any meridian plane (plane with constant ϕ). The continuity equation in spherical coordinates thus becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right)=0\,. \qquad (5\text{-}19)$$

A Stokes stream function, $\psi(r,\theta)$, is defined through

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \tag{5-20}$$

such that the continuity equation, (5-19), is satisfied automatically. The vorticity is thus calculated in terms of $\psi(r,\theta)$ as

$$\boldsymbol{\omega} = \omega_\phi \hat{\mathbf{e}}_\phi = \nabla \times \mathbf{u} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin\theta\hat{\mathbf{e}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ u_r & ru_\theta & 0 \end{vmatrix} = \left[\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi,$$

or

$$\boldsymbol{\omega} = \omega_\phi \hat{\mathbf{e}}_\phi = -\frac{1}{r\sin\theta}\left(D^2\psi\right)\hat{\mathbf{e}}_\phi, \tag{5-21}$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

The alternative form of the Stokes equation, (5-4b), implies that

$$D^4\psi = D^2 D^2 \psi = 0. \tag{5-22}$$

The boundary conditions for (5-22) are the no-slip condition at the surface of the sphere,

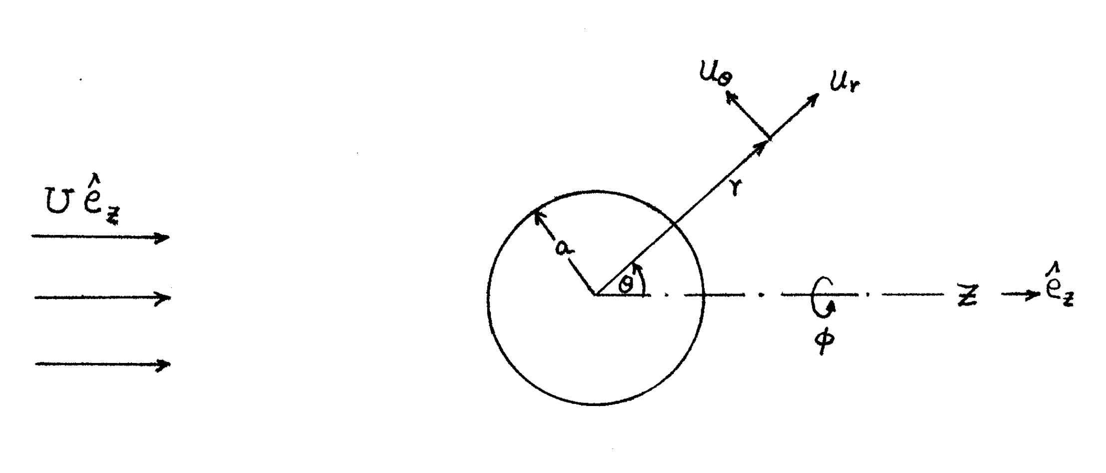


Figure 5-5 : The coordinates for uniform flow past a sphere.

$$u_r = 0 \quad \text{and} \quad u_\theta = 0 \quad \text{at} \quad r = a\,, \tag{5-23a}$$

and the far field condition,

$$u_r \hat{e}_r + u_\theta \hat{e}_\theta = U\hat{e}_z \quad \text{as} \quad r \to \infty\,. \tag{5-23b}$$

With (5-20), these conditions can be expressed in terms of $\psi(r,\theta)$ as

$$\psi = \text{constant} = 0 \quad \text{at} \quad r = a\,, \tag{5-24a}$$

$$\frac{\partial \psi}{\partial r} = 0 \quad \text{at} \quad r = a\,, \tag{5-24b}$$

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} = U\cos\theta \quad \text{as} \quad r \to \infty\,, \tag{5-24c}$$

and

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} = U\sin\theta \quad \text{as} \quad r \to \infty\,. \tag{5-24d}$$

The constant in (5-24a) is taken to be zero without loss of generality because it is the derivative of the stream function (i.e., the velocity) but not the stream function itself is of physical interest. Equations (5-24c) and (5-24d) together imply that

$$\psi \to \frac{1}{2}Ur^2 \sin^2\theta \quad \text{as} \quad r \to \infty\,. \tag{5-24e}$$

The solution procedure of (5-22) subject to (5-24a), (5-24b) and (5-24e) is as follows. Equation (5-24e) suggests that the solution should be of the form (may be regarded as a special form of "separation of variables")

$$\psi(r,\theta) = \frac{1}{2}Uf(r)\sin^2\theta\,. \tag{5-25}$$

By substituting (5-25) into (5-22), we have

$$\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)^2 f = \left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)f = 0, \qquad (5\text{-}26)$$

By substituting $f(r)=r^n$ into the above equation, we found $n=-1$, 2, 1 and 4. Thus

$$f(r)=\frac{A}{r}+Br+Cr^2+Dr^4, \qquad (5\text{-}27)$$

where A, B, C and D are constants for integration. By comparing (5-24e) with (5-25), we have $f(r)\to r^2$ as $r\to\infty$, thus $C=1$ and $D=0$. With (5-25), equations (5-24a) and (5-24b) imply that $f=0$ and $df/dr=0$ at $r=a$, it follows that $A=a^3/2$ and $B=-3a/2$. Finally, by substituting (5-27) into (5-25), we obtain

$$\psi(r,\theta) = \frac{1}{2}Ua^2\sin^2\theta\left(\frac{1}{2}\frac{a}{r}-\frac{3}{2}\frac{r}{a}+\left(\frac{r}{a}\right)^2\right). \qquad (5\text{-}28)$$

It is interested to note that $\psi(r,\theta)$ is symmetric with respect to $\theta=\pi/2$, which implies that there is no wake region behind the body for low Reynolds number flow, as indicated in Figure 4-4 before. The velocity components and the vorticity can be calculated via (5-20) and (5-21). They are

$$u_r=\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}=U\cos\theta\left[\frac{1}{2}\left(\frac{a}{r}\right)^3-\frac{3}{2}\frac{a}{r}+1\right], \qquad (5\text{-}29a)$$

$$u_\theta=-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}=U\sin\theta\left[\frac{1}{4}\left(\frac{a}{r}\right)^3+\frac{3}{4}\frac{a}{r}-1\right], \qquad (5\text{-}29b)$$

and

$$\boldsymbol{\omega}=\omega\hat{\mathbf{e}}_\phi=-\frac{1}{r\sin\theta}\left[\frac{\partial^2\psi}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right]\hat{\mathbf{e}}_\phi=-\frac{3}{2}\frac{U}{a}(\sin\theta)\left(\frac{a}{r}\right)^2\hat{\mathbf{e}}_\phi. \qquad (5\text{-}29c)$$

The pressure is evaluated by using the momentum equation, i.e., the Stokes equation, which is written in component form as

$$0=-\frac{\partial(P/\rho)}{\partial r}+\nu\left(\nabla^2u_r-\frac{2u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-2u_\theta\frac{\cot\theta}{r^2}\right) \qquad (5\text{-}30a)$$

and

$$0=-\frac{1}{r}\frac{\partial(P/\rho)}{\partial\theta}+\nu\left(\nabla^2 u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2\sin^2\theta}\right), \qquad \text{(5-30b)}$$

where

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}.$$

By substituting (5-29a) and (5-29b) into (5-30a) and (5-30b), we found

$$\frac{\partial(P/\rho)}{\partial r}=\frac{3\nu aU}{r^3}\cos\theta \qquad \text{(5-31a)}$$

and

$$\frac{\partial(P/\rho)}{\partial\theta}=\frac{3\nu aU}{2r^2}\sin\theta. \qquad \text{(5-31b)}$$

The pressure can be obtained by carrying out the integration of (5-31a) and (5-31b). The result is

$$\frac{P}{\rho}=\frac{P_\infty}{\rho}-\frac{3}{2}\frac{\mu aU}{r^2}\cos\theta, \qquad \text{(5-32)}$$

where P_∞ is the pressure far from the body where the velocity is $U\hat{e}_z$. On the surface of the sphere, P attains a minimum value at $\theta=0$ but a maximum value at $\theta=\pi$. The stress tensor for the present problem has the following non-zero components,

$$T_{r\theta}\ =\ T_{\theta r}\ =\ \mu\left(r\ \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\ +\ \frac{1}{r}\ \frac{\partial u_r}{\partial\theta}\right) \qquad \text{(5-33a)}$$

and

$$T_{rr}\ =\ -P\ +\ 2\mu\,\frac{\partial u_r}{\partial r}. \qquad \text{(5-33b)}$$

The traction is

$$\mathbf{t}\ =\ T_{rr}\hat{\mathbf{e}}_r\ +\ T_{r\theta}\hat{\mathbf{e}}_\theta. \qquad \text{(5-34)}$$

The magnitude of the force exerted by the fluid on the sphere along the z-direction is calculated as (see Figure 5-6)

$$F\ =\ \int_0^\pi \left(T_{rr}\ \cos\theta\ -\ T_{r\theta}\ \sin\theta\right)\Big|_{r=a}2\pi a(\sin\theta)ad\theta. \qquad \text{(5-35)}$$

The stress components at the wall,

$$T_{rr}\Big|_{r=a} = -\left(P_\infty - \frac{3}{2}\frac{\mu U}{a}\cos\theta\right) + 2\mu U\cos\theta\left(-\frac{3}{2}\frac{a^3}{r^4} + \frac{3}{2}\frac{a}{r^2}\right)\Bigg|_{r=a}$$

$$= -\left(P_\infty - \frac{3}{2}\frac{\mu U}{a}\cos\theta\right), \qquad (5\text{-}36a)$$

and

$$T_{r\theta}\Big|_{r=a} = -\frac{3}{2}\frac{\mu U}{a}\sin\theta. \qquad (5\text{-}36b)$$

It is interested to find that the contribution from the term $2\mu\partial u_r / \partial r$ in (5-33b) to $T_{rr}\big|_{r=a}$ is identically zero, and thus $T_{rr}\big|_{r=a}$ is due only to the pressure. By substituting (5-36a) and (5-36b) into (5-35), and carrying out the integration, we have

$$F = 2\pi a\mu U(1 + 2) = 6\pi a\mu U. \qquad (5\text{-}37)$$

One third of the force is contributed from the pressure (resulting from the first term of (5-35)), and two third of the force is associated with the shear stress. The moving stream drags the sphere with a force $F\hat{\mathbf{e}}_z$, and (5-37) is thus called the Stokes drag law. The drag may be expressed in a dimensionless form in terms of the so-called drag coefficient,

$$C_D \equiv \frac{F}{\frac{1}{2}\rho U^2 \pi a^2}. \qquad (5\text{-}38)$$

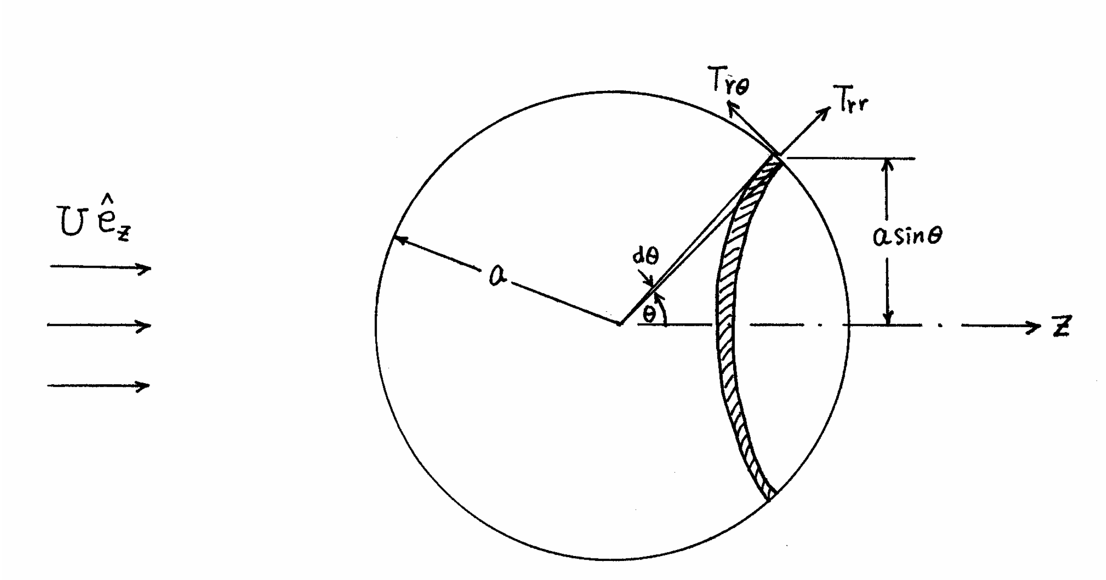


Figure 5-6: Illustration for calculating the force on the sphere. The differential area for integration is a circular "ring" with radius $a\sin\theta$ and thickness $ad\theta$, and thus equals $2\pi a(\sin\theta)ad\theta$.

With (5-37), we found

$$C_D = \frac{24}{\mathbf{R}}, \tag{5-38a}$$

where $\mathbf{R} = 2a\rho U / \mu$ is the Reynolds number. Equation (5-38a) is plotted in Figure 5-7 together with the experimental result. It is found that (5-38a) coincides with the observation when $\mathbf{R} \le 10^{-1}$, and is approximately valid when $\mathrm{R} \le 1$.

The above results are obtained by setting the convection term to zero, i.e., by totally neglecting the inertia effect. This is a good approximation for flow near the sphere, but is not valid for flow sufficiently far from the body as was pointed out by Oseen (1910). To see this, lets estimate the relative importance of the convection term to the viscous term using the results in (5-29a) and (5-29b). By using the scaling analysis, we found

$$\frac{\rho \mathbf{u} \cdot \nabla \mathbf{u}}{\mu \nabla^2 \mathbf{u}} \sim \frac{\rho U^2 / r}{\mu U / r^2} \sim \mathbf{R} \frac{r}{a}. \tag{5-39}$$

For region near the sphere, $r \sim a$, and thus the convection is indeed negligible in comparing with the viscous term as $\mathbf{R} << 1$. However, for region at $r \sim O(a / \mathbf{R})$, the convection term is comparable to the viscous term, which implies that the convection term is not negligible for region sufficiently far from the body. In such a far field region, the velocity approximately equals to $U\hat{e}_z$. Thus Oseen proposed to account partially the convective term by replacing the Stokes equation with the following linearized momentum equation,

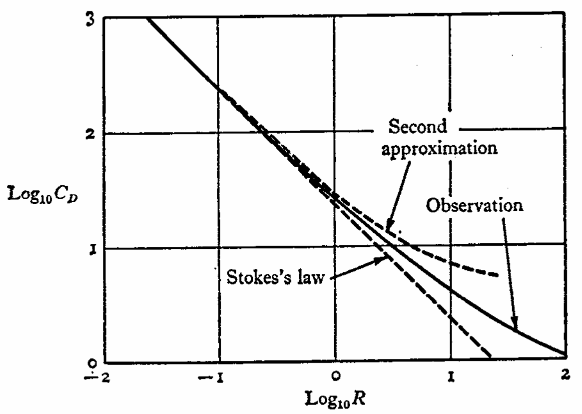


Figure 4.9.2. Comparison of measured values of the drag on a sphere (taken from Castleman 1925) and two theoretical estimates, Stokes's law $C_D = 24/R$, and a second approximation $C_D = 24R^{-1}(1 + \frac{3}{16}R)$, where $R = 2a\rho U/\mu$.

Figure 5-7: Comparison of the Stokes drag law with the observation (this figure is adopted from Batchelor (1967)).

$$\rho U \frac{\partial}{\partial z} \mathbf{u} = -\nabla P + \mu \nabla^2 \mathbf{u}, \qquad (5\text{-}40a)$$

or

$$U \frac{\partial}{\partial z} \boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega}, \qquad (5\text{-}40b)$$

which is obtained by taking curl operation of (5-40a). Similar as before for deriving (5-22), we found

$$D^4 \psi = \frac{\mathbf{R}}{a} \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \right) D^2 \psi, \qquad (5\text{-}41)$$

which is solved subject to the no-slip boundary conditions, (5-24a) and (5-24b), and the far field condition, (5-24e), at upstream. The result is (see Yih's book)

$$\psi = \frac{1}{2} U r^2 \sin^2\theta + \frac{1}{4} U \frac{a^3}{r} \sin^2\theta - \frac{3}{2} \nu a (1 + \cos\theta) \left(1 - e^{-\frac{U}{2\nu} r (1 - \cos\theta)} \right). \qquad (5\text{-}42)$$

This stream pattern is not symmetric with respect to $\theta = \pi / 2$, which is different from that in (5-28) for Stokes flow. The flow corresponding to the result described by (5-42) is called the Oseen flow. The stream pattern for the Oseen flow is plotted in Figure 5-8 together with the result of the Stokes flow. There exists a wake region behind the sphere for the Oseen flow. The drag based on the Oseen flow can be obtained in a way similar to that in Stokes flow before. The result is

$$F = 6\pi\mu a U \left(1 + \frac{3}{16} \mathbf{R} \right), \qquad (5\text{-}43)$$

which was also plotted in Figure 3-7 (the line denoted by "second approximation"). Equation (5-43) is called the Oseen's drag law in literatures. Neither the first approximation (Stokes drag) nor the second approximation (Oseen's drag) predicts the experimental data (the line denoted by "observation") nicely as **R** increases.

In fact, the linearized assumption in Oseen flow over predicts the convective effect near the sphere. The Oseen flow is a better approximation only for region sufficiently far from the sphere, while the Stokes flow is a more appropriate representation of the flow near the sphere. Thus neither the Stokes nor the Oseen solution is uniformly valid

(valid for the whole field). An approximate uniformly valid solution is obtained by Produman and Pearson (J. Fluid Mechanics, V.2, pp.237-262, 1957) using the perturbation method. They found the drag as

$$F = 6\pi\mu a U\left(1 + \frac{3}{16}\mathbf{R} + \frac{9}{160}\mathbf{R}^2 \ln\left(\frac{\mathbf{R}}{2}\right) + O\left(\mathbf{R}^2\right)\right), \qquad (5\text{-}44)$$

which agrees nicely with the observation in Figure 3-7 for $\mathbf{R} < 1$. For Reynolds number greater than unity, the drag can be estimated by using the experimental result as shown in Figure 3-9. One useful correlation for the experimental data of the drag shown in Figure 3-9 is

$$\begin{aligned} F &= 6\pi\mu a U\left(1 + 0.15\mathbf{R}^{0.687}\right) && \text{for} \quad \mathbf{R} < 1{,}000 \qquad (5\text{-}45) \\ &= 0.22\rho U^2 \pi a^2 && \text{for} \quad 1{,}000 < \mathbf{R} < 200{,}000, \end{aligned}$$

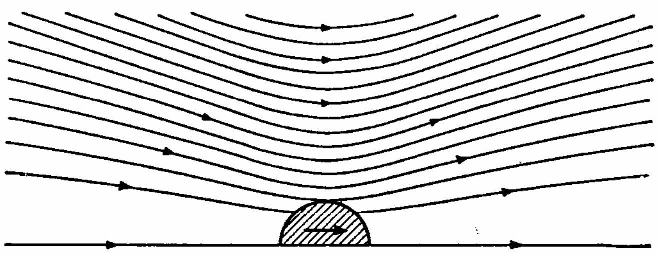


Figure 4.9.1. Streamlines, in an axial plane, for flow due to a moving sphere at $R \ll 1$ (with complete neglect of inertia forces).

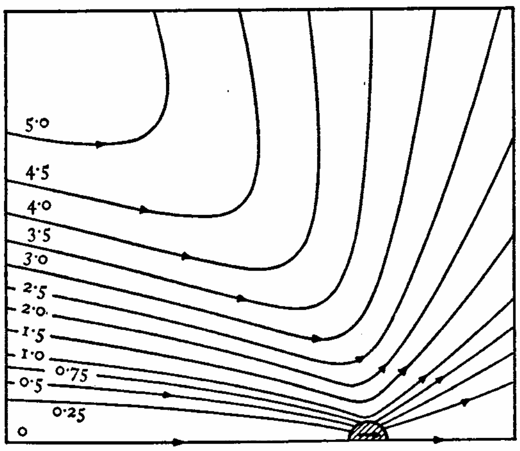


Figure 4.10.1. Streamlines in an axial plane for the outer part of the flow field due to a moving sphere, according to the Oseen equations. ψ is equal to some constant times the numbers shown on the streamlines.

