CHAPTER 29
Interest Rate Derivatives: The Standard Market Models

Practice Questions

29.1
An amount

$$\$20,000,000 \times 0.02 \times 0.25 = \$100,000$$

would be paid out 3 months later.

29.2
A swap option (or swaption) is an option to enter into an interest rate swap at a certain time in the future with a certain fixed rate being used. An interest rate swap can be regarded as the exchange of a fixed-rate bond for a floating-rate bond. A swaption is therefore the option to exchange a fixed-rate bond for a floating-rate bond. The floating-rate bond will be worth its face value at the beginning of the life of the swap. The swaption is therefore an option on a fixed-rate bond with the strike price equal to the face value of the bond.

29.3
In this case, $F_0 = (125-10)e^{0.1\times 1} = 127.09$, $K = 110$, $P(0,T) = e^{-0.1\times 1}$, $\sigma_B = 0.08$, and $T = 1.0$.

$$d_1 = \frac{\ln(127.09/110) + (0.08^2/2)}{0.08} = 1.8456$$

$$d_2 = d_1 - 0.08 = 1.7656$$

From equation (29.2), the value of the put option is

$$110e^{-0.1\times 1}N(-1.7656) - 127.09e^{-0.1\times 1}N(-1.8456) = 0.12$$

or \$0.12.

29.4
When spot volatilities are used to value a cap, a different volatility is used to value each caplet. When flat volatilities are used, the same volatility is used to value each caplet within a given cap. Spot volatilities are a function of the maturity of the caplet. Flat volatilities are a function of the maturity of the cap.

29.5
In this case, L=1,000, $\delta_k = 0.25$, $F_k = 0.12$, $R_K = 0.13$, $r = 0.115$, $\sigma_k = 0.12$, $t_k = 1.25$, $P(0, t_{k+1}) = 0.8416$.

$$L\delta_k = 250$$

$$d_1 = \frac{\ln(0.12/0.13) + 0.12^2 \times 1.25/2}{0.12\sqrt{1.25}} = -0.5295$$

$$d_2 = -0.5295 - 0.12\sqrt{1.25} = -0.6637$$

The value of the option is

$$250 \times 0.8416 \times [0.12N(-0.5295) - 0.13N(-0.6637)]$$

$$= 0.59$$

or \$0.59.

29.6

The implied volatility measures the standard deviation of the logarithm of the bond price at the maturity of the option divided by the square root of the time to maturity. In the case of a five year option on a ten year bond, the bond has five years left at option maturity. In the case of a nine year option on a ten year bond, it has one year left. The standard deviation of a one year bond price observed in nine years can be normally be expected to be considerably less than that of a five year bond price observed in five years. (See Figure 29.1.) We would therefore expect the price to be too high.

29.7

The present value of the principal in the four year bond is $100e^{-4\times 0.1} = 67.032$. The present value of the coupons is, therefore, $102 - 67.032 = 34.968$. This means that the forward price of the five-year bond is

$$(105 - 34.968)e^{4\times 0.1} = 104.475$$

The parameters in Black's model are therefore $F_B = 104.475$, $K = 100$, $r = 0.1$, $T = 4$, and $\sigma_B = 0.02$.

$$d_1 = \frac{\ln 1.04475 + 0.5\times 0.02^2 \times 4}{0.02\sqrt{4}} = 1.1144$$

$$d_2 = d_1 - 0.02\sqrt{4} = 1.0744$$

The price of the European call is

$$e^{-0.1\times 4}[104.475N(1.1144) - 100N(1.0744)] = 3.19$$

or \$3.19.

29.8

The option should be valued using Black's model in equation (29.2) with the bond price volatility being

$$4.2 \times 0.07 \times 0.22 = 0.0647$$

or 6.47%.

29.9

A 5-year zero-cost collar where the strike price of the cap equals the strike price of the floor is the same as an interest rate swap agreement to receive floating and pay a fixed rate equal to the strike price. The common strike price is the swap rate. Note that the swap is actually a forward swap that excludes the first exchange. (See Business Snapshot 29.1)

29.10

There are two way of expressing the put–call parity relationship for bond options. The first is in terms of bond prices:

$$c + I + Ke^{-RT} = p + B_0$$

where c is the price of a European call option, p is the price of the corresponding European put option, I is the present value of the bond coupon payments during the life of the option, K is the strike price, T is the time to maturity, B_0 is the bond price, and R is the risk-free interest rate for a maturity equal to the life of the options. To prove this, we can consider two portfolios. The first consists of a European put option plus the bond; the second consists of

the European call option, and an amount of cash equal to the present value of the coupons plus the present value of the strike price. Both can be seen to be worth the same at the maturity of the options.
The second way of expressing the put–call parity relationship is

$$c + Ke^{-RT} = p + F_B e^{-RT}$$

where F_B is the forward bond price. This can also be proved by considering two portfolios. The first consists of a European put option plus a forward contract on the bond plus the present value of the forward price; the second consists of a European call option plus the present value of the strike price. Both can be seen to be worth the same at the maturity of the options.

29.11
The put–call parity relationship for European swap options is

$$c + V = p$$

where c is the value of a call option to pay a fixed rate of s_K and receive floating, p is the value of a put option to receive a fixed rate of s_K and pay floating, and V is the value of the forward swap underlying the swap option where s_K is received and floating is paid. This can be proved by considering two portfolios. The first consists of the put option; the second consists of the call option and the swap. Suppose that the actual swap rate at the maturity of the options is greater than s_K. The call will be exercised and the put will not be exercised. Both portfolios are then worth zero. Suppose next that the actual swap rate at the maturity of the options is less than s_K. The put option is exercised and the call option is not exercised. Both portfolios are equivalent to a swap where s_K is received and floating is paid. In all states of the world, the two portfolios are worth the same at time T. They must therefore be worth the same today. This proves the result.

29.12
Suppose that the cap and floor have the same strike price and the same time to maturity. The following put–call parity relationship must hold:

$$\text{cap} + \text{swap} = \text{floor}$$

where the swap is an agreement to receive the cap rate and pay floating over the whole life of the cap/floor. If the implied Black volatilities for the cap equal those for the floor, the Black formulas show that this relationship holds. In other circumstances, it does not hold and there is an arbitrage opportunity.

29.13
Yes. For example, if a zero-coupon bond price at some future time is lognormal, there is some chance that the price will be above par. This, in turn, implies that the yield to maturity on the bond is negative.

29.14
In equation (