Figure 5-8: Sketch of the stream patterns for Stokes (upper figure) and Oseen (lower figure) flows. Note that the frame of reference is set in the fluid instead at the center of the sphere, which implies that the stream function corresponding to the uniform flow is subtracted from equation (5-28) for the Stokes flow and (5-42) for the Oseen flow, respectively.

which is employed frequency in the field of two-phase suspension flow. When the sphere is translating with $U_B\hat{\mathbf{e}}_z$ instead of that fixed in Figure 5-5, the speed U in the above expressions for the drag is replaced by the relative speed, $U - U_B$. If the center of the sphere is translating with a velocity $\mathbf{U}_B$, which has direction different from $\hat{\mathbf{e}}_z$, the drag law should be expressed in vector form. For example, the Stokes drag law in vector form is

$$\mathbf{F} = 6\pi\mu a\left(U\hat{\mathbf{e}}_z - \mathbf{U}_B\right). \tag{5-56}$$

The direction of the drag force is parallel to the direction of the relative velocity, $\left(U\hat{\mathbf{e}}_z - \mathbf{U}_B\right)$.

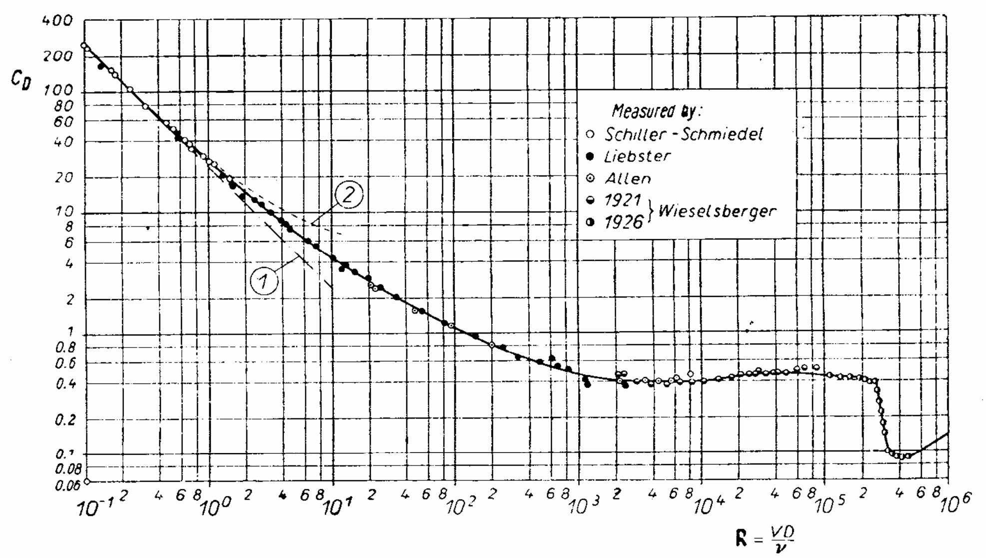


Fig. 1.5. Drag coefficient for spheres as a function of the Reynolds number
Curve (1): Stokes's theory, eqn. (6.10); curve (2): Oseen's theory, eqn. (6.13)

Figure 5-9: Experimental data for the variation of the drag coefficient with the Reynolds number. This figure is adopted from Schlichting (1979).

Homework

(1) Consider the low Reynolds number flow around a sphere. (i) Calculate the drag by first finding the dissipation of energy in the fluid-filled space. (ii) Compute $\mathbf{n} \cdot \nabla \boldsymbol{\omega}$ at the surface of a sphere. Find its integral over the surface.

(2) For the lubrication problem, find the pressure drop versus flow rate relation for a slot with width given by $h = h_0 + A \sin(2\pi x / L)$.

(3) Work out the Stokes-flow drag for low Reynolds number flow around a fluid sphere with viscosity, μ_i, which is different from the viscosity of the fluid outside the sphere, μ. The boundary conditions at the surface of the fluid sphere are (same notations as those in the Stokes flow around a solid sphere discussed above)

$$u_r = 0, \qquad T_{r\theta} \text{ and } u_\theta \text{ are continuous.}$$

It is not possible to make T_{rr} continuous; assume any imbalance is taken up by surface tension. The flow inside is described by the same equation as that outside the sphere, i.e.,

$$D^4\psi = 0 \qquad \text{for flow outside the sphere,}$$

and

$$D^4\psi_i = 0 \qquad \text{for flow inside the sphere.}$$

Take the solutions of form

$$\psi = \frac{1}{2} U \sin^2 \theta F(r),$$

and

$$\psi_i = \frac{1}{2} U \sin^2 \theta f(r).$$

Solve for the flow field, the pressure field, and the drag on the fluid sphere. The result of the drag is

$$F = 6\pi\mu a U \frac{1 + 2\mu/(3\mu_i)}{1 + \mu / \mu_i}.$$

Note that as $\mu_i \to \infty$, the result reduces to that of a solid sphere as discussed before; and as $\mu_i \to 0$, the result is the drag on a bubble.

CHAPTER 6: POTENTIAL FLOWS

As discussed in Chapter 4, the flow is essentially invisicd for most of the flow region at high Reynolds number, except in the thin boundary layer next to the body and in the wake downstream behind the body, as shown in Figure 4-1(b). The governing equations for such inviscid fluid are the continuity and Euler equations. For incompressible flow, the continuity and Euler equations are

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{(6-1a)}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P , \qquad \text{(6-1b)}$$

where P is the modified pressure which absorbs the effect of gravity. The corresponding vorticity equation for incompressible, inviscid fluid according to (2-104) is

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} , \qquad \text{(6-2)}$$

which states that the rate of change of vorticity following the fluid motion is due only to the vortex stretching mechanism shown on the right hand side of (6-2). The baroclinic, the compression and the viscous diffusion mechanisms in (2-103) are all absent in the present inviscid, incompressible flow. Also the no-slip condition is not applied here for inviscid fluid, and thus there is no vorticity generation at the solid boundary. If $\boldsymbol{\omega} = 0$ initially throughout the field, $\boldsymbol{\omega} = 0$ later according to (6-2).

We call the flow irrotational if $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u} = 0$ throughout the field. By using the vector identity, $\nabla \times \nabla \phi = 0$, for any scalar function $\phi = \phi(\mathbf{x}, t)$, we thus may define a scalar function called the velocity potential, $\phi(\mathbf{x}, t)$, such that

$$\mathbf{u} = \nabla \phi(\mathbf{x}, t) . \qquad \text{(6-3)}$$

According to the continuity equation, (6-1a),

$$\nabla^2 \phi = 0, \tag{6-4}$$

which is solved subject to the no cross flow boundary condition at solid surface,

$$\mathbf{u} \cdot \mathbf{n} = \nabla \phi \cdot \mathbf{n} = \frac{\partial \phi}{\partial n} = \mathbf{U}_B \cdot \mathbf{n}, \tag{6-4a}$$

where $\mathbf{U}_B$ is the velocity of the body associated with the solid surface, and **n** is the local unit outward normal vector of the surface. Once $\phi(\mathbf{x}, t)$ is known, we may calculate the velocity, **u**, through (6-3). It is interested to note that here we have obtained **u** by using only the irrotational condition and the continuity equation, without reference to the conservation of momentum, i.e., the Euler equation. As in the low Reynolds flow in Chapter 5, we may integrate the Euler equation to obtain the pressure field, provided **u** is known. However, it is simpler here to use the Bernoulli equation for irrotational flow, (2-109), instead of integrating the Euler equation. The Bernoulli equation is rewritten below for convenience,

$$\frac{\partial \varphi}{\partial t} + \frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{p}{\rho} + gz = f(t). \tag{6-5}$$

The incompressible, inviscid and irrotational flow is called the potential flow, because the governing equation, (6-4), is the Laplace equation. The results of the potential theory in mathematics can be applied here for potential flow,

(I) Some simple three dimensional potential flows

We first consider several simple potential flows.

(i) Point source or sink

The potential

$$\phi(x, y, z, t) = \frac{C(t)}{r}, \quad \text{with} \quad r = \sqrt{x^2 + y^2 + z^2}, \tag{6-6}$$

is a solution of the Laplace equation, (6-4), which can be validated by direct substitution. The velocity field expressed in spherical coordinates according to (6-3) is

$$u_r = -\frac{C(t)}{r^2}, \qquad u_\theta = 0, \qquad u_\phi = 0, \tag{6-7}$$

which is singular as $r \to 0$. The fluid is flowing radial outward or inward, depending on the sign of $C(t)$, and is thus called the source or sink flow. The strength of the source or sink is characterized by using the total volume flow rate, $Q(t)$, from or to the singularity at $r = 0$. For the present purely radial flow, the total volume flow rate in spherical coordinates is calculated via

$$Q(t) = \int_0^\pi u_r(r, t) 2\pi r^2 \sin\theta d\theta = -4\pi C(t). \tag{6-8}$$

Thus

$$C(t) = -\frac{Q(t)}{4\pi},$$

or

$$\phi = -\frac{Q(t)}{4\pi r}, \qquad \text{and} \qquad u_r = \frac{Q(t)}{4\pi r^2}. \tag{6-9}$$

For a source flow, $Q(t) > 0$; but $Q(t) < 0$ for a sink flow. For steady flow, $Q(t)$ is constant. Equation (6-9) is the flow induced by a source/sink located at the origin of the spherical coordinates. If the source/sink is located at $r = r_{ref}$, (6-9) is replaced by

$$\phi = -\frac{Q(t)}{4\pi(r - r_{ref})}, \qquad \text{and} \qquad u_r = \frac{Q(t)}{4\pi(r - r_{ref})^2}. \tag{6-10}$$

(ii) Source in a uniform stream – Rankine half body

The velocity potential of a uniform stream, $U_\infty \mathbf{i}$, is

$$\phi = U_\infty x, \tag{6-11}$$

which is also a solution of the Laplace equation, (6-4). As the Laplace equation is linear, the superposition of the first equation in (6-9) and (6-11) is also a solution of (6-4). The result is

$$\phi = U_\infty x - \frac{Q}{4\pi\sqrt{x^2 + y^2 + z^2}}, \qquad (6\text{-}12)$$

in a Cartesian coordinates, (x, y, z). Here the x-axis is parallel to the incoming stream, and the source is located at the origin. The velocity components along are

$$u = \frac{\partial\phi}{\partial x} = U_\infty + \frac{Q}{4\pi}\frac{x}{\left(x^2 + y^2 + z^2\right)^{3/2}}, \qquad (6\text{-}13a)$$

$$v = \frac{\partial\phi}{\partial y} = \frac{Q}{4\pi}\frac{y}{\left(x^2 + y^2 + z^2\right)^{3/2}}, \qquad (6\text{-}13b)$$

and

$$w = \frac{\partial\phi}{\partial z} = \frac{Q}{4\pi}\frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}}. \qquad (6\text{-}13c)$$

The flow is symmetric with respect to the x-axis as seen by examining the functional relationships of (6-12) and (6-13a~c). For axial symmetric flow, it is convenient to work with cylindrical coordinates, $(\hat{r}, \theta, \hat{x})$, with

$$\hat{x} = x \qquad \text{and} \qquad \hat{r} = \sqrt{y^2 + z^2}. \qquad (6\text{-}14)$$

For axial symmetric flow, all the flow properties are independent of θ, and the flow pattern in the $\hat{x}r$-plane is enough for describing the whole field. To study this further, lets first rewrite (6-12) in terms of the cylindrical coordinates,

$$\phi = U_\infty x - \frac{Q}{4\pi\sqrt{\hat{x}^2 + \hat{r}^2}}. \qquad (6\text{-}15)$$

The velocity components along the $\hat{r}$ and $\hat{x}$ directions are evaluated as

$$u_{\hat{x}} = \frac{\partial\phi}{\partial\hat{x}} = U_\infty + \frac{Q}{4\pi}\frac{\hat{x}}{\left(\hat{x}^2 + r^2\right)^{3/2}}, \qquad (6\text{-}16a)$$

and

$$u_{\hat{r}} = \frac{\partial\phi}{\partial\hat{r}} = \frac{Q}{4\pi}\frac{\hat{r}}{\left(\hat{x}^2 + \hat{r}^2\right)^{3/2}}. \qquad (6\text{-}16b)$$

The streamline in the $\hat{r}\hat{x}$-plane (actually an axial symmetric stream surface in space generated by rotating the streamline about the $\hat{x}$-axis) is evaluated by

$$\frac{d\hat{r}}{d\hat{x}} = \frac{u_{\hat{r}}}{u_{\hat{x}}} = \frac{\dfrac{Q}{4\pi}\dfrac{\hat{r}}{\left(\hat{x}^2 + \hat{r}^2\right)^{3/2}}}{U_\infty + \dfrac{Q}{4\pi}\dfrac{\hat{x}}{\left(\hat{x}^2 + \hat{r}^2\right)^{3/2}}} = \frac{\hat{r}}{\dfrac{4\pi U_\infty}{Q}\left(\hat{x}^2 + \hat{r}^2\right)^{3/2} + \hat{x}}. \qquad (6\text{-}17)$$

On the $\hat{x}$-axis, $\hat{r} = 0$, $\hat{u}_{\hat{r}}$ is identically zero, and $\hat{u}_{\hat{x}} = 0$ when

$$x = \hat{x} \equiv x_s = -\sqrt{\frac{Q}{4\pi U_\infty}} \qquad (6\text{-}18)$$

according to (6-16a) and (6-16b). The point where the velocity is zero is called the stagnation point. In the present problem, $(x, y, z) = (x_s,0,0)$, or $(\hat{x}, \hat{r}) = (x_s,0)$, is the only stagnation point in the flow. The streamline starting from the stagnation point can be obtained by solving (6-17) with boundary condition $\hat{r} = 0$ at $\hat{x} = x_s$. Such streamline is called the separating streamline because it separates the fluid from the free stream and that emitted from the source, and is sketched in Figure 6-1. As the fluid from the free stream cannot flow across the separating streamline, we may replace the flow region enclosed by such separating streamline by a solid body without affecting the flow outside the separating streamline. The solid body is called the Rankine half body in the literature. The flow field for the uniform flow past a Rankine half body is the same as that outside the separating streamline of the flow field constructed by the superposition of a uniform flow and a source.

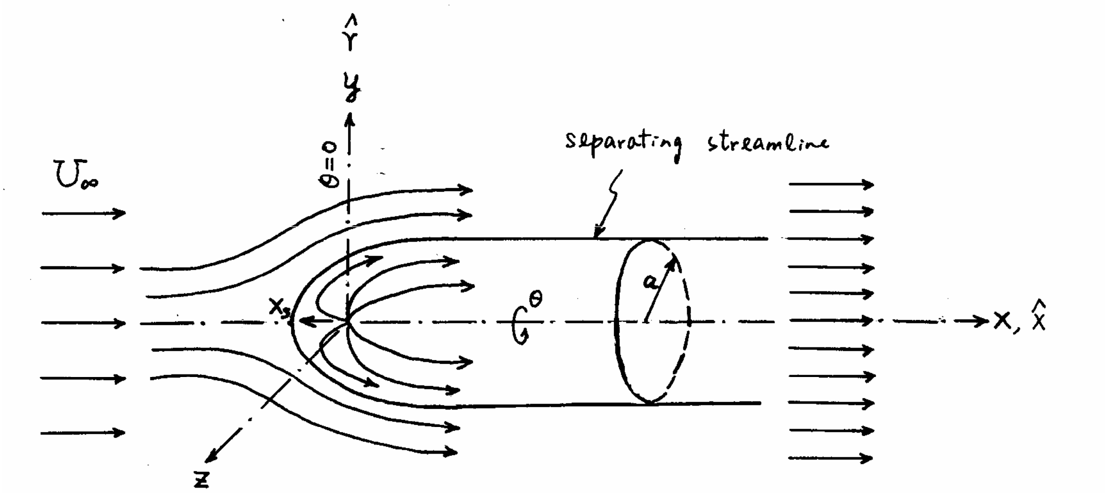


Figure 6-1: Sketch of the streamlines for the flow, which is the superposition of a uniform flow and a source located at the origin. The flow outside the separating streamline is the same as that over a Rankine half body.

For region sufficiently far from the source downstream, it is expected that the flow emitted from the source matches smoothly with the uniform stream. Thus a downstream radius, a, (see figure 6-1) is defined through

$$Q = \pi a^2 U_\infty , \qquad (6\text{-}19)$$

to represent the radial extent from the x-axis of effect of the source, i.e., the radius of the Rankine half body downstream. By substituting (6-19) into (6-18) and (6-17), we have

$$x_s = -\frac{a}{2}, \qquad (6\text{-}20a)$$

and

$$\frac{d\hat{r}}{d\hat{x}} = \frac{\hat{r}}{\frac{4}{a^2}\left(\hat{x}^2 + \hat{r}^2\right)^{3/2} + \hat{x}}, \qquad (6\text{-}20b)$$

which express the results in terms of a.

(iii) Source and sink in a uniform flow – Rankine ovoid

Similar as before in the last section, the flow generated by the superposition of a uniform flow, a source at $x = -d$ and a sink at $x = d$ as sketched in figure 6-2(a) is also a solution of the Laplace equation, the corresponding velocity potential is

$$\phi = U_\infty x - \frac{Q}{4\pi}\frac{1}{\sqrt{(x+d)^2 + y^2 + z^2}} + \frac{Q}{4\pi}\frac{1}{\sqrt{(x-d)^2 + y^2 + z^2}}. \qquad (6\text{-}21)$$

It is a good exercise to calculate the velocity field and the stream function associated with the above velocity potential using similar procedures as for the problem of Rankine half body. The flow is symmetric about the x-axis, as observed from (6-21) and the associated velocity field. The stream pattern in a symmetric plane is sketched in Figure 6-2(b). The fluid emitting from the source goes to the sink, and the free stream is distorted by the existence of the source and the sink. As the strength of the source equals to that of the sink, all the fluid emitting from the source enters the sink; the flow is also symmetric with respect to the yz-plane. The separating streamline in the symmetric plane (or stream surface in space) is an ellipse. Thus the flow outside the separating streamline is the same as that for a uniform flow past an ellipsoid. The ellipsoid is called the Rankine ovoid.

It is expected that the major axis of the ellipsoid becomes smaller as the spacing between the source and the sink is reduced, so that the ellipsoid tends to approach a sphere. However, it is observed from (6-21) that the velocity potential approaches that of a uniform flow as $d \to 0$ if Q is held fixed. Thus the limiting process for an ovoid to approach a sphere as $d \to 0$ requires $Q \to \infty$ such that Qd is finite. The flow resulting from the superposition of a source and a sink of equal strength under such a limiting process is called a dipole. According to (6-21), the velocity potential for a uniform flow over a dipole, or doublet, is

$$\phi = U_\infty x + \lim_{d \to 0}\left\{ \frac{2dQ}{4\pi}\left(\frac{\dfrac{-1}{\sqrt{(x+d)^2+\hat{r}^2}} + \dfrac{1}{\sqrt{(x-d)^2+\hat{r}^2}}}{2d} \right) \right\}$$

$$= U_\infty x - \frac{Qd}{2\pi}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+\hat{r}^2}} \right) = U_\infty x + \frac{Qd}{2\pi}\frac{x}{\left(x^2+\hat{r}^2\right)^{3/2}}, \qquad (6\text{-}22)$$

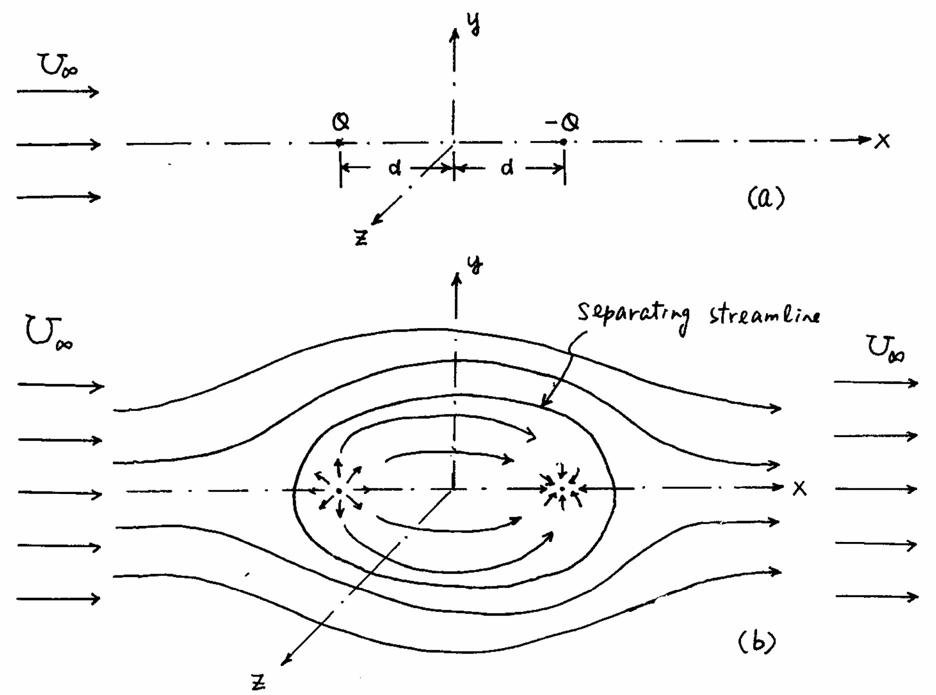


Figure 6-2: (a) The flow is composed by the superposition of a uniform flow, a source with strength Q located at $x = -d, y = z = 0$, and a sink with strength –Q located at $x = d, y = z = 0$. (b) Sketch of the streamlines of the flow. The ellipsoid enclosed by the stream surface generated by rotating the elliptic separating streamline about the x-axis is called the Rankine ovoid.

which is still a solution of (6-4). Here $\hat{r}^2 = y^2 + z^2$ is the radial distance from the x-axis. It is more convenient to work with the spherical coordinates (r, θ, ϕ), as shown in Figure 6-3, with

$$x = r\cos\theta, \qquad \hat{r} = \sqrt{y^2 + z^2} = r\sin\theta, \qquad \text{and} \qquad r = \sqrt{x^2 + \hat{r}^2}$$

for the present axial symmetric problem. Equation (6-22) can be written in terms of the spherical coordinates as

$$\phi = U_\infty r\cos\theta + \frac{Qd}{2\pi}\frac{\cos\theta}{r^2}. \tag{6-23}$$

The velocity components in spherical coordinates are

$$u_r = \frac{\partial\phi}{\partial r} = U_\infty\cos\theta - \frac{Qd}{2\pi}\frac{2\cos\theta}{r^3}, \tag{6-24a}$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U_\infty\sin\theta - \frac{Qd}{2\pi}\frac{\sin\theta}{r^3}, \tag{6-24b}$$

and $u_\omega = 0$. On the x-axis, we have $u_r = 0$ when

$$r \equiv a = \left(\frac{Qd}{\pi U_\infty}\right)^{1/3}. \tag{6-25}$$

With (6-25), the velocity potential in (6-23) and the velocity components in (6-24a,b) become

$$\phi = U_\infty\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)\cos\theta, \tag{6-26a}$$

$$u_r = U_\infty\left(1 - \frac{a^3}{r^3}\right)\cos\theta, \tag{6-26b}$$

and

$$u_\theta = -U_\infty\left(1 + \frac{1}{2}\frac{a^3}{r^3}\right)\sin\theta. \tag{6-26c}$$

The radial component of the velocity, u_r, is zero when $r = a$, which implies that the flow in the region outside $r = a$ does not mix with that inside. a is the radius of the sphere, which encloses the fluid associated with the doublet. The flow described by (6-26a-c) in the region $r \ge a$

thus also describes a uniform flow past a sphere with radius a. The stream pattern in a symmetric plane for the flow is sketched in Figure 6-3. On the surface of the sphere, $u_r = 0$ and $u_\theta = -(3/2)U_\infty \sin\theta$. At $\theta = \pm\pi/2$, $u_\theta = -(3/2)U_\infty$, which is the maximum speed throughout the flow. On the other hand, $u_\theta = 0$ at $\theta = 0$ and π. Thus $(r,\theta) = (a, \pm\pi/2)$ are the two stagnation points in the flow. The pressure on the surface of the sphere is evaluated by using the Bernoulli equation,

$$\frac{P}{\rho} = \frac{P_\infty}{\rho} + \frac{1}{2}U_\infty^{\ 2}\left(1 - \frac{9}{4}\sin^2\theta\right), \tag{6-27}$$

where the dynamic pressure is employed, or the gravity is neglected. P attains a same maximum value at $\theta = 0$ and π, but is a minimum at $\theta = \pm\pi/2$. The minimum pressure at $(r,\theta) = (a, \pm\pi/2)$ is

$$\frac{P_{\min}}{\rho} = \frac{P_\infty}{\rho} - \frac{5}{8}U_\infty^{\ 2}, \tag{6-27a}$$

and the maximum pressure at $(r,\theta) = (a, 0)$ and $(r,\theta) = (a, \pi)$ is

$$\frac{P_{\max}}{\rho} = \frac{P_\infty}{\rho} + \frac{1}{2}U_\infty^{\ 2}. \qquad \text{(6-27a)}$$

The pressure variation is associated with the local change of the fluid velocity. If the fluid under consideration is liquid, "cavitation" will occur when the local pressure is reduced to a value, which is less than the vapor pressure of the liquid. Bubbles are generated when the cavitation occurs, and serious damage of the body surface may be resulted due to the collapse of the bubbles. Cavitation is an interesting and important topic in hydraulic engineering and naval architecture.

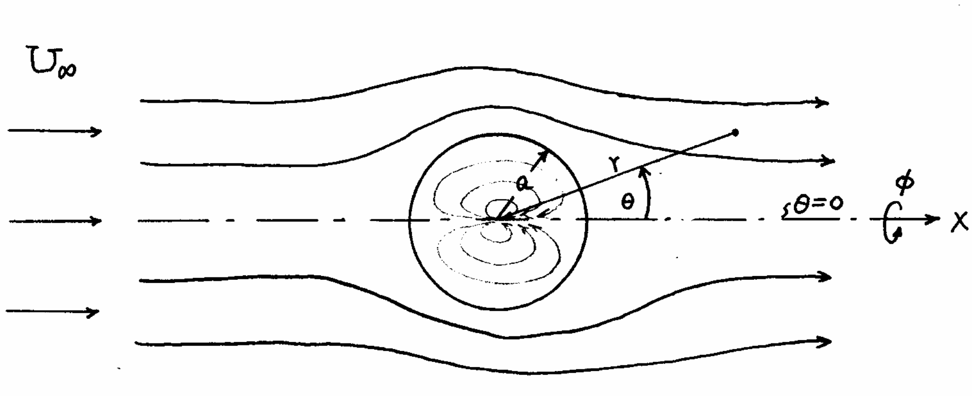


Figure 6-3: Sketch of the flow patterns for uniform flow and a doublet.

The pressure is symmetric with respect to $\theta = \pi/2$, and thus the sphere experiences no drag for the present potential flow. This is not true for the real (viscous) situation. The drag equals to zero for a symmetric body in a potential flow is called the D'Alembert paradox.

(iv) General solution of Laplace equation – Mathematical approach

So far we have proposed solutions for some particular potential flows, say, the source/sink, the uniform flow, the flow past a sphere, and etc., without actually solving the governing equation. Such solutions can certainly be obtained through a rigorous mathematical approach as follows. The Laplace equation for axial symmetric flow written in spherical coordinates is,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0 . \qquad (6\text{-}28)$$

By using the method of separation of variables, we set

$$\phi(r,\theta) = R(r)T(\theta) .$$

Equation (6-28) implies

$$\frac{r^2}{R}\left(R'' + \frac{2}{R}R'\right) = l(l+1) , \qquad (6\text{-}28a)$$

and

$$-\frac{1}{T}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) = l(l+1) . \qquad (6\text{-}28b)$$

The constant associated with the method of separation of variables is written as $l(l+1)$ so that (6-28b) is in the form of the Legendre equation. The general solution of (6-28a) is

$$R(r) = Ar^l + Br^{-(l+1)} ,$$

and the general solution of (6-28b), the Legendre equation, which is bounded at $\theta = 0$ and π, is

$$T(\theta) = P_l(\theta) ; \qquad l = 0,\ 1,\ 2,\ \ldots..$$

where

$$P_0(\cos\theta) = 1,$$

$$P_1(\cos\theta) = \cos\theta,$$

$$P_2(\theta) = \frac{1}{2}\left(1 - 3\cos^2\theta\right)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$P_l(x) = \frac{1}{2^l l!}\frac{d}{dx^l}\left(x^2 - 1\right)^l; \qquad x = \cos\theta.$$

Thus the general solution of (6-28) is

$$\phi(r,\theta) = \sum_{l=0}^{\infty} P_l(\cos\theta)\left(A_l r^l + B_l r^{-l-1}\right). \qquad (6\text{-}29)$$

Different flows result from different choices of the coefficients. For example, the only solution, which is independent of θ, is

$$\phi(r,\theta) = P_0(\cos\theta)\left[A_0 + \frac{B_0}{r}\right] = A_0 + \frac{B_0}{r}, \qquad (6\text{-}30)$$

with $A_1 = B_1 = A_2 = B_2 = \ldots\ldots = 0$. The constant A_0 in (6-30) does not affect the velocity, and can be taken as zero without loss of generality; the term B_0 / r represents the source/sink flow. Another example is for flow around a sphere. We may choose A_l and B_l such that

$$\nabla\phi \to U_\infty \hat{\mathbf{i}} \quad \text{as} \quad r \to \infty, \qquad (6\text{-}31a)$$

and

$$\frac{\partial\phi}{\partial r} = 0 \quad \text{at} \quad r = a, \qquad (6\text{-}31b)$$

in a spherical coordinates. Equation (6-31a) requires

$$\phi \to U_\infty x + \text{constant} = U_\infty r\cos\theta + \mathrm{C},$$

which implies

$$A_0 = C, \qquad A_1 = U_\infty, \qquad A_2 = A_3 = \ldots\ldots = 0.$$

Then (6-29) becomes

$$\phi(r,\theta) = C + U_\infty \cos\theta + \sum_{l=0}^{\infty} P_l(\cos\theta) B_l r^{-l-1}. \qquad \text{(6-32)}$$

By applying (6-31b), we have

$$0 = \left.\frac{\partial\phi}{\partial r}\right|_{r=0} = U_\infty \cos\theta + \sum_{l=0}^{\infty} P_l(\cos\theta)(-l-1) B_l r^{-l-1}.$$

It follows that

$$0 = U_\infty + (-1-1) B_1 a^{-3},$$

and

$$B_0 = B_2 = B_3 = \ldots\ldots = 0.$$

Thus

$$B_1 = \frac{1}{2} a^3 U_\infty.$$

Finally, (6-32) becomes

$$\phi(r,\theta) = U_\infty \cos\theta \left(r + \frac{1}{2}\frac{a^3}{r^2} \right) + C, \qquad \text{(6-32a)}$$

which is the same as that in (6-26a) since the constant C may be taken as zero without loss of generality.

(v) Flow past a slender body of revolution represented as distribution of sources and sinks

We have learned that the uniform flow past a sphere under the potential theory can be obtained by superposition of a uniform flow and a doublet. Similarly, the flow field for a uniform flow past a slender body of revolution as shown in Figure 6-4(a) can also be obtained by superposition of a uniform flow and a suitable distribution of line source and sink (as source with negative strength) as shown in Figure 6-4(b). Let $q(x)$ be the source strength per unit length, the potential at $(x,\hat{r})$ with $\hat{r}^2 = y^2 + z^2$ due to a source of strength $q(x_0)dx_0$ is (refer to (6-10))

$$\frac{-q_0(x_0)dx_0}{4\pi\sqrt{(x-x_0)^2+\hat{r}^2}}.$$

The velocity potential due to a distribution of such sources in a uniform stream as shown in Figure 6-4(b) is

$$\phi(x,\hat{r}) = U_\infty x - \frac{1}{4\pi}\int_0^l \frac{q_0(x_0)dx_0}{\sqrt{(x-x_0)^2+\hat{r}^2}}. \tag{6-33}$$

The flow is symmetric with respect to the x-axis as expected from the symmetric nature of the elements composed of the flow in Figure 6-4(b). If the total strength of the line source (sink) is

$$\int_0^l q(x_0)dx_0 = 0, \tag{6-34}$$

the body will close up behind as that shown in Figure 6-4(a). Thus the rest issue is to relate the artificial source (sink) strength, $q(x_0)$, to the actual body shape, say, the variation of the cross sectional area, $A(x_0)$. An approximate method to determine $q(x_0)$ for a slender body of

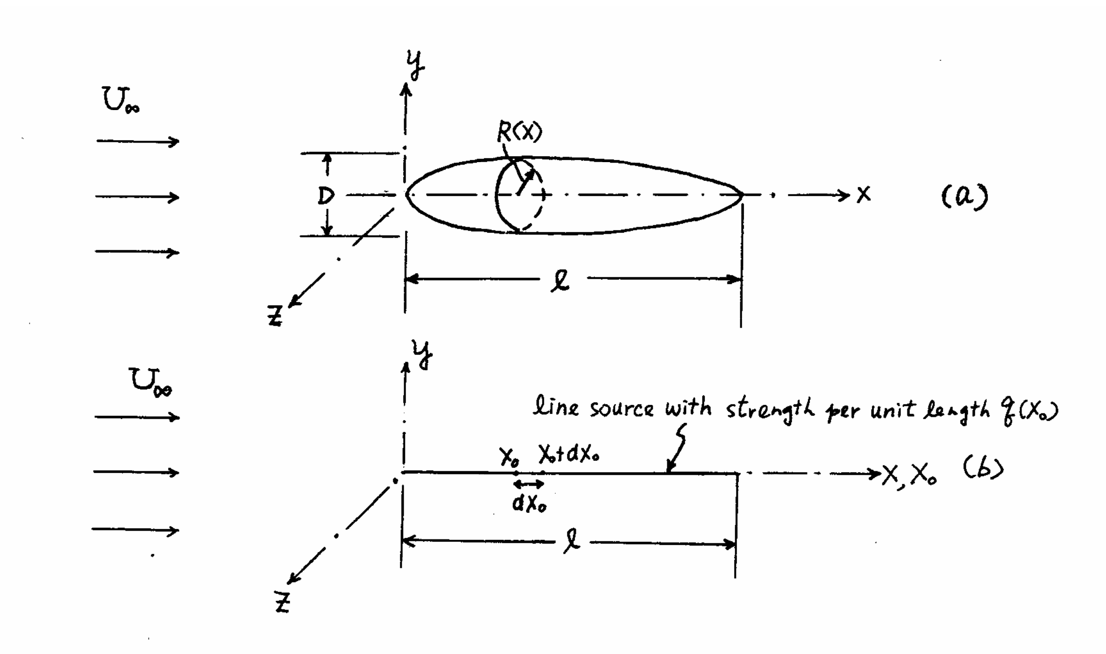


Figure 6-4: A slender body of revolution in (a) can be represented by a suitable distribution of line source (sink) in (b) for studying the uniform flow past the slender body.

revolution is as follows. As $D << l$ for slender body (see Figure 6-5), the x-component of the velocity can be expressed as

$$u(x,\hat{r}) = U_\infty + u'(x,\hat{r}) \qquad \text{with} \qquad u' << U_\infty . \tag{6-35}$$

Consider a small control volume shown in Figure 6-5. With $R(x)$ the local radius of the slender body of revolution at x, the local cross sectional area is $A(x) = \pi R(x)^2$. By applying the conservation of mass to the control volume, we have

$$U_\infty A(x + dx) - U_\infty A(x) = q(x)dx ,$$

under the condition $u' << U_\infty$. Thus as $dx \to 0$,

$$q(x) = U_\infty \frac{dA(x)}{dx} \qquad \text{or} \qquad q_0(x_0) = U_\infty \frac{dA(x_0)}{dx_0} . \tag{6-36}$$

$q(x_0)$ is positive in the front part of the body while $A(x_0)$ is increasing, and is negative in the rear part of the body while $A(x_0)$ is decreasing.

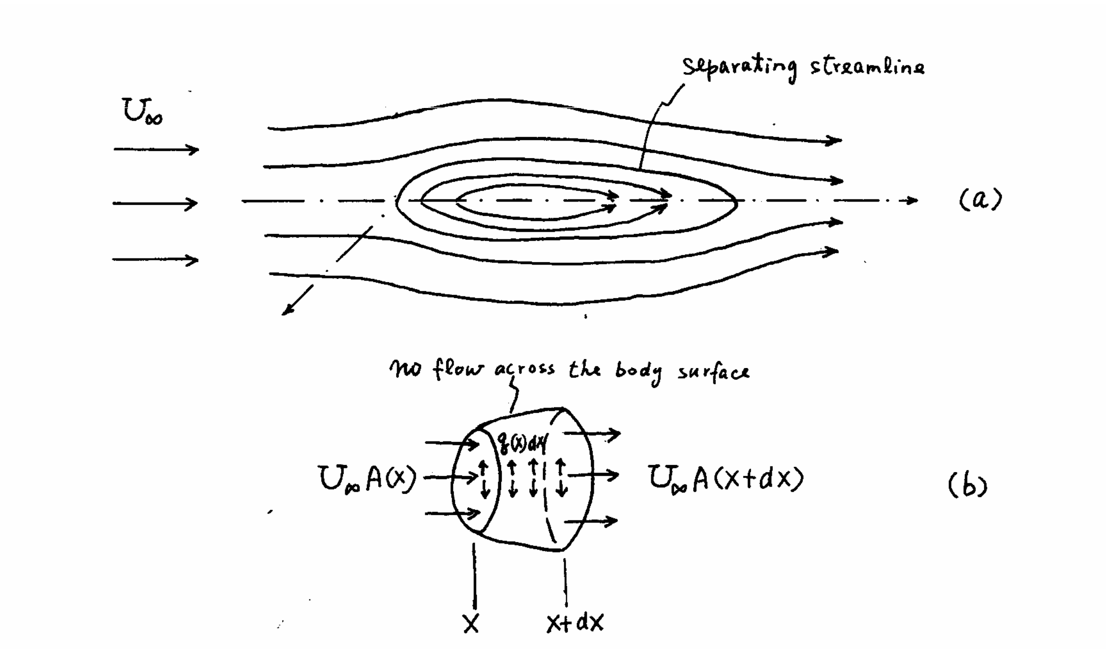


Figure 6-5: (a) Sketch of the streamlines in a symmetric plane for uniform flow past a line source (sink) with total net strength equals to zero. (b) A control volume for relating the strength, $q(x)$, to the shape of the body.

(vi) Method of image

One powerful technique in potential flow that we have employed so far is the application of superposition of simple solutions to build up more complicated flows in an unbounded fluid. We may employ such technique also to the case of bounded region by using the method of images, which is illustrated as follows. Consider the flow due to a source located above an infinite flat surface as shown in Figure 6-6(a). The flow is required to satisfy the no cross flow boundary condition at the flat surface, i.e., $\partial\phi / \partial y = 0$ at $y = 0$. Instead of solving actually the Laplace equation,

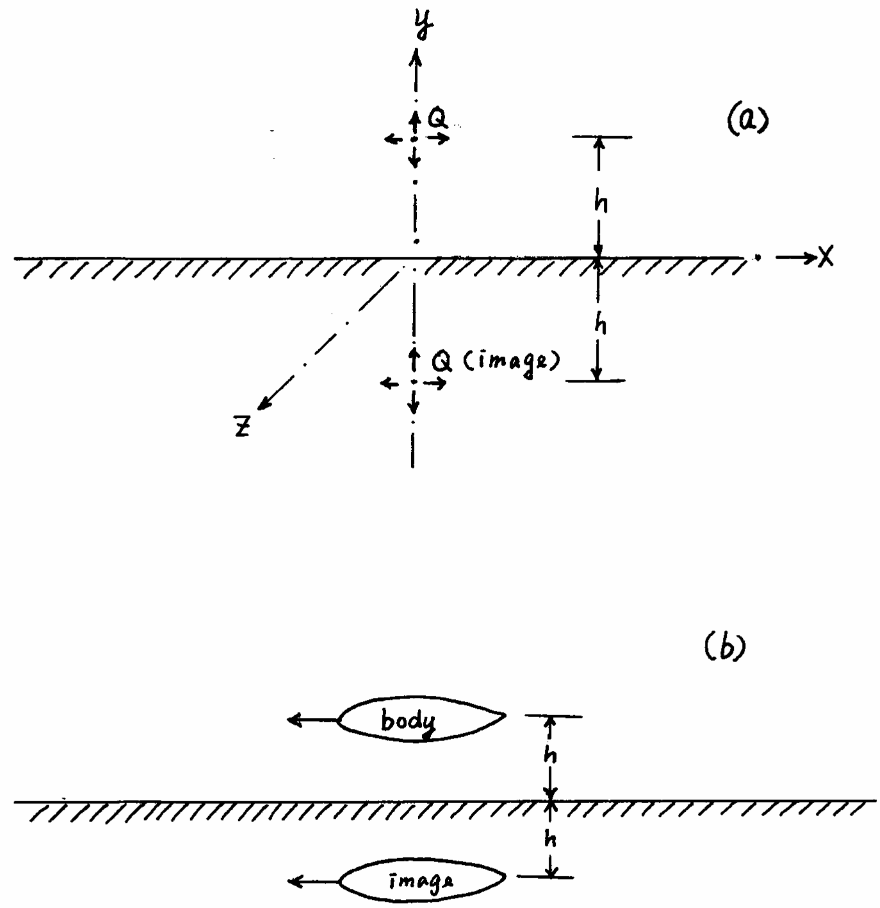


Figure 6-6: Illustration for the method of images. (a) A source is located at $y = h$ above an infinite flat surface. The flow field can be studied by the superposition of the source and its image by regarding that the infinite flat surface as a mirror. (b) A body of revolution is translating parallel to an infinite flat surface.

(6-4), subject to the appropriate boundary conditions, the method of images is to construct the solution by the superposition of the potential due to the source at $x = 0, y = h, z = 0$ with strength Q and that due to the image of the source at $x = 0, y = h, z = 0$ with the same strength. The resulting potential is

$$\phi = -\frac{Q(t)}{4\pi\sqrt{x^2 + (y-h)^2 + z^2}} - \frac{Q(t)}{4\pi\sqrt{x^2 + (y+h)^2 + z^2}}. \qquad (6\text{-}37)$$

Note that

$$v = \frac{\partial \phi}{\partial y} = \frac{Q(t)(y-h)}{4\pi\left(x^2 + (y-h)^2 + z^2\right)^{3/2}} + \frac{Q(t)(y+h)}{4\pi\left(x^2 + (y+h)^2 + z^2\right)^{3/2}},$$

which is zero when $y = 0$. Thus the flow with its velocity potential described by (6-37) indeed satisfies the no cross flow condition at the flat surface $y = 0$. Similarly, the translation of a slender body of revolution parallel to a flat surface as shown in Figure 6-6(b) can also be studied by the method of images.

(II) Two-dimensional potential flow

The major characteristic of the two-dimensional potential flow is that we can employ the powerful method of complex variables for solving problems. For the incompressible two-dimensional flow in a plane, the continuity equation is satisfied automatically by defining a stream function, ψ, in a Cartesian coordinates, (x, y), such that

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x}. \qquad (6\text{-}38)$$

The stream function has the following properties: (1) $\psi = \text{constant}$ is a streamline, (2) the difference in the values of the stream function for two streamlines is the volume flow rate per unit depth between them, and (3) the streamlines and equipotentials (lines of constant velocity potential) are perpendicular to each other. To prove the third issue, we have

$$\nabla\phi \cdot \nabla\psi = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} = u(-v) + v(u) = 0, \qquad (6\text{-}39)$$

according to (6-38) and (6-3). Thus line of constant ϕ is perpendicular to that of constant ψ. In summary, we have

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}, \tag{6-40}$$

according to the definition of the velocity potential and stream function. As the two equations in (6-40) are the Cauchy-Riemann equations, we may construct a complex function,

$$F(z) \equiv \phi(x, y) + i\psi(x, y), \tag{6-41}$$

which is an analytic function. Here $z = x + iy$, and $i = \sqrt{-1}$. For an analytic function, we have: (1) the complex derivative, *dF/dz*, exists independently of how $\Delta x \to 0$ and $\Delta y \to 0$ for any point in the complex plane, and (2) analytic functions can be expressed as a convergent power series.

The complex velocity is evaluated according to

$$W(z) = \frac{dF}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv, \tag{6-42}$$

and its complex conjugate

$$\overline{W}(z) = u + iv,$$

such that

$$W(z)\overline{W}(x) = (u - iv)(u - iv) = u^2 + v^2 = \mathbf{u} \cdot \mathbf{u}.$$

The complex variables can be expressed in terms of both the Cartesian coordinates, (x, y), and the polar coordinates, (r, θ), as shown in Figure 6-7. The relations between these two coordinates are

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}, \tag{6-43a}$$

and

$$\begin{aligned} W &= u - iv = \left(u_r \cos\theta - u_\theta \sin\theta\right) - i\left(u_r \sin\theta + u_\theta \cos\theta\right) \\ &= \left(u_r - iu_\theta\right)\left(\cos\theta - i\sin\theta\right) = \left(u_r - iu_\theta\right)e^{-i\theta} \quad . \end{aligned} \tag{6-43b}$$

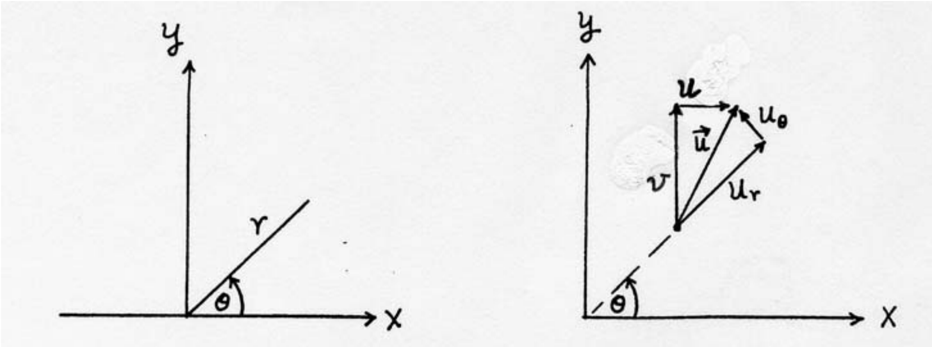


Figure 6-7: The coordinates for studying two-dimensional potential flow.

(i) Some simple flows

Here we consider some simple two-dimensional potential flows.

(1) Uniform flow

The uniform flow parallel to the *x*-axis in a plane can be described by

$$\phi = Ux, \qquad \text{or} \qquad \psi = Uy, \tag{6-44a}$$

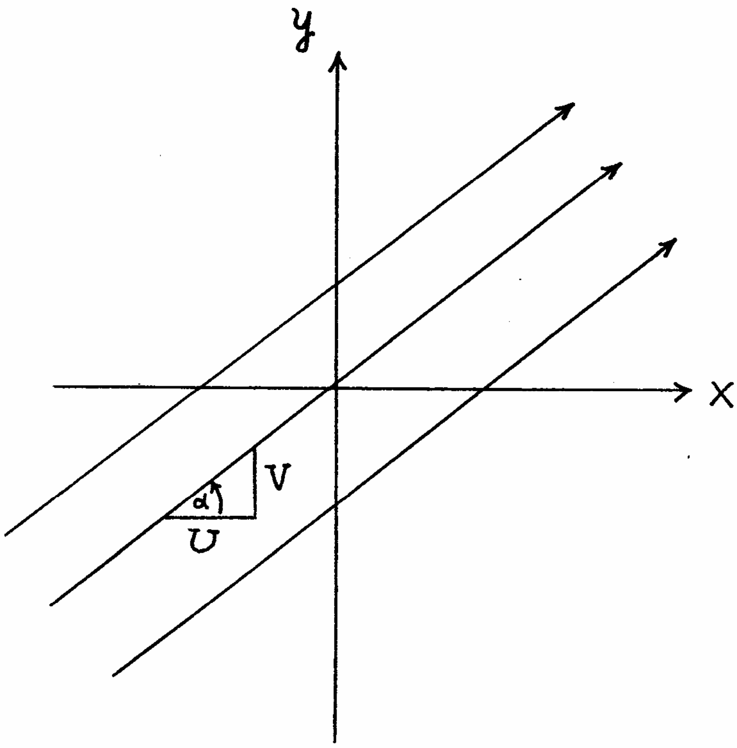


Figure 6-8: Sketch of a uniform flow, which makes an angle α with the *x*-axis.

where U is the uniform speed. The corresponding complex potential is

$$F(z) = \phi + i\psi = (x + iy)U = Uz. \qquad \text{(6-44b)}$$

If the uniform flow is not parallel to the x-axis as that shown in Figure 6-8, we have

$$F(z) = (U - iV)z, \qquad \text{(6-45)}$$

and the complex velocity

$$W(z) = \frac{dF}{dz} = U - iV$$

as expected. The stream function and the velocity potential are then

$$\psi = \text{Im}[(U - iV)(x + iy)] = Uy - Vx, \qquad \text{(6-45a)}$$

and

$$\phi = \text{Re}[(U - iV)(x + iy)] = Uy + Vx. \qquad \text{(6-45a)}$$

As

$$U = \sqrt{U^2 + V^2}\cos\alpha, \qquad V = \sqrt{U^2 + V^2}\sin\alpha,$$

(6-45) may also be written as

$$F(z) = (U - iV)z = \sqrt{U^2 + V^2}e^{-i\alpha}z. \qquad \text{(6-45c)}$$

(2) Source and sink

The complex potential for a source or sink at the origin of the complex plane, $z = x + iy = 0$, is

$$F(z) = \frac{Q}{2\pi}\ln z, \qquad \text{(6-46)}$$

which is analytic except at $z = 0$. By substituting $z = re^{i\theta}$ (see (6-43)) in (6-46), we have

$$F(z) = \frac{Q}{2\pi}(\ln r + i\theta).$$

Then

$$\phi = \frac{Q}{2\pi}\ln r \qquad \text{and} \qquad \psi = \frac{Q}{2\pi}\theta \qquad \text{(6-46a)}$$

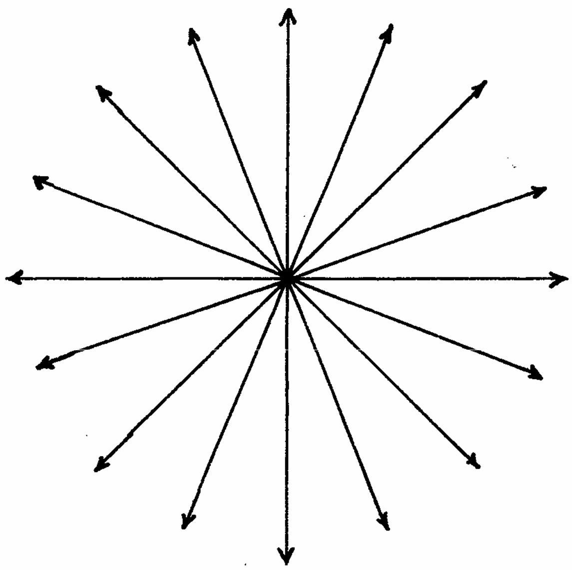


Figure 6-9: Sketch of the streamlines for flow emitted from a point source.

as sketched in Figure 6-9. The corresponding velocity components in polar coordinates are

$$u_r = \frac{\partial \phi}{\partial r} = \frac{Q}{2\pi}\frac{1}{r} \qquad \text{and} \qquad u_\theta = 0\,. \tag{6-47}$$

Here Q is the volume flow rate per unit depth as observed from

$$\int_0^{2\pi} u_r r d\theta = \int_0^{2\pi} \frac{Q}{2\pi} d\theta = Q\,.$$

For source or sink is at z_0 instead of at the origin, the complex potential in (6-46) is replaced by

$$F(z) = \frac{Q}{2\pi}\ln(z - z_0)\,. \tag{6-48}$$

(3) Plane vortex

The plane vortex is a particular flow of interest, which has no counterpart in three-dimensional potential flow. The complex potential is

$$F(z) = -\frac{i\Gamma}{2\pi}\ln z\,. \tag{6-49}$$

With $z = re^{i\theta}$,

$$F(z) = -\frac{i\Gamma}{2\pi}\left(\ln r + i\theta\right),$$

and thus

$$\phi = \frac{\Gamma\theta}{2\pi} \qquad \text{and} \qquad \psi = -\frac{\Gamma}{2\pi}\ln r\,, \tag{6-49a}$$

as sketched in Figure 6-10. The corresponding velocity components are

$$u_r = 0 \qquad \text{and} \qquad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r}\,. \tag{6-50}$$

The strength of the vortex, Γ, is called the circulation, which is defined through

$$\text{circulation} \equiv \oint \mathbf{u}d\mathbf{r} = \int_0^{2\pi} u_\theta r d\theta = \Gamma\,. \tag{6-51}$$

The circulation is not zero when the contour of integration encloses the singularity, i.e., the origin. The flow is irrotational except at origin.

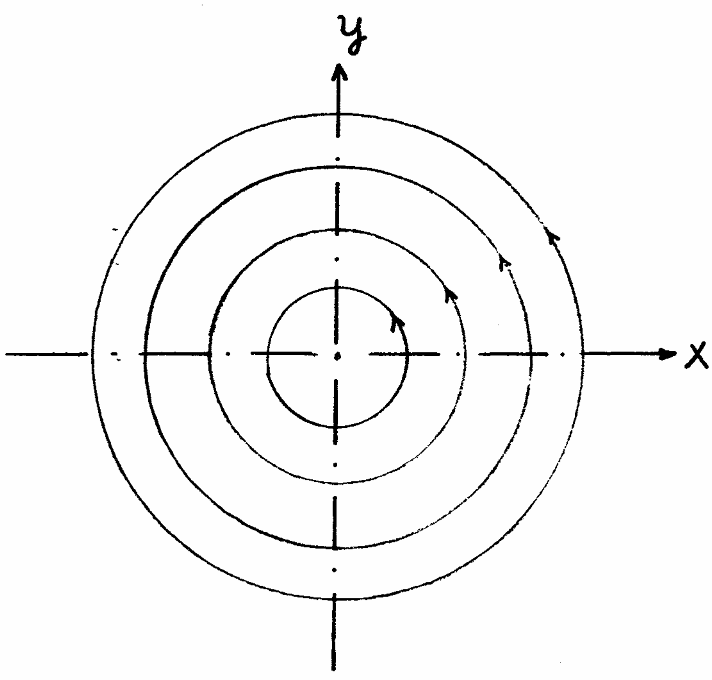


Figure 6-10: Sketch of the flow for a plane vortex.

(4) Flow in a sector

The complex potential for the flow in a sector as shown in Figure 6-11 is

$$F(z) = Uz^n = Ur^n e^{in\theta} = Ur^n \cos n\theta + iUr^n \sin n\theta, \qquad (6\text{-}52)$$

for real constants, n and U. The velocity potential and the stream function are thus

$$\phi = Ur^n \cos n\theta \qquad \text{and} \qquad \psi = Ur^n \sin n\theta. \qquad (6\text{-}52a)$$

Any streamline of the flow is a possible solid boundary for steady flow. In particular, the streamline with $\psi = 0$ is chosen for

$$\theta = 0 \qquad \text{and} \qquad \theta = \pi / n.$$

The complex velocity

$$W = \frac{dF}{dz} = nUz^{n-1} = nUr^{n-1}e^{i(n-1)\theta} = u - iv = (u_r - iu_\theta)e^{-i\theta}. \qquad (6\text{-}53)$$

Then

$$u_r = nUr^{n-1} \cos n\theta \qquad \text{and} \qquad u_\theta = -nUr^{n-1} \sin n\theta, \qquad (6\text{-}53a)$$

or

$$u = nUr^{n-1} \cos(n - 1)\theta \qquad \text{and} \qquad v = -nUr^{n-1} \sin(n - 1)\theta. \qquad (6\text{-}53b)$$

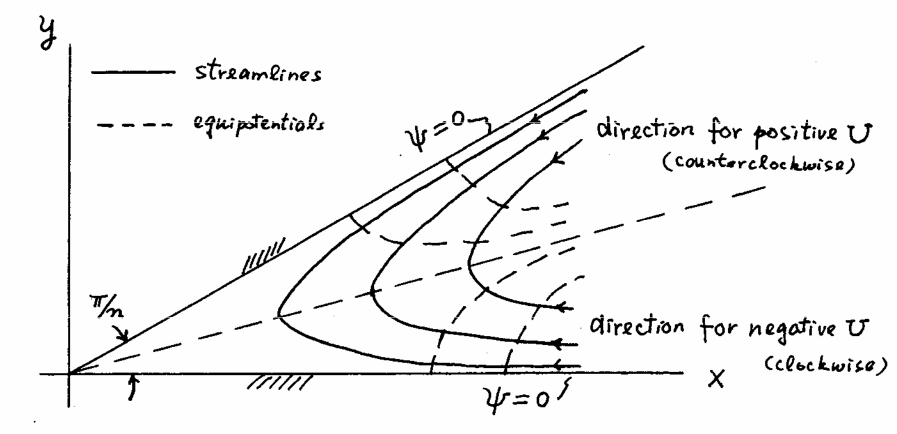


Figure 6-11: Sketch of the streamlines and equipotentials for flow in a sector.

The variations of u_r and u_θ with θ for $n > 1$ and positive U are shown in Figure 6-12, which is illustrative for plotting Figure 6-11. The magnitude of the complex velocity

$$|W| = \sqrt{u^2 + v^2} = \sqrt{W\overline{W}} = n|U|r^{n-1}, \tag{6-54}$$

is independent of θ. When $r \to 0$, $|W| \to 0$ if $n > 1$, which shows a stagnation point at $r = 0$ for the case with a vertex angle less than π. However, $|W| \to \infty$ if $n < 1$, which shows a singular point at $r = 0$. The cases corresponding to different values of n are sketched in Figure 6-13.

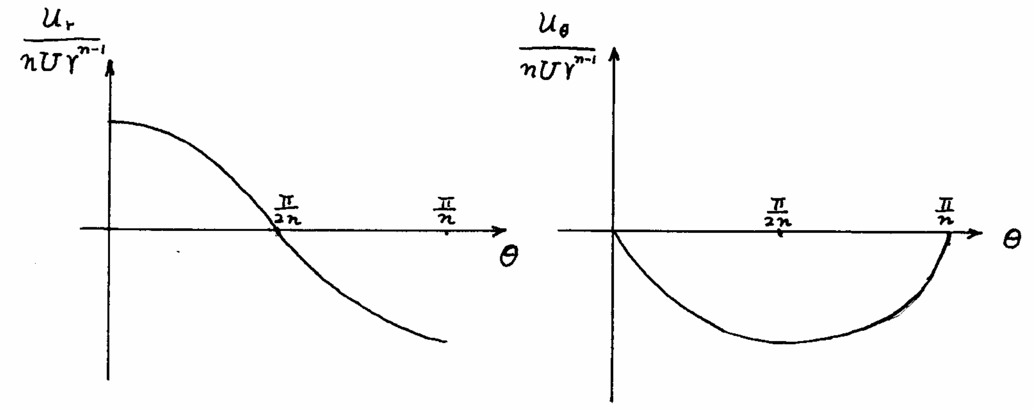


Figure 6-12: The variations of the velocity components with θ.

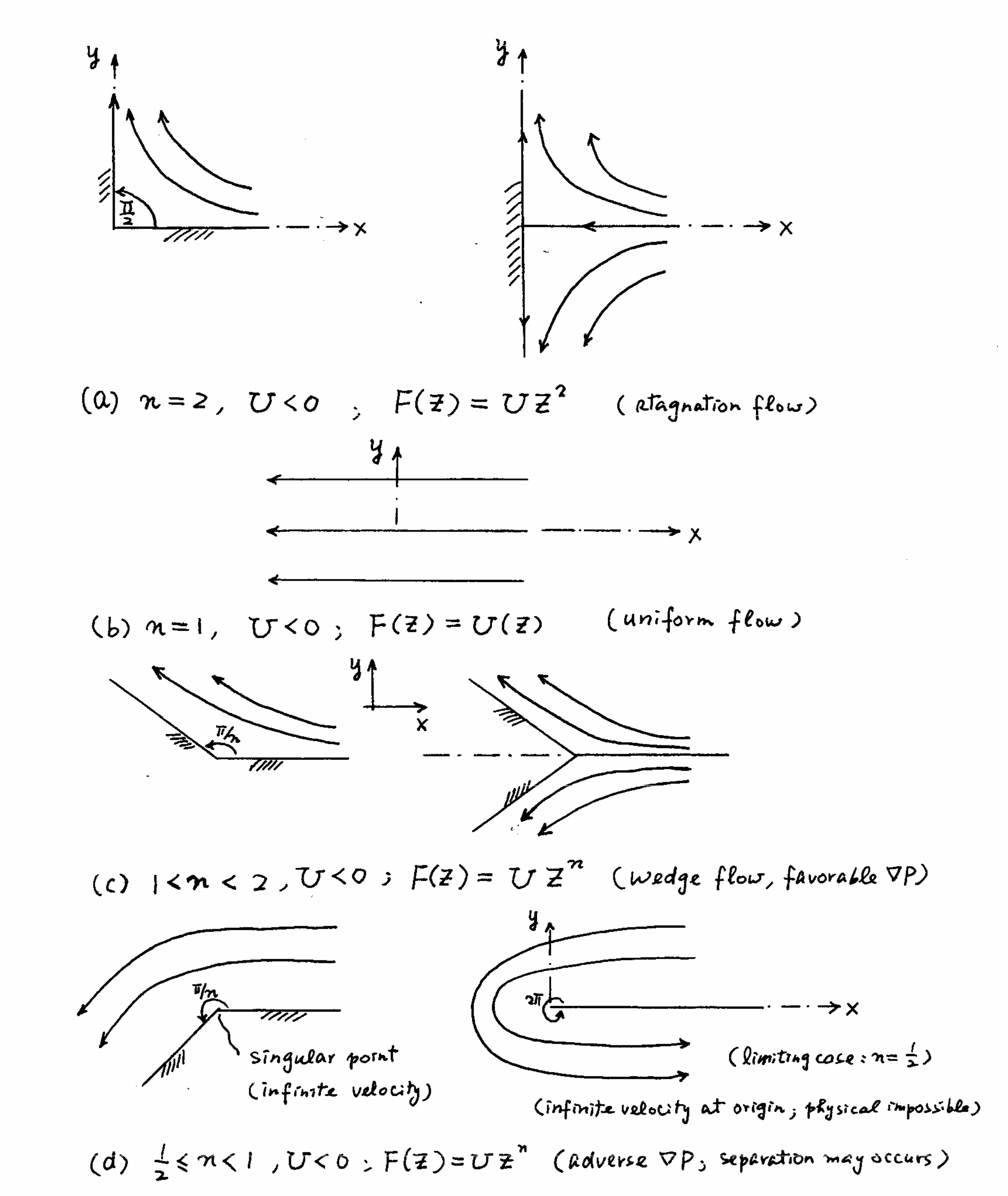


Figure 6-13: Several special cases for flows in a sector.

(5) Flow due to a doublet or dipole

As in three-dimensional potential flow, the flow due to a doublet or a dipole can be generated by the superposition of the flow due to a source and that due to a sink of equal strength at the same position. Consider the flow due to a source with strength Q at $x=-\varepsilon, y=0$ and a sink with strength Q at $x=\varepsilon, y=0$ as shown in Figure 6-14. The complex potential is

$$
\begin{aligned}
F(z) &= \frac{Q}{2\pi}\ln(z+\varepsilon) - \frac{Q}{2\pi}\ln(z-\varepsilon) \\
&= \frac{Q}{2\pi}\ln\left(\frac{z+\varepsilon}{z-\varepsilon}\right) \\
&= \frac{Q}{2\pi}\ln\left(\frac{1+\varepsilon/z}{1-\varepsilon/z}\right).
\end{aligned} \tag{6-55a}
$$

With the Taylor series expansion,

$$
\begin{aligned}
(1+\delta)^{\alpha} &= 1+\alpha\delta+...... , \\
(1+\delta)^{-1} &= 1-\delta+......,
\end{aligned}
$$

and

$$
\ln(1+\delta) = \delta + ...,
$$

for small values of δ, equation (6-55a) becomes

$$
\begin{aligned}
F(z) &= \frac{Q}{2\pi}\ln\left[\left(1+\frac{\varepsilon}{z}\right)\left(1+\frac{\varepsilon}{z}+.......\right)\right] \\
&= \frac{Q}{2\pi}\ln\left[1+2\frac{\varepsilon}{z}+......\right] \\
&= \frac{Q}{2\pi}2\frac{\varepsilon}{z}+O\left(\varepsilon^{2}\right).
\end{aligned} \tag{6-55b}
$$

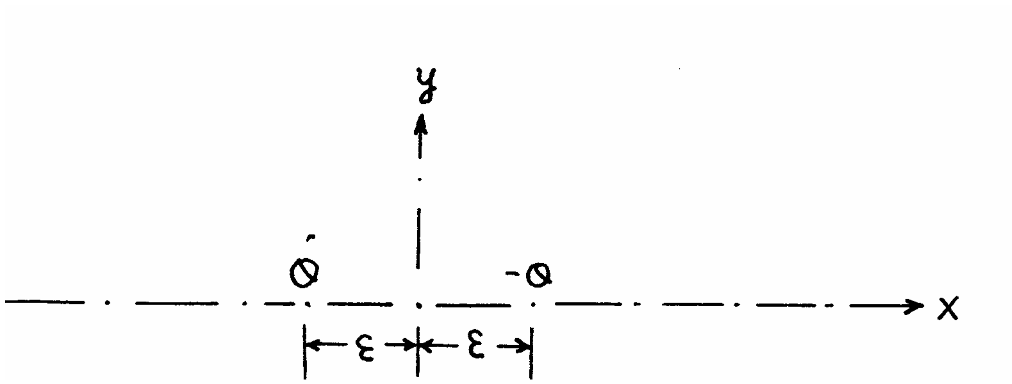


Figure 6-14: A doublet or dipole is generated by the superposition of a source and a sink with equal strength by letting $\varepsilon \rightarrow 0$.

Taking the limit as $\varepsilon \to 0$, $Q \to \infty$, such that $Q\varepsilon = \pi\mu$, where μ is a finite value, equation (6-55b) becomes

$$F(z) = \frac{\mu}{z}, \tag{6-56}$$

which is the complex potential due to a doublet or dipole with strength μ. The complex potential, $F(z)$, is analytic in (6-56) except at $z = 0$. By substituting $z = x + iy$ into (6-56), we have

$$F(z) = \frac{\mu}{x + iy} = \frac{\mu(x - iy)}{x^2 + y^2}. \tag{6-56a}$$

Thus

$$\phi = \frac{\mu x}{x^2 + y^2} \qquad \text{and} \qquad \psi = -\frac{\mu y}{x^2 + y^2}. \tag{6-57}$$

The streamlines are

$$\psi = -\frac{\mu y}{x^2 + y^2} = \text{constant},$$

or

$$x^2 + y^2 = -\left(\frac{\mu}{\psi}\right) y,$$

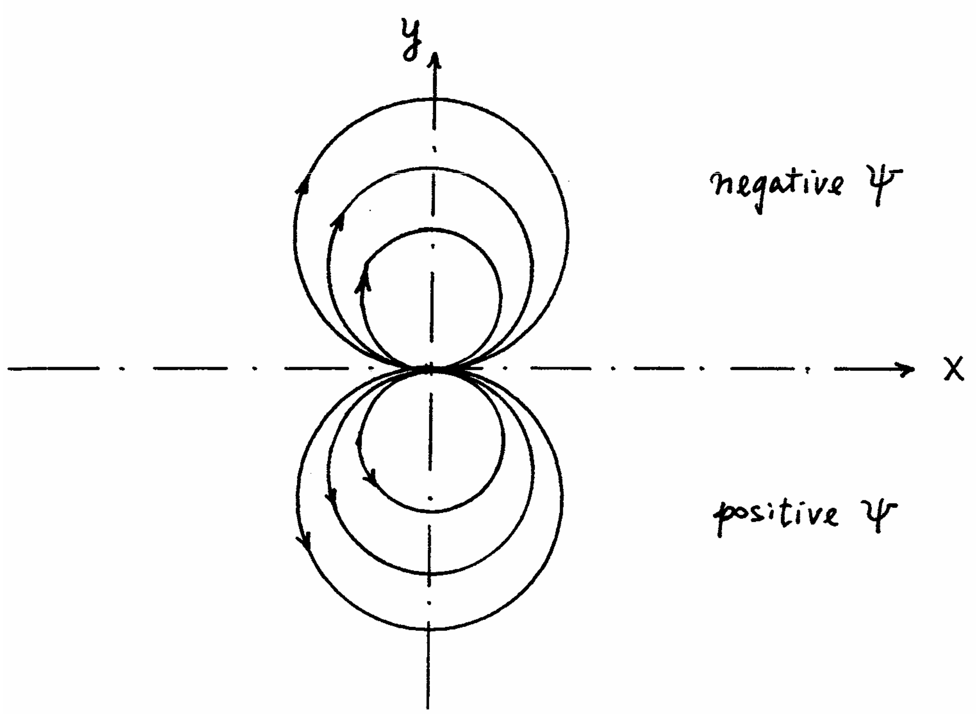


Figure 6-15: Sketch of the stream patterns due to a doublet.

or

$$x^2 + \left(y + \frac{\mu}{2\psi} \right)^2 = \left(\frac{\mu}{2\psi} \right)^2, \qquad (6\text{-}58)$$

which describes a circle with radius $|\mu/(2\psi)|$ at center $(x, y) = (0, -\mu/(2\psi))$. The circle passes through the origin; and the stream patterns are sketched in Figure 6-15. The complex velocity

$$W(z) = u - iv = u_r - iu_\theta = \frac{dF}{dz} = -\frac{\mu}{z^2} = -\frac{\mu}{r^2} e^{-2i\theta}, \qquad (6\text{-}59)$$

and thus

$$u_r = -\frac{\mu \cos\theta}{r^2} \qquad \text{and} \qquad u_\theta = -\frac{\mu \sin\theta}{r^2}. \qquad (6\text{-}59a)$$

It is interest to note that the doublet has the directional property. If the doublet is generated by superposition of a source and a nearby sink along a line, which makes an angle α with respect to the x-axis, as shown in Figure 6-16, the complex potential

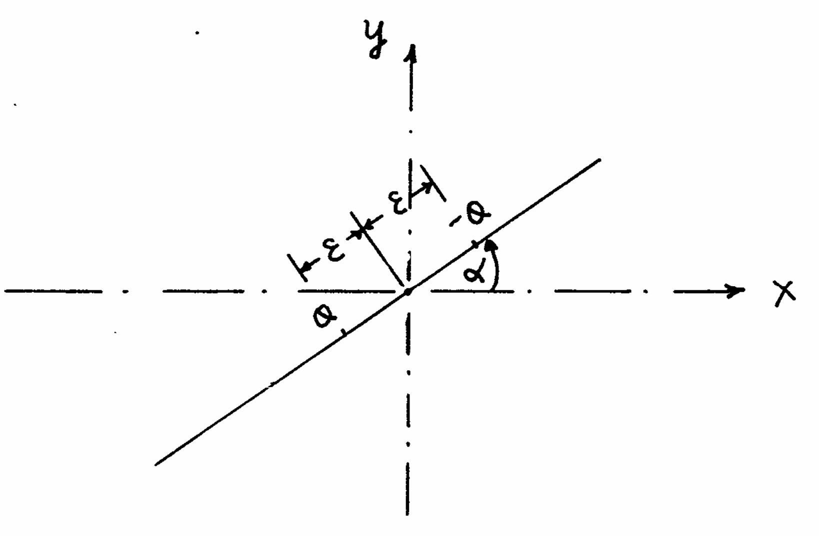


Figure 6-16: A sketch showing the directional property of a doublet.

$$F(z) = \frac{Q}{2\pi}\ln\left(z - \varepsilon e^{i(\pi+\alpha)}\right) - \frac{Q}{2\pi}\ln\left(z - \varepsilon e^{i\alpha}\right)$$
$$= \frac{Q}{2\pi}\ln\left(z + \varepsilon e^{i\alpha}\right) - \frac{Q}{2\pi}\ln\left(z - \varepsilon e^{i\alpha}\right)$$
$$= \frac{Q}{2\pi}\ln\left(\frac{1+\frac{\varepsilon}{z}e^{i\alpha}}{1-\frac{\varepsilon}{z}e^{i\alpha}}\right)$$
$$= \frac{Q}{2\pi}\ln\left[\left(1+\frac{\varepsilon}{z}e^{i\alpha}\right)\left(1+\frac{\varepsilon}{z}e^{i\alpha}+......\right)\right]$$
$$= \frac{Q}{2\pi}\ln\left(1+2\frac{\varepsilon}{z}e^{i\alpha}+......\right)$$
$$= \frac{Q}{2\pi}2\frac{\varepsilon}{z}e^{i\alpha}+......,$$

or

$$F(z) = \frac{\mu}{z}e^{i\alpha}, \tag{6-60}$$

if the strength of the doublet, μ, is defined through $Q\varepsilon = \pi\mu$ when $\varepsilon \to 0$ and $Q \to \infty$. If we apply the method of image to study the flow generated by a doublet at $z_0 = x_0 + iy_0$ in the vicinity of a solid flat wall as shown in Figure 6-17, the resulting complex potential is

$$F(z) = \frac{\mu e^{i\alpha}}{z-(x_0+iy_0)} + \frac{\mu e^{i(\pi-\alpha)}}{z-(-x_0+iy_0)} = \frac{\mu e^{i\alpha}}{z-(x_0+iy_0)} - \frac{\mu e^{-i\alpha}}{z-(-x_0+iy_0)} \tag{6-61}$$

by the superposition of the flow due to the doublet at $z_0 = x_0 + iy_0$ with orientation α and that due to its image at $-x_0 + iy_0$ with orientation $\pi - \alpha$.

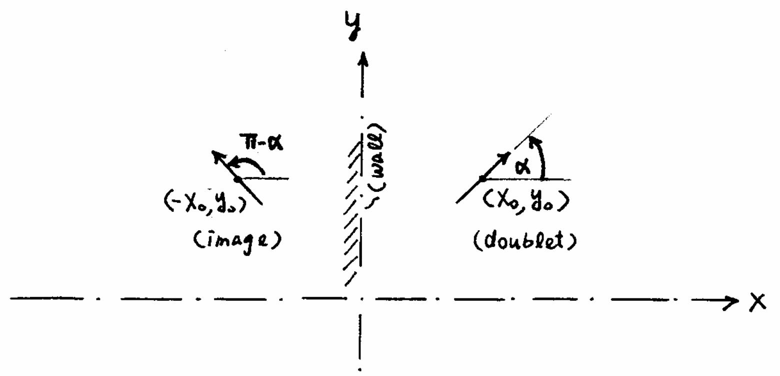


Figure 6-17: The image of the doublet makes an angle $\pi - \alpha$ with respect to the x-axis, which is different from α, the angle between the doublet axis and the x-axis.

(ii) Uniform flow past a circular cylinder

As in three-dimensional potential flow, the complex potential for uniform flow past a circular cylinder can be obtained by the superposition of a uniform flow and a doublet,

$$F(z) = Uz + \frac{\mu}{z} = \left(Ur\cos\theta + \frac{\mu}{r}\cos\theta\right) + i\left(Ur\sin\theta - \frac{\mu}{r}\sin\theta\right). \quad \text{(6-62a)}$$

Thus

$$\phi = Ur\cos\theta + \frac{\mu}{r}\cos\theta \qquad \text{and} \qquad \psi = Ur\sin\theta - \frac{\mu}{r}\sin\theta. \quad \text{(6-62b)}$$

By taking $\psi = 0$ on a circle of radius a, we have

$$aU - \frac{\mu}{a} = 0$$

according to the second equation of (6-63). Thus the radius of the cylinder, a, is related to the strength of the doublet for generating the flow as

$$a = \sqrt{\mu / U}. \quad \text{(6-63)}$$

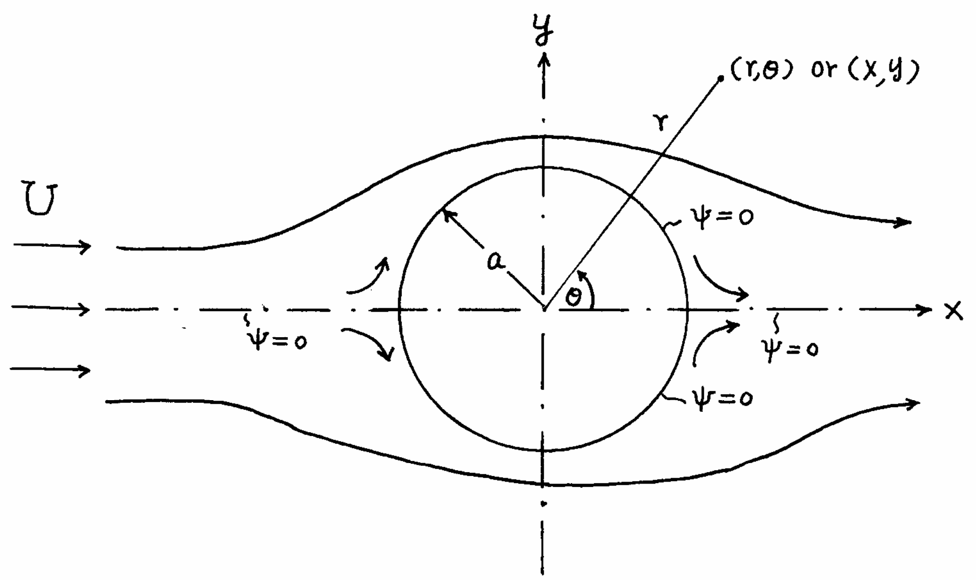


Figure 6-18: Sketch for uniform flow past a circular cylinder.

It follows that

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta, \tag{6-64a}$$

and

$$F(z) = U\left(z + \frac{a^2}{z}\right). \tag{6-64b}$$

The velocity components

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \tag{6-65a}$$

and

$$u_\theta = -\frac{\partial\psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta. \tag{6-65b}$$

The radial velocity component is identically zero on the surface of the cylinder, $r = a$. The fluid slips on the surface of the cylinder for the present inviscid flow, and the circumferential velocity component

$$u_\theta = -2U\sin\theta, \tag{6-65c}$$

which attains a maximum value, $2U$, at $\theta = \pi/2$. Note that the flow is symmetric with respect to both the x-axis and y-axis. There is no net force on the cylinder for potential flow.

(iii) Flow around a circular cylinder with circulation

If we superimpose a plane vortex with its center at the origin of the coordinates on the uniform flow past a circular cylinder, the complex potential

$$F(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z + C. \tag{6-66}$$

Here a constant, C, is added without affecting the velocity field. Such constant is chosen such that $\psi = 0$ on the surface of the cylinder, i.e.,

$$C = \frac{i\Gamma}{2\pi}\ln a\,.$$

Then

$$F(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln\left(\frac{z}{a}\right). \qquad (6\text{-}67)$$

By substituting $z = re^{i\theta}$,

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right), \qquad (6\text{-}68)$$

which indeed satisfies $\psi = 0$ at $r = a$ as expected. The stream function in (6-68) is not unique, and depends on the value of Γ as follows. To see this, the velocity components

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta\,, \qquad (6\text{-}69a)$$

and

$$u_\theta = -\frac{\partial\psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}. \qquad (6\text{-}69b)$$

On the surface of the cylinder, $r = a$, the radial component $u_r = 0$ identically, and the circumferential component

$$u_\theta = -2U\sin\theta + \frac{\Gamma}{2\pi a}. \qquad (6\text{-}69c)$$

Stagnation points on the surface of the cylinder are the values at $\theta = \theta_s$ and $r = a$ where $u_r = 0$ and $u_\theta = 0$. According to (6-69c),

$$\sin\theta_s = \frac{\Gamma}{4\pi Ua}. \qquad (6\text{-}70)$$

There are four different cases as illustrated in Figure 6-19. When $\Gamma = 0$, $\theta_s = 0$ and π. The flow is symmetric as before in the last section, and there is no net force on the cylinder due to the flow. When $\Gamma < 0$ and $|\Gamma/(4\pi Ua)| < 1$, the stagnation points shift "downward" as shown in Figure 6-19(b) according to (6-70). The velocity is larger on the top of the cylinder, the pressure is reduced there according to the Bernoulli equation, and thus a lift force is generated by the flow associated with the effect of circulation. A negative lift is generated if $\Gamma > 0$. When $\Gamma < 0$, $|\Gamma/(4\pi Ua)| = 1$ and $\theta_s = -3\pi/2$; there exists only one stagnation point on the surface of the cylinder. A lift associated with the negative

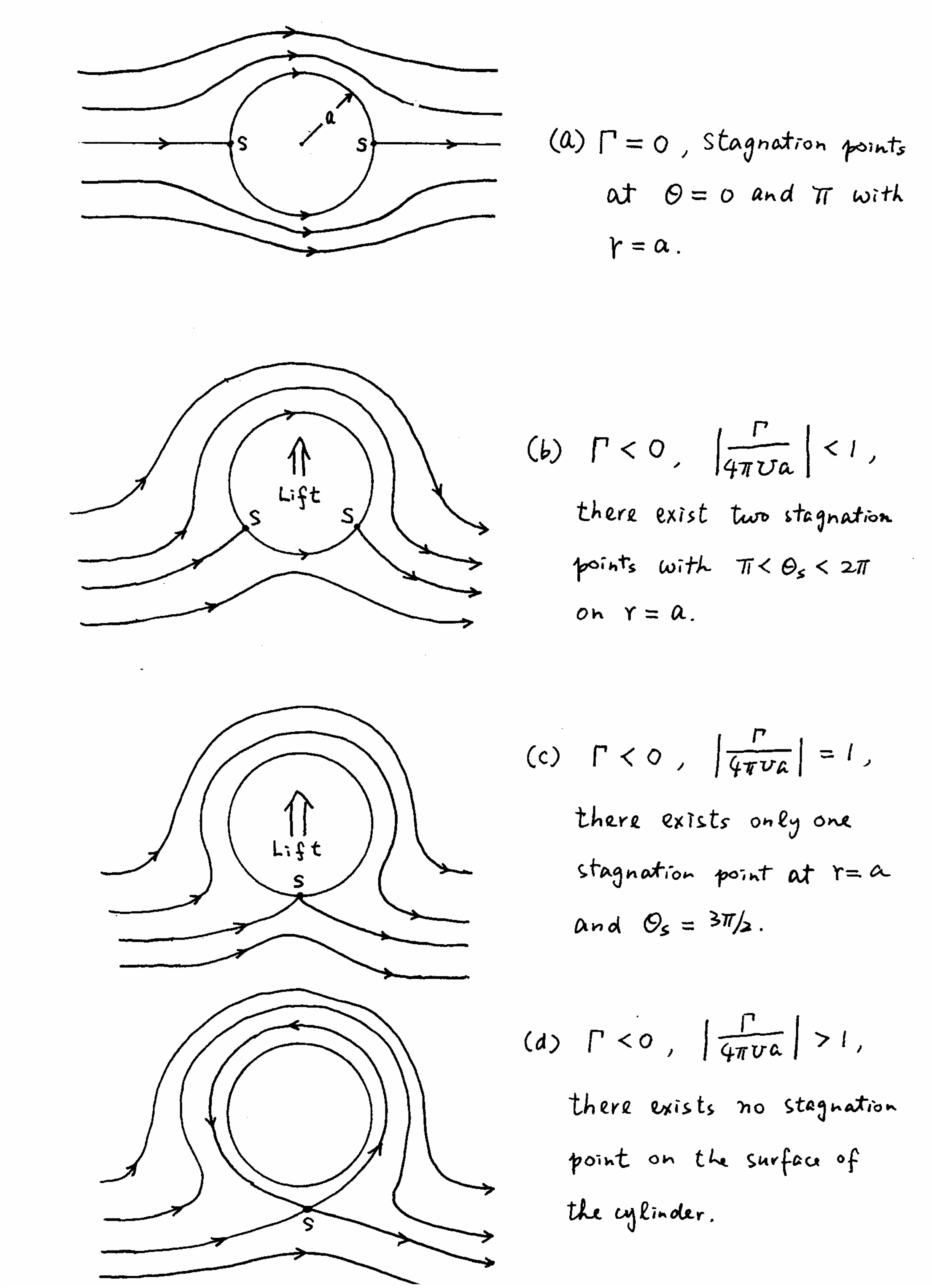


Figure 6-19: Sketch of the flow for different situation. The point "s" represents the location of the stagnation point.

circulation is also generated as in case (b). When $\Gamma < 0$ and $|\Gamma/(4\pi Ua)| > 1$, there is no stagnation point on the body surface. However, we may find a stagnation point in the flow field as follows. For stagnation point, the complex velocity

$$W(z) = \frac{dF}{dz} = U\left(1 - \frac{a^2}{z^2}\right) - \frac{i\Gamma}{2\pi z} = 0. \tag{6-71}$$

Thus the stagnation point occurs at

$$\frac{z}{a} = \frac{i\Gamma}{4\pi Ua} \pm \sqrt{1 - \left(\frac{\Gamma}{4\pi Ua}\right)^2}.$$

Both roots are pure imaginary when

$$\left(\frac{\Gamma}{4\pi Ua}\right)^2 > 1.$$

One root occurs within the cylinder, and the other is

$$\frac{z}{a} = -i\left|\frac{\Gamma}{4\pi Ua}\right| - i\sqrt{\left(\frac{\Gamma}{4\pi Ua}\right)^2 - 1}, \tag{6-72}$$

which implies that the stagnation point outside the cylinder occurs at

$$r_s = -\left(\left|\frac{\Gamma}{4\pi Ua}\right| + \sqrt{\left(\frac{\Gamma}{4\pi Ua}\right)^2 - 1}\right) \quad \text{and} \quad \theta_s = \frac{3\pi}{2},$$

as illustrated in Figure 6-19(d). With the velocity known, we may evaluate the pressure through the Bernoulli equation, and then calculate the lift force on the cylinder. For two-dimensional flow, there exists a Blasius theorem for calculating the force and moment on the body directly once the complex velocity is known, as discussed in Chapter 4 of Currie's book.

(iv) Method of images

As in three-dimensional potential flow, the method of images can be employed for studying a variety of problems.

(v) Flow for complicated geometries

In the text of complex variables, method of changes of variables is usually employed for solving problems with more complicated geometries, for example, the conformal mapping. In two-dimensional potential flows, two transformations, the Joukowski transformation and the Schwarz-Christoffel transformation, are employed quite successfully in studying certain problems, as discussed in Chapter 4 of Currie's book.

(vi) Water waves

Although the potential flow assumption is not quite successfully in determining the force on the body. The potential theory is a suitable tool for studying the water waves (see Chapter 6, Currie).

Homework

(1) Consider the uniform potential flow over a sphere. Obtain the stream function in a symmetric plane, plot the stream patterns, and calculate the force on the sphere.

(2) Consider the two-dimensional potential flow due to a doublet next to an infinite solid wall as that shown in Figure 6-17 of the course note. Obtain the velocity potential, the stream function, the velocity field, the pressure field, and the force acting on the solid wall.

(3) Consider uniform flow past a circular cylinder with negative circulation as that shown in Figure 6-19 of the course note. Calculate the pressure field, and the force on the cylinder for the four cases shown in the figure.

(4) Consider three-dimensional potential flow. The following figure shows a doublet of strength μ located at $x = l$ and a doublet of strength μ^* located at $x = a^2 / l$. Show that the surface $r = a$ corresponds to $\psi = 0$ if $\mu^* = -a^3\mu / l^3$, and hence deduce that the stream function for a doublet of strength μ located a distance *l* from the center of a sphere of radius *a* is

$$\psi(r, \theta) = -\frac{\mu}{4\pi\xi}\sin^2\beta + \frac{\mu a^3}{4\pi l^3\eta}\sin^2\alpha .$$

Also deduce that the corresponding velocity potential is

$$\phi(r, \theta) = \frac{\mu}{4\pi\xi^2}\cos\beta - \frac{\mu a^3}{4\pi l^3\eta^2}\cos\alpha$$

Show that the force, which acts on a sphere of radius *a* owing to a doublet of strength μ located a distance *l* from the center of the sphere along the *x*-axis is

$$\mathbf{F} = \frac{3\rho\mu^2 a^3 l}{2\pi\left(l^2 - a^2\right)^4}\mathbf{e}_x .$$

(from Currie's book, Ch.5)

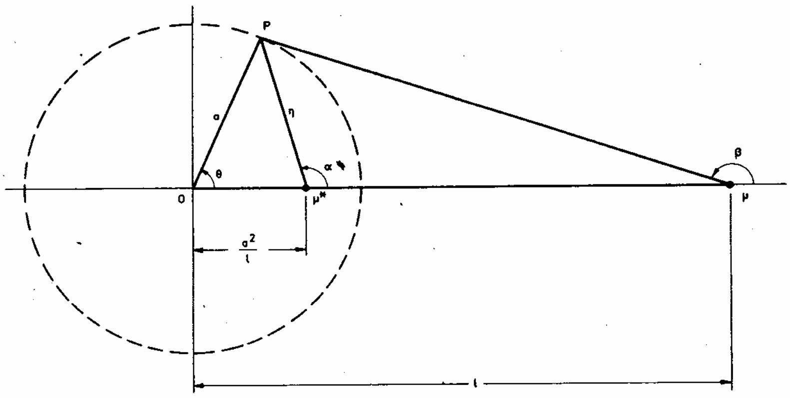


FIGURE 5.13
Superposition of doublet of strength μ and doublet of strength μ^* which leads to doublet of strength μ outside a sphere of radius a.

(5) Determine the complex potential for a circular cylinder of radius a in a flow field which is produced by a counterclockwise vortex of strength Γ located a distance l from the axis of the cylinder. This may be done by writing down the complex potential for a clockwise vortex of strength Γ located at $z = -b$, a counterclockwise vortex of strength Γ located at $z = -a^2/b$, a clockwise vortex of strength Γ located at $z = a^2/l$, a counterclockwise vortex of strength Γ located at $x = l$, and a constant $-[i\Gamma/(2\pi)]\log b$. Then let $b \to \infty$ and show that the circle of radius a is a streamline. Obtain the force acting on the cylinder.

CHAPTER 7: LAMINAR BOUNDARY LAYER THEORY

The major drawback of the potential flow is that the force on the body cannot be predicted correctly due to the absence of viscosity. A remedy to include the viscous effect is to introduce a "thin" boundary layer next to the body surface as discussed in chapter 4. For high Reynolds number flow, the flow region is separated into two regions, the inviscid main flow and the boundary layer region (including the boundary layer next to the solid surface and the wake downstream of the body), as illustrated in Figure 4-1(b). The governing equations for the two-dimensional boundary layer flows have been derived in chapter 4 (see equations (4-18a-b)). Here in this chapter we shall consider some laminar boundary layer flows. The laminar boundary layer flow becomes unstable and transits to turbulent boundary layer flow when the Reynolds number exceeds some critical values (see Schlichting's book).

(I) Boundary layer on a flat plate, a wedge, and a perpendicular wall (stagnation flow)

Consider the two-dimensional boundary layer flow over a flat plate, a wedge, and a perpendicular wall (i.e., stagnation flow) as shown in Figure 7-1. The governing equations and the associated boundary conditions are equations 4-17(a-d), and are stated as follows.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{4-17a}$$

and

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_s\frac{dU_s}{dx}+\nu\frac{\partial^2 u}{\partial y^2}\,, \tag{4-17b}$$

subject to

$$u=0,\quad v=0\qquad \text{at}\qquad y=0\,, \tag{4-17c}$$

and

$$u=U_s(x)\qquad \text{as}\qquad y\to\infty\,. \tag{4-17d}$$

The surface velocity of the inviscid main flow, $U_s(x)$, may be evaluated from the potential flow theory (refer to section (i)-(4) of chapter 6), as

$$U_s(x) = Ax^n, \tag{7-1}$$

where n is related to the wedge angle, 2θ, or $\alpha(= \pi - \theta)$ through

$$n = \frac{\pi}{\alpha} - 1 = \frac{\theta}{\pi - \theta}. \tag{7-1a}$$

The values for n for different cases are illustrated in Figure 7-1. The scale of the thickness of the boundary layer, δ, can be estimated as

$$\frac{\delta}{x} \sim \frac{1}{\sqrt{R_x}}, \tag{7-2}$$

through a scaling analysis (see chapter 4), where R_x is the Reynolds number defined as

$$R_x = \frac{U_s x}{\nu}. \tag{7-2a}$$

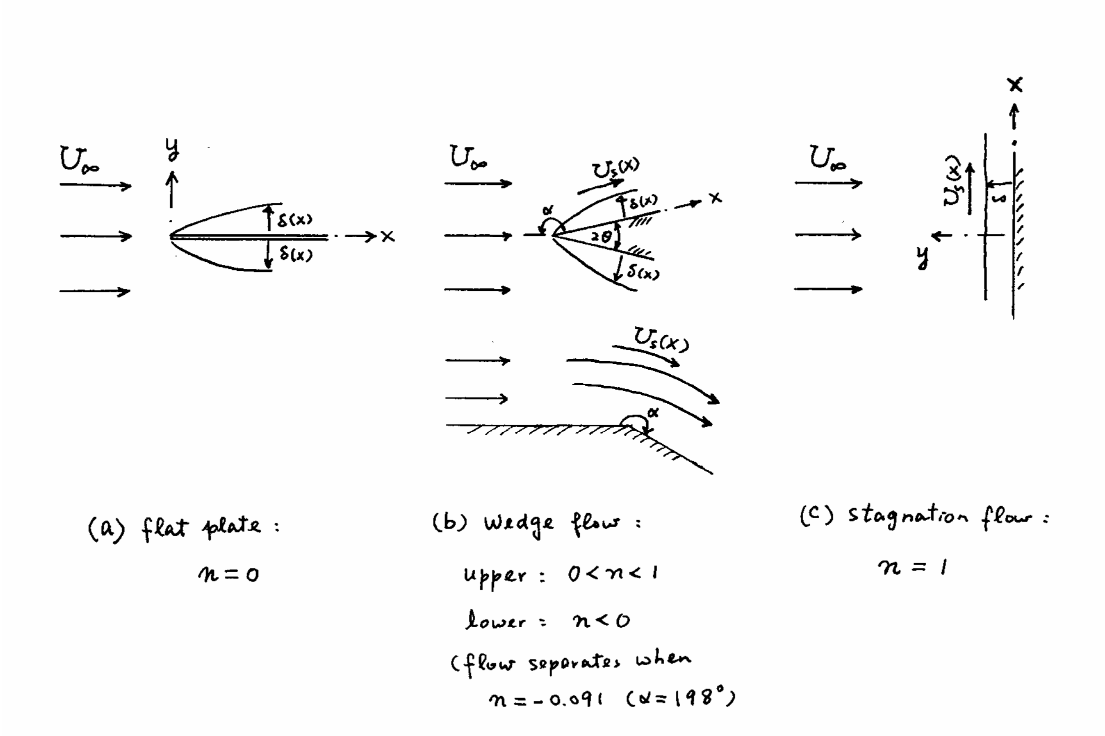


Figure 7-1: Sketch of the two-dimensional boundary layer flow over (a) a flat plate, (b) a wedge, and (c) a perpendicular wall. The cases for the flat plate and the perpendicular wall can be regarded as special cases for the wedge flow. The surface velocity of the inviscid main flow for these cases can be expressed as $U_s(x) = Ax^n$, where A and n are constants.

On setting

$$\delta = \frac{x}{\sqrt{R_x}}, \tag{7-2b}$$

we now try to solve the present problem by using the method of similarity transformation. Let

$$u(x, y) = U_s(x)f'(\eta) \qquad \text{and} \qquad \eta = y / \delta(x), \tag{7-3}$$

where $f'(\eta) = df / d\eta$. With $U_s(x)$ and $\delta(x)$ given by (7-1) and (7-2b), we may deduce the equation governing $f(\eta)$ from (4-17a) and (4-17b) as follows.

$$f''' + \frac{n+1}{2} ff'' + n(1 - f'^2) = 0, \tag{7-4}$$

which is called the Falkner-Skan equation. The corresponding boundary conditions for (7-4) are

$$f'(\infty) - 1 = f'(0) = f(0) = 0. \tag{7-5}$$

Equation (7-4) subject to (7-5) is solved via numerical integration, say, using the computer program discussed before in chapter 3 for the stagnation flow with suitable modifications for the functions. The numerical results are plotted in Figure 7-2. The line with $n = 0$ corresponds to the boundary layer flow over a flat plate at zero angle of attack; there is no applied pressure gradient from the external inviscid main flow, i.e., $\partial P / \partial x = -\rho U_s dU_s / dx = 0$. The applied pressure gradient is favorable for $n > 0$, but is adverse for negative *n*. The shapes of the velocity profiles for different values are direct consequences of the applied pressure gradient. The slope of $f'(\eta)$ tends to approach vertical as $|n|$ increases for negative *n*. According to the numerical solution, the slope of $f'(\eta)$ becomes vertical, the shear stress at wall vanishes, and the flow separates from the surface when $n = -0.091$, which corresponds to $\alpha = 198^o$ in Figure 7-1. It is generally defined that the separation occurs when the shear stress at wall equals to zero. Reversed flow is observed locally next to the wall when $n < -0.091$. According to the profiles plotted in Figure 7-2, $f'(\eta)$ approaches unity as η increases. For example, $f'(\eta) \approx 1$ when $\eta \approx 5$ for $n = 0$, which implies that the actual boundary layer thickness is about 5δ. The boundary layer becomes thinner as *n* increases, but increases rapidly as –*n* increases.

With $f'(\eta)$ known, we may calculate the shear stress at wall

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\rho U_s^{\,2}}{\sqrt{R_x}} f''(0), \tag{7-6}$$

the displacement thickness,

$$\delta^* \equiv \int_0^\infty \left(1 - \frac{u}{U_s}\right) dy = \frac{x}{\sqrt{R_x}} \int_0^\infty \left(1 - f'(\eta)\right) d\eta, \tag{7-7}$$

and other flow properties. The validity of the above similarity solution is studied as follows. We have assumed that

$$\frac{\partial u}{\partial x} << \frac{\partial u}{\partial y}$$

in deriving the boundary layer equation as discussed in chapter 4, which implies that

$$\frac{\delta}{x} \sim \frac{1}{\sqrt{R_x}} << 1. \tag{7-8}$$

According to (7-2a), (7-8) is not valid at small x. Thus the similarity solution is not valid near the leading edge of the flat plate or the wedge for $0 < n < 1$. However, for stagnation flow, $n = 1$, and

$$\delta = \frac{x}{\sqrt{\frac{(Ax)x}{\nu}}} = \sqrt{\frac{\nu}{A}},$$

which is a constant (independent of x !). Thus the above similarity solution is still valid. In fact, the viscous diffusion term along the x-direction is identically zero for the stagnation flow; the solution of the stagnation flow is an exact solution as discussed in chapter 3.

For the case of boundary layer flow over a flat plate, $n = 0$, equation (7-4) reduces to

$$f''' + \frac{1}{2} ff'' = 0, \tag{7-9}$$

which is called the Blasius equation. In particular, $f''(0) \cong 0.332$,

$$\frac{\tau_w}{\frac{1}{2}\rho U_s^{\ 2}} = \frac{0.664}{\sqrt{R_x}}, \tag{7-9a}$$

and

$$\frac{\delta^*}{x} = \frac{1.73}{\sqrt{R_x}}. \tag{7-9b}$$

The drag coefficient on one side of the flat plate with length L,

$$C_D = \frac{\int_0^L \tau_w(x)dx}{\frac{1}{2}\rho U_s^{\ 2} L} = \frac{1}{L}\int_0^L \frac{0.664}{\sqrt{R_x}}dx = \frac{1.328}{\sqrt{R_x}}. \tag{7-9c}$$

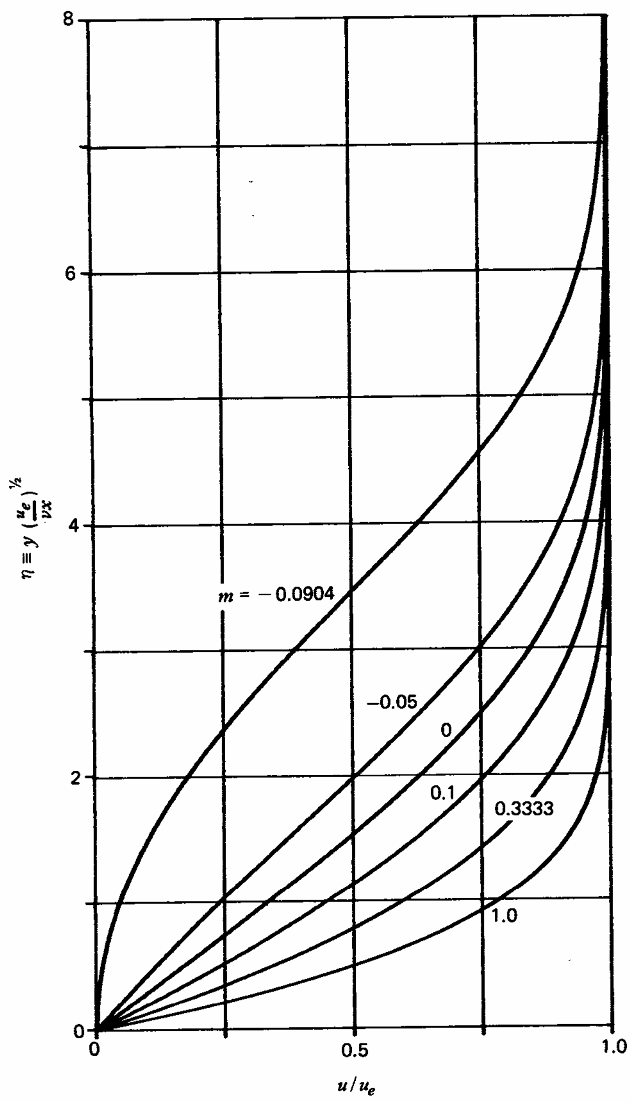


Figure 7-2: Numerical solutions of the Falkner-Skan equation for different values of n. The figure is adopted from Panton's book. Here $m = n$, and $u_e = U_s$.

(II) Interaction between the inviscid main flow and the boundary layer flow

A more rigorous analysis for steady flow at high Reynolds number using the boundary layer theory is as follows.

(1) Solve the inviscid main flow (external flow) with no cross flow boundary condition. Potential flow theory is employed if the flow is incompressible and irrotational.
(2) Obtian $U_s(x)$ and $U_s dU_s / dx$, i.e., $-(1/\rho)(dP_s/dx)$ for the streamwise pressure gradient inside the boundary layer.
(3) Solve the boundary layer flow using the boundary layer equations.
(4) Update the inviscid main flow by taking into account the boundary layer effect, usually through the displacement effect. The original shape is replaced the modified shape as shown in Figure 7-3. The modified shape of the body is "thicker" than that of the original body by a displacement thickness.
(5) Update the boundary layer flow using the new values of $U_s(x)$ and $U_s dU_s / dx$ associated with the updated external flow.
(6) Iteration between the external and the boundary layer flow as above until convergence is attained.

Similar iteration procedures can be employed if the external inviscid flow is rotational and/or compressible. Of course, the potential theory cannot be applied for such cases, and solutions of the continuity and Euler equations are required. Usually the above iteration is not needed since the displacement effect in item (4) is small, and the procedure (4) always calls for numerical calculation.

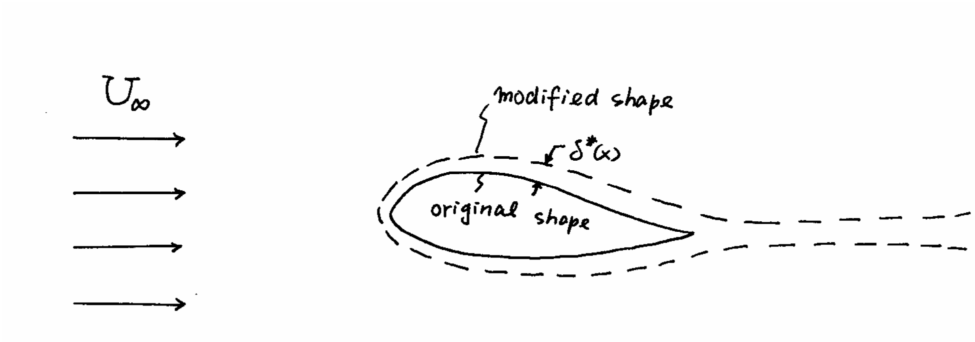


Figure 7-3: The original shape of the body is replaced by the modified shape, which takes into account the displacement effect, for updating the inviscid main flow.

(III) Integral method

The integral method is an approximate method for solving the boundary layer equation. We shall first define some length scales for the boundary layers, then derive the von Karman's momentum integral equation (1921), and finally illustrate the method through an example.

(i) Some scales for boundary layers

As in the undergraduate fluid mechanics, the boundary layer thickness, δ, is usually defined as the distance from the wall (or from the centerline for boundary-free shear flows such as wake or jet flows, see next section) where the local velocity equals 99% of the free stream velocity (or 50% of the centerline velocity for jet flow), i.e., when $u(x, \delta) = 0.99U_s(x, t)$. As U_s is function of *x* and *t* in general, the boundary layer thickness δ is also function of *x* and *t*. Of course, one may define δ as the distance from the wall where the local velocity equals another fraction other than 99% of the free stream velocity, say, when $u(x, \delta) = 0.9U_s(x, t)$, for example.

The displacement thickness, $\delta^*(x, t)$, which was defined before in (7-7), can be rewritten as

$$U_s\delta^* = \int_0^\infty (U_s - u)dy\,. \tag{7-10}$$

The physical interpretation of the displacement thickness is illustrated in Figure 7-4(a) and (b). The term $U_s\delta^*$ in (7-10) represents the volume flow rate defect. The name "displacement thickness" comes from the following reason. Since $u = \partial\psi / \partial y$, we have

$$\psi = \int_0^y u(x, y)dy = U_s y - \int_0^y (u - U_s)dy^* \tag{7-11}$$

for a given *x*. At $y >> \delta$ (outside the boundary layer),

$$\psi \to U_s y + \int_0^\infty (u - U_s)dy = U_s y - U_s\delta^*\,. \tag{7-11a}$$

It follows that

$$\psi = 0 \qquad \text{at} \qquad y = \delta^* . \qquad (7\text{-}11b)$$

Thus the effect of the boundary layer is to make the body appear to be at $y = \delta^*$ as seen from an observer outside of the boundary layer, which is termed the displacement effect, and has been employed in the last section for updating the external inviscid main flow.

The momentum thickness, $\theta(x, t)$, is defined as

$$\theta \equiv \int_0^\infty \frac{u}{U_s}\left(1 - \frac{u}{U_s}\right) dy , \qquad (7\text{-}12)$$

which can be rewritten as

$$U_s^{\,2}\theta = \int_0^\infty u(U_s - u)dy . \qquad (7\text{-}12a)$$

The term $U_s^{\,2}\theta$ represents the momentum defect of the boundary layer flow, as illustrated in Figure 7-4(c) and (d).

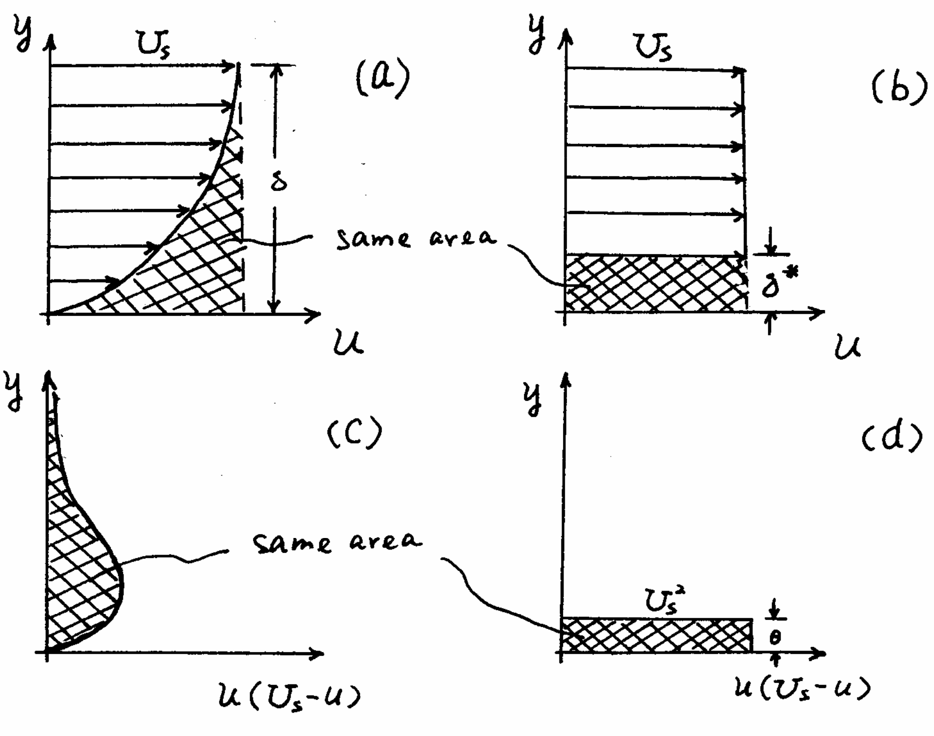


Figure 7-4: Illustration for the boundary layer and displacement thickness (a and b), and for the momentum thickness (c and d).

(ii) von Karman's momentum integral equation (1921)

Here we shall derive the momentum integral equation for unsteady incompressible two-dimensional boundary layer flow with suction (negative normal velocity) or blowing (positive normal velocity) at the wall. The mechanism for blowing and/or suction is usually employed for boundary layer control. The governing equations for the unsteady incompressible two-dimensional boundary layer flow with constant viscosity are (4-18a,b), and are expressed as follows for convenience.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \qquad \text{(7-13a)}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U_s}{\partial t} + U_s \frac{\partial U_s}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} . \qquad \text{(7-13b)}$$

By multiplying (7-13a) with u and adding to (7-13b), we obtain

$$\frac{\partial (u - U_s)}{\partial t} + \frac{\partial}{\partial x}\left(u^2\right) + \frac{\partial}{\partial y}(uv) = U_s \frac{\partial U_s}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} . \qquad \text{(7-14)}$$

With the application of continuity equation,

$$\frac{\partial}{\partial y}(uv) = \frac{\partial}{\partial y}\left[v(u - U_s)\right] + U_s \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[v(u - U_s)\right] - U_s \frac{\partial u}{\partial x}$$

$$= \frac{\partial}{\partial y}\left[v(u - U_s)\right] - \frac{\partial}{\partial x}(U_s u) + u \frac{\partial U_s}{\partial x} ,$$

it follows from (7-14) that

$$\frac{\partial (u - U_s)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 - U_s u\right) + (u - U_s)\frac{\partial U_s}{\partial x} + \frac{\partial}{\partial y}\left[v(u - U_s)\right] = \nu \frac{\partial^2 u}{\partial y^2} .$$

By integrating the above equation with respect to y from $y = 0$ to ∞, we obtain

$$\frac{\partial}{\partial t}\int_0^\infty (u - U_s)dy + \frac{\partial}{\partial x}\int_0^\infty \left(u^2 - U_s u\right)dy + \left[\int_0^\infty (u - U_s)dy\right]\frac{\partial U_s}{\partial x} + v_w U_s = -\nu \left.\frac{\partial u}{\partial y}\right|_{y=0} ,$$

where $v_w = v_w(x)$ is the specified suction/or blowing velocity, which may varies along the wall. On using the definitions

$$U_s\delta^* = \int_0^\infty (U_s - u)dy ,$$

$$U_s^2\theta = \int_0^\infty u(U_s - u)dy ,$$

and

$$-\nu\frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\tau_w(x, t)}{\rho},$$

where $\tau_w(x, t)$ is the shear stress at wall, we finally obtain

$$\frac{\partial}{\partial t}\left(U_s\delta^*\right) + \frac{\partial}{\partial x}\left(U_s^2\theta\right) + U_s\frac{\partial U_s}{\partial x}\delta^* + v_w U_s = \frac{\tau_w}{\rho}, \tag{7-15}$$

which is called the von Karman's momentum integral equation. The terms in (7-15) represent the unsteady effect, the effect of momentum defect, the effect of the imposed pressure gradient, the suction/blowing effect, and the effect of the wall shear.

(iii) Example for the application of the integral method

Here we consider a simple example, which is the steady uniform flow past a flat plate without suction and blowing. $U_s = U_\infty$, which is a constant here and thus there is no imposed pressure gradient. Equation (7-15) reduces to

$$\frac{d}{dx}\left(U_\infty^2\theta\right) = \frac{\tau_w}{\rho}. \tag{7-15a}$$

If a velocity profile, which satisfying the boundary conditions, is assumed across the boundary layer, i.e.,

$$\frac{u(x, y)}{U_\infty} = f(\eta) \quad \text{for} \quad 0 \le \eta = \frac{y}{\delta(x)} \le 1$$

$$= 1 \quad \text{for} \quad \eta > 1, \tag{7-16}$$

with a given functional relationship of $f(\eta)$, we can evaluate the spatial variation of the boundary layer thickness, $\delta(x)$, by solving (7-15a). Once $\delta(x)$ is known, we can evaluate $\tau_w(x)$, and thus the drag on the flat plate. The simplest profile, which satisfying the no-slip condition at wall and the free stream condition at $y = \delta$, is the linear profile, i.e.,

$$\frac{u(x, y)}{U_\infty} = \eta \quad \text{for} \quad 0 \le \eta = \frac{y}{\delta(x)} \le 1$$

$$= 1 \quad \text{for} \quad \eta > 1. \tag{7-16a}$$

We have

$$\delta^* = \frac{\delta}{2} , \qquad \theta = \frac{\delta}{6} \qquad \text{and} \qquad \frac{\tau_w}{\rho} = \frac{\nu U_\infty}{\delta} .$$

Thus (7-15a) implies

$$\frac{d}{dx}\left(\frac{U_\infty{}^2\delta}{6}\right) = \frac{\nu U_\infty}{\delta}, \tag{7-17}$$

which is solved subject to $\delta = 0$ at $x = 0$. The solution is

$$\frac{\delta}{x} = \frac{3.46}{\sqrt{R_x}} . \tag{7-17a}$$

In particular, the shear stress at wall,

$$\frac{\tau_w}{\frac{1}{2}\rho U_s{}^2} = \frac{2\nu}{U\delta} = \frac{0.58}{\sqrt{R_x}}, \tag{7-17b}$$

which is only 12.65% less than the Blasius solution in (7-9a). It is worth to point out that the approximate solution in (7-17b) is fairly accurate although the assumed linear profile is a very crude assumption as the shear stress is constant across the whole boundary layer and is not continuous at the outer edge of the boundary layer. A better result can be obtained by using a smooth profile. For example, one may assume

$$\frac{u(x, y)}{U_\infty} = a + b\eta + c\eta^2 + d\eta^3 + e\eta^4 \tag{7-18}$$

for $\eta \leq 1$ for general steady boundary layer flow with an imposed pressure gradient but without suction and blowing. The coefficients *a*, *b*, *c*, *d* and *e* are obtained by satisfying

$$u = U_s \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{at} \quad y = \delta \qquad \text{(7-18a)}$$

and

$$u = 0 \quad \text{and} \quad \nu \frac{\partial^2 u}{\partial y^2} = -U_s \frac{\partial U_s}{\partial x} \quad \text{at} \quad y = 0. \qquad \text{(7-18b)}$$

Here the last condition in (7-18b) is derived from the steady form of equation (7-13b) with the imposed no-slip boundary condition at wall.

For steady flow, it is noted that the application of the momentum method is to reduce the problem governed by a set of partial differential equations, say, (7-13a) and (7-13b) without the unsteady terms, by that governed by an ordinary differential equation, say, (7-15) without the unsteady term. The integral method can also employed for studying the unsteady boundary layer flows, although the resulting integral equation is a partial differential equation with independent variables *x* and *t*.

(IV) Boundary-free shear flow

So far in this chapter we have discussed the boundary layer flow next to the solid boundary. As discussed in chapter 4, we may also apply boundary layer assumption in solving the boundary-free shear flow, which is a kind of the boundary layer flow governed by the same boundary layer equations as before, but without solid boundary. Examples of the boundary-free shear flows are the wakes, the jets and the shear layers (or called the mixing layers). For the laminar case, the analysis of the shear layer can be found from Batchelor's book; the jet problem will be discussed in the homework set; and the rest of this section will concentrate on the study of the wake problem. The discussion of the corresponding turbulent boundary-free shear flow can be found from the book "A first course in turbulence", by H. Tennekes and J. L. Lumley, MIT press, 1972.

The velocity field of a steady two-dimensional laminar far wake is sketched in Figure 7-5. There exists a velocity defect downstream of the body, which is a direct consequence of the drag exerted by the fluid on the body, as illustrated in the following integral analysis. Consider a sufficiently large control volume, V, enclosing the body as shown in Figure 7-6(a). The surface of the control volume having unit outward normal $\hat{\mathbf{n}}$ consists of two parts, the inner part, S_1, and the outer part, S_2. Here S_1 is the surface of the body having unit outward normal $\hat{\mathbf{N}} = -\hat{\mathbf{n}}$. With the application of the continuity equation, the steady incompressible Navier-Stokes equation with negligible body force can be expressed in a conservative form as

$$\nabla \cdot (\rho \mathbf{uu} - \mathbf{T}) = 0\,, \qquad (7\text{-}19a)$$

or in index form as

$$\frac{\partial}{\partial x_i}\left(\rho u_i u_j - T_{ij}\right) = 0\,, \qquad (7\text{-}19b)$$

where $\mathbf{T}$ is the stress tensor. On taking the volume integral of (7-19a) with respect to the control volume, and using the divergence theorem, we have

$$\int_V \nabla \cdot (\rho \mathbf{uu} - \mathbf{T}) = \int_{S_1} \hat{\mathbf{n}} \cdot (\rho \mathbf{uu} - \mathbf{T}) dS + \int_{S_2} \hat{\mathbf{n}} \cdot (\rho \mathbf{uu} - \mathbf{T}) dS = 0\,.$$

With $\mathbf{u} = 0$ on S_1, the above equation becomes

$$\int_{S_1} \hat{\mathbf{n}} \cdot \mathbf{T} dS = -\int_{S_1} \hat{\mathbf{N}} \cdot \mathbf{T} dS = \int_{S_2} \hat{\mathbf{n}} \cdot (\rho \mathbf{uu} - \mathbf{T}) dS\,.$$

The force exerted on the body by the fluid is

$$\mathbf{F} \equiv \int_{S_1} \hat{\mathbf{N}} \cdot \mathbf{T} dS = \int_{S_2} \hat{\mathbf{n}} \cdot (\mathbf{T} - \rho \mathbf{uu}) dS\,. \qquad (7\text{-}20)$$

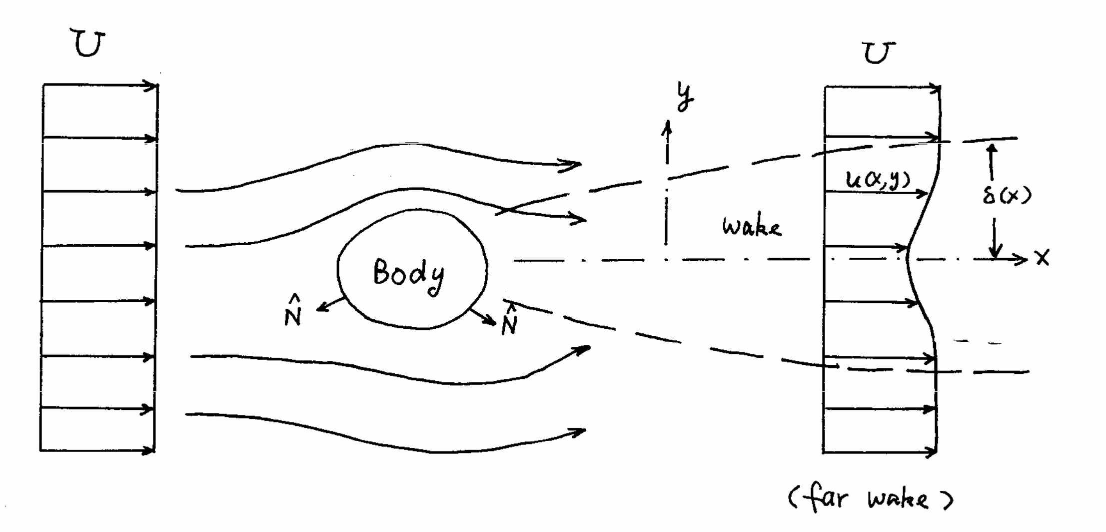


Figure 7-5: Sketch of the wake flow.

The drag on the body, D, is defined as $D = \mathbf{F} \cdot \hat{\mathbf{i}}$, where $\hat{\mathbf{i}}$ is the unit normal vector along the direction of the uniform stream as shown in Figure 7-5, and thus

$$D = \int_{S_2} \hat{\mathbf{n}} \cdot (\mathbf{T} - \rho \mathbf{u}\mathbf{u}) \cdot \hat{\mathbf{i}} dS . \qquad (7\text{-}21)$$

Recall that the stress tensor,

$$\mathbf{T} = -p\mathbf{I} + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \lambda \mathbf{I} \nabla \cdot \mathbf{u} , \qquad (7\text{-}22a)$$

for Newtonian fluid, where **I** is the unit tensor, or expressed in index form as

$$T_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} .$$

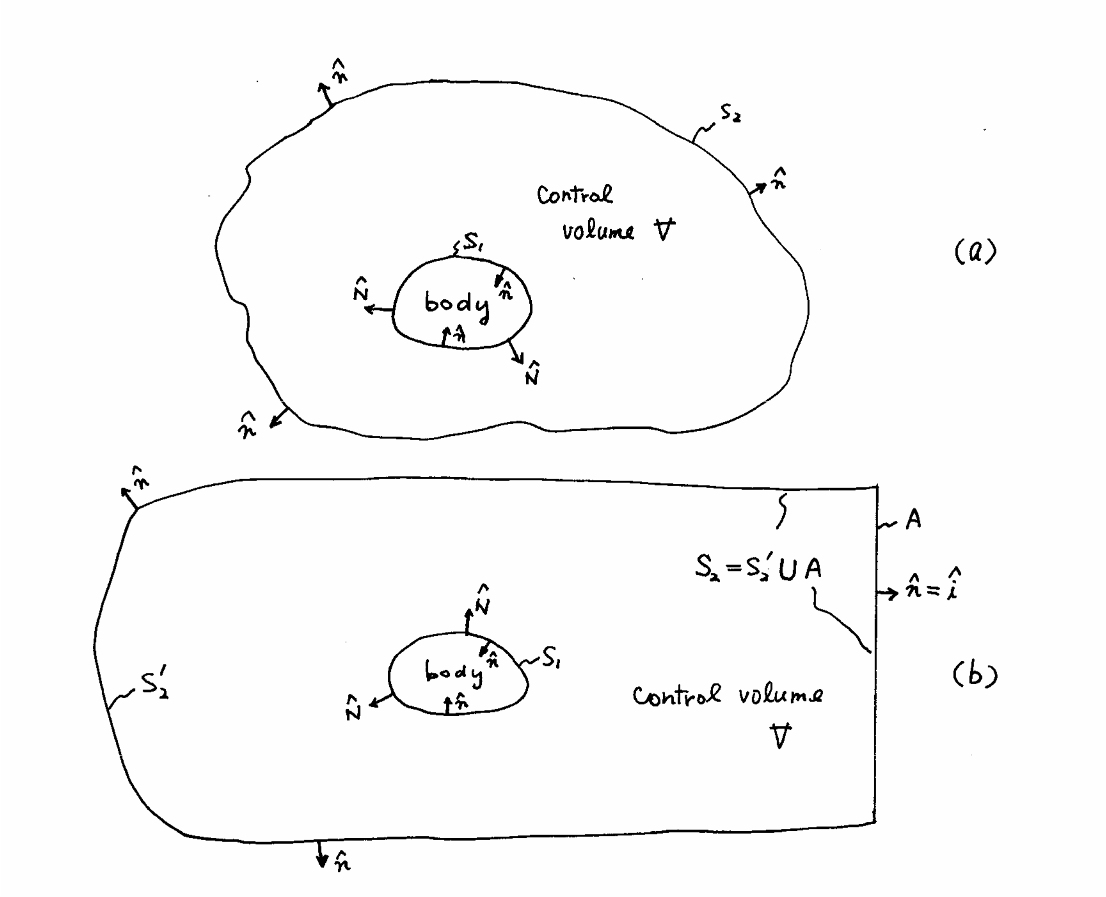


Figure 7-6 : Sketch of the control volumes for evaluating the drag.

If S_2 is sufficiently far from the body, $\mathbf{T} \cong -p_\infty \mathbf{I}$, with p_∞ the pressure associated with the free stream *U*,

$$\int_{S_2} \hat{\mathbf{n}} \cdot \mathbf{T} ds = \int_{S_2} - p_\infty \hat{\mathbf{n}} dS = -p_\infty \int_{S_2} \hat{\mathbf{n}} dS = 0,$$

and thus (7-21) implies

$$D \cong -\int_{S_2} \rho(\hat{\mathbf{n}} \cdot \mathbf{u})(\mathbf{u} \cdot \hat{\mathbf{i}}) dS . \qquad (7\text{-}23)$$

The *x*-component of the velocity *u* can be expressed in terms of the velocity defect (*u*-*U*) as

$$u = \mathbf{u} \cdot \hat{\mathbf{i}} = U + (u - U), \qquad (7\text{-}24)$$

equation (7-23) then becomes

$$D \cong -\int_{S_2} \rho(\hat{\mathbf{n}} \cdot \mathbf{u})[U + (u - U)] dS . \qquad (7\text{-}25)$$

The first term on the right hand side of (7-25) is zero since there is no net mass efflux from S_2, i.e.,

$$\int_{S_2} \rho \hat{\mathbf{n}} \cdot \mathbf{u} dS = 0, \qquad (7\text{-}26)$$

which can be proved by integrating the continuity equation for steady flow , $\nabla \cdot (\rho \mathbf{u}) = 0$, over the control volume, i.e.,

$$0 = \int_V \nabla \cdot (\rho \mathbf{u}) dV = \int_{S_1} \hat{\mathbf{n}} \cdot \rho \mathbf{u} dS + \int_{S_2} \rho \hat{\mathbf{n}} \cdot \mathbf{u} dS = \int_{S_2} \rho \hat{\mathbf{n}} \cdot \mathbf{u} dS ,$$

by using the no-cross flow condition at S_1. Thus (7-25) becomes

$$D \cong -\int_{S_2} \rho(\hat{\mathbf{n}} \cdot \mathbf{u})[u - U] dS . \qquad (7\text{-}27)$$

As the shape of the large control volume is arbitrary, we may choose a special control volume in Figure 7-6(b) such that $S_2 = S_2' \cap A$. Here the downstream part of the control surface, *A*, is a flat surface perpendicular to the flow with unit normal vector $\hat{\mathbf{n}} = \hat{\mathbf{i}}$. For the integration over the part of S_2', the velocity defect $(u - U) \to 0$, and thus its contribution to the drag is zero. Therefore, equation (7-27) finally becomes

$$D \cong -\int_A \rho u(u - U)dS, \qquad (7\text{-}28)$$

which relates the velocity defect to the drag.

As the far wake (the wake that is sufficiently downstream from the body) is a slender region of flow variation, it is adequate to use the boundary layer assumption and thus the boundary layer equations to study the flow field inside the far wake. As the pressure outside the wake is constant, the pressure gradient for the boundary layer flow inside the wake is identically zero, and thus the governing boundary layer equations for the steady wake in the Cartesian coordinates shown in Figure 7-5 are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (7\text{-}29a)$$

and

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}. \qquad (7\text{-}29b)$$

Denote the velocity defect by

$$u' = u - U, \qquad (7\text{-}30)$$

equation (7-29b) implies

$$U\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x} + v\frac{\partial u'}{\partial y} = \nu\frac{\partial^2 u'}{\partial y^2}. \qquad (7\text{-}31)$$

As

$$u' \sim v << u \sim U, \qquad (7\text{-}32)$$

the second and the third terms in (7-31) are negligible in comparing with the first term, and thus (7-31) is linearized as

$$U\frac{\partial u'}{\partial x} = \nu\frac{\partial^2 u'}{\partial y^2}, \qquad (7\text{-}33)$$

which is in the form of the heat equation. Equation (7-33) is solved subject to

$$u' \to 0 \quad \text{as} \quad y \to \pm\infty, \tag{7-33a}$$

and the integral constraint according to the "drag-velocity defect" relation in (7-28) with the application of (7-32), i.e.,

$$\int_{-\infty}^{\infty} u'\, dy = -\frac{D}{\rho U}, \tag{7-33b}$$

for given value of D. It is worth to point out that the integral constraint in (7-33b) is employed instead of specifying a given condition at a fixed value of x for solving the problem. Also according to (7-33b), the total velocity defect of the velocity profile across the wake is constant (independent of x!). Let

$$u'(x, y) = -A(x) f(\eta), \tag{7-34}$$

with

$$\eta = \frac{y}{\delta(x)}, \tag{7-34a}$$

where $\delta = \sqrt{\nu x / U}$ is a scale for the boundary layer thickness, which is obtained through a scaling analysis of (7-33). On substituting (7-34) into (7-33), we have

$$-\frac{xA'(x)}{A(x)} f(\eta) + \frac{1}{2}\eta f'(\eta) = -f''(\eta). \tag{7-35}$$

For self preserving (i.e., similarity solution exists), it is required that $xA'(x) / A(x)$ to be independent of x. With (7-34), (7-33b) implies that

$$-A\sqrt{\frac{\nu x}{U}} \int_{-\infty}^{\infty} f(\eta) d\eta = -\frac{D}{\rho U}.$$

On setting

$$\int_{-\infty}^{\infty} f(\eta) d\eta = 1, \tag{7-36}$$

we found

$$A(x) = \frac{D}{\rho} \frac{1}{\sqrt{U\nu x}}, \tag{7-37}$$

and thus

$$\frac{xA'(x)}{A(x)} = -\frac{1}{2}. \qquad (7\text{-}38)$$

With (7-38), (7-35) becomes

$$f''(\eta) + \frac{1}{2}\eta f'(\eta) + \frac{1}{2}f(\eta) = 0, \qquad (7\text{-}39)$$

which is solved subject to the boundary conditions according to (7-33a),

$$f(\pm\infty) = 0, \qquad (7\text{-}39a)$$

and the integral constraint, (7-36). To solve the problem, we first re-write (7-39) as

$$f'' + \frac{1}{2}(\eta f)' = 0, \qquad (7\text{-}40a)$$

and thus obtain

$$f' + \frac{1}{2}\eta f = C_1, \qquad (7\text{-}40b)$$

by carry out the integration. The constant of integration $C_1 = 0$ if $f' \to 0$ and $\eta f \to 0$ as $\eta \to \pm\infty$, which is consistent with (7-39a), as validated from the solution later. Thus equation (7-40b) becomes

$$f' + \frac{1}{2}\eta f = 0, \qquad (7\text{-}40c)$$

which has solution

$$f = C_2 e^{-\frac{1}{4}\eta^2}. \qquad (7\text{-}40d)$$

The constant of integration, C_2, is determined by substituting (7-40d) into (7-36), the result is $C_2 = 1/\sqrt{4\pi}$. Thus

$$f = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}\eta^2}, \qquad (7\text{-}41a)$$

or

$$u'(x, y) = -\frac{D}{\rho U}\sqrt{\frac{U}{\nu x}}\frac{1}{\sqrt{4\pi}} e^{-\frac{y^2 U}{4\nu x}}. \qquad (7\text{-}41b)$$

Homework

(1) For steady boundary layer flow over an axial symmetric body as shown below. The governing equations are

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0, \tag{1a}$$

and

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_s\frac{dU_s}{dx} + \nu\frac{\partial^2 u}{\partial y^2}. \tag{1b}$$

The Mangler's transformation (1948) is defined through the following relations,

$$\bar{x} = \frac{1}{L^2}\int_0^x r^2(x)dx, \qquad \bar{y} = \frac{r(x)}{L}y, \qquad \bar{U}_s(\bar{x}) = U_s(x),$$

$$\bar{u}(\bar{x},\bar{y}) = u(x,y), \quad \text{and} \quad \bar{v}(\bar{x},\bar{y}) = \frac{L}{r(x)}\left[v(x,y) + \frac{r'(x)}{r(x)}yu(x,y)\right], \tag{2}$$

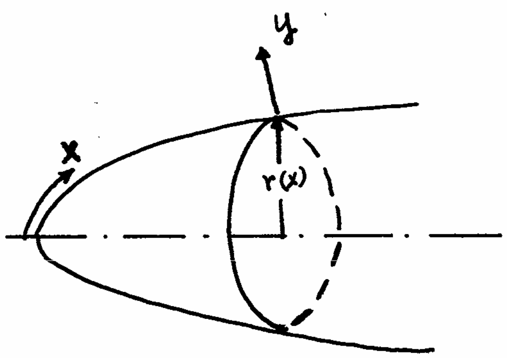
where L is a reference length, and $r(x)$ is the radius of the cross sectional area at x. Derive

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{3a}$$

and

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \bar{U}_s\frac{d\bar{U}_s}{dx} + \nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{3b}$$

which are of the same form as those in planar boundary layer flow, and thus the method employed before in this chapter can be applicable for solving the problems.



(2) Carry out the integral momentum analysis of the uniform flow over a flat plate for the assumed velocity profile

$$\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3. \tag{4}$$

Compute the boundary layer thickness δ, the displacement thickness, the momentum thickness, and $C_f = \tau_w /(\rho U_\infty^2 / 2)$. (from White, p.329)

(3) The quantity $(\delta^* / \tau_w)(dp / dx)$ is called the Clauser's parameter. It compares external pressure gradient to wall friction and is very useful for turbulent boundary layers. Show that this parameter is a constant for a given laminar Falkner-Skan wedge-flow boundary layer. What value does this parameter have at the separation condition? (from White, p.330)

(4) Consider the two-dimensional laminar jet as shown in the following figure. The fluid is ejected from a slit with width b in a wall into infinite space filled with the same fluid. The mass and momentum discharge per unit depth are $Q_m = b\rho U$ and $M = b\rho U^2$, respectively. On setting $b \to 0$ and $U \to \infty$ simultaneously such that M = constant, we have $Q_m = M / U \to 0$. Thus a constant momentum source can be generated approximately by a weak mass source. A jet is a flow generated by a constant momentum source, i.e.,

$$\int_{-\infty}^{\infty} \rho u^2 dy = M = \text{constant} \tag{5}$$

for any x in the Cartesian coordinates shown in the figure. As the jet is a slender region of flow variation, and the pressure is uniform outside the jet, we may apply the boundary layer equations,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6a}$$

and

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{6b}$$

to describe the jet flow. These equations are solved subject to the boundary conditions

$$u = 0 \quad \text{at} \quad y \to \infty, \tag{7a}$$

and $$\frac{\partial u}{\partial y} = v = 0 \quad \text{at} \quad y = 0 \quad \text{(symmetric condition)}, \tag{7b}$$

together with the integral constraint in (5). Equation (6a) is satisfied automatically by introducing the stream function, ψ, such that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \tag{8}$$

Equation (6b) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}, \tag{9a}$$

and (5) becomes

$$\int_{-\infty}^{\infty} \rho \left(\frac{\partial \psi}{\partial y} \right)^2 dy = M = \text{constant}. \tag{9b}$$

Let x, δ and Q be the length scale in the x-direction, the length scale in the y-direction and the scale for ψ. Show that

$$\delta \sim \left(\frac{\rho \nu^2 x^2}{M} \right)^{1/3} \quad \text{and} \quad Q \sim \left(\frac{M \nu x}{\rho} \right)^{1/3} = \nu \left(\frac{Mx}{\rho \nu^2} \right)^{1/3} \tag{10}$$

by carrying out a scaling analysis of equations (9a) and (9b). Thus let

$$\psi(x, y) = \left(\frac{M \nu x}{\rho} \right)^{1/3} f(\eta) \quad \text{with} \quad \eta = \left(\frac{M}{\rho \nu^2 x^2} \right)^{1/3} y, \tag{11}$$

prove that $f(\eta)$ is governed by

$$3f''' + ff'' + f'^2 = 0, \tag{12}$$

which is solved subject to the boundary conditions

$$f(0) = f''(0) = f'(\infty) = 0, \tag{12a}$$

according to (9a), (7a) and (7b). Show that the solution is

$$f = \sqrt{2} C \tanh\left(\frac{C\eta}{3\sqrt{2}} \right). \tag{13a}$$

The constant C is determined by the integral constraint. Show that

$$C = \left(\frac{9}{4\sqrt{2}} \right)^{1/3}.$$

With (13a), calculate the velocity components, u and v. The results are

$$u = \left(\frac{3M^2}{32 \rho^2 \nu x} \right)^{1/3} \sec h^2(\xi) \tag{14a}$$

and

$$v = \left(\frac{M \nu}{6 \rho x^2} \right)^{1/3} \left(2\xi \sec h^2 \xi - \tanh \xi \right), \tag{14b}$$

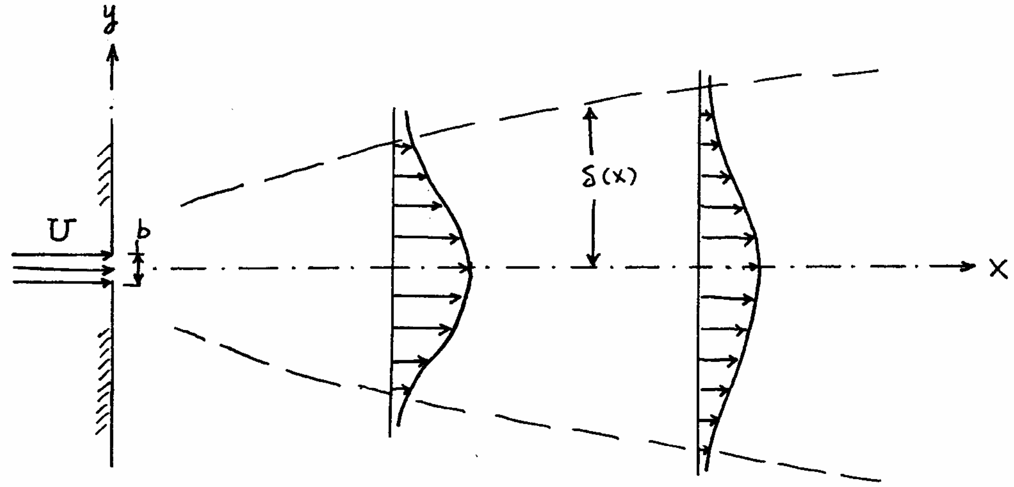
where $\xi = \eta / (48)^{1/3}$. Note that

$$v(x, y \to \infty) = -\left(\frac{M \nu}{6 \rho x^2} \right)^{1/3}, \tag{15}$$

which is not zero, and this phenomenon is called the entrainment. As there exists entrainment at the outer edge of the jet, the total volume flow rate, Q_x, across the jet also varies with x. Show that

$$Q_x \equiv \int_{-\infty}^{\infty} u(x, y) dy = 2 \left(\frac{9}{2} \right)^{1/3} \left(\frac{M \nu}{\rho} \right)^{1/3} x^{1/3} \int_0^{\infty} \sec h^2 \xi d\xi, \tag{16}$$

which increases with x.


y
U
b
δ(x)
x

CHAPTER 8: GENERAL DISCUSSION ON OTHER TOPICS IN FLUID MECHANICS

First let us review what we have learned previously in this course. We have laid down the basis of continuum fluid mechanics in chapter 2, and illustrated vary mechanisms and some analytical techniques via selected exact solutions of the governing equations in chapter 3. As the governing equations of fluid mechanics are nonlinear in nature, exact solutions for the governing equations are limited, and thus the fluid mechanics problems are always studied under certain approximations, which forms the topics in chapter 4-7. Since the Reynolds number is the most important dimensionless parameters for general fluid mechanics problems, approximation is made here under the conditions of low and large Reynolds number. The flow physics from small to large Reynolds number is discussed in chapter 4. When the Reynolds number is sufficiently small, the nonlinear convection term can be neglected, and the flow is governed by linear equations. Two types of low Reynolds number flows, lubrication (internal flow) and flow past a sphere (external flow), are discussed in chapter 5. When the Reynolds number is sufficiently large, the flow region can be separated into two regions, the inviscid main flow and the thin boundary layer regions. Each of these two regions is solved by simplified equations derived from the Navier-Stokes equation. The inviscid main flow contains most of the flow region, which is governed by the continuity and the Euler equation. If the inviscid main flow is further irrotational, the governing equation of the inviscid main flow becomes the Laplace equation, and the resulting flow is called the potential flow. Elementary potential flow is discussed in chapter 6. The boundary layer region is a slender region of flow variation, with its streamwise extent much greater than that in the cross streamwise direction. It contains the boundary layer next to the solid boundary (for satisfying the no-slip condition) and the wake region behind the body. The equations governing the flow inside the boundary layer region are called the boundary layer equations, which were derived in chapter 4. Some simple planar boundary layer flows were discussed in chapter 7.

The present course is an introductory course for the student who wishes to learn further fluid mechanics beyond the undergraduate fluid mechanics. There are three succeeding courses, namely, the “viscous flow”, the “compressible flow”, and the “introduction to turbulence”, available in the Institute of Applied Mechanics at National Taiwan University. The detailed contents of these courses may vary for different instructors, but it is expected that the major topics should remain essentially the same. According to the author’s own experiences and realization, the essential topics of these three courses are as follows.

(I) Viscous flow

If the course “viscous flow” is treated as a follow-up course of the present course, it is suggested that the following topics are adequate.

- (i) Unsteady low Reynolds number flows
  - (a) Arbitrary motion of a sphere in an unbounded fluid – a natural discussion of the extension of the steady Stokes flow to unsteady situation (Yih’s book).
  - (b) Swimming of microscopic organisms – illustration of a propulsion mechanism via the viscous effect (Taylor’s papers).
  - (c) Squeeze film problem – a type of lubrication problem, which finds direct applications to MEMS (selected papers).
- (ii) Some complicated laminar viscous flows
  - (a) Ekman flow – illustration for studying problems in a rotating frame (Batchelor’s book).
  - (b) Entrance flow – illustration for the interaction of inviscid and boundary layer flows (Schlichting’s book).
  - (c) Flow in curved and/or rotating channels/pipes – illustration of the secondary flows (Schlichting’s book, selected papers).
- (iii) Axisymmetric and three-dimensional boundary layers (Yih’s book, Schlichting’s book, and Rosenhead’s book)
- (iv) Unsteady boundary layers (Schlichting’s book, Rosenhead’s book)
- (v) Introduction to hydrodynamic stability (selected materials from chapter 1 to 4 of the book by Drazin & Reid)

(II) Compressible flow

The author has taught this course several times, and included:

(i) Most of the materials in the book by Anderson,

(ii) The compressible boundary layers in Schlichting's book, and

(iii) Elementary acoustics (refer to chapter 4 of Thompson's book; chapter 1 and 2 of Lighthill's book).

Of course, the serious students should read the famous classical texts by Shapiro (1953) and Liepmann & Roshko (1957).

(III) Introduction to turbulence

The author had taught this course several times and followed basically the text by Tennekes & Lumley (1972), together with some preliminary discussions on experimental methods based on Bradshaw's book.

However, the author believed that a rigorous understanding on fluid mechanics should included some further materials on potential flow, which includes: (1) more details on two-dimensional potential flow (chapter 4 of Currie's book), (2) Unsteady potential flow (Yih's book), and (3) Elementary water waves (chapter 3 and 4 of Lighthill's book). Unfortunately, these topics cannot be included naturally in the above three courses offered by the Institute of Applied Mechanics.

Refererences:

The books by Batchelor, Schlichting and Yih have already been cited in the sheet of the course outline, other references are listed as follows.

(1) Anderson, J. D. Jr., "Modern compressible flow," McGraw-Hill, 1990.
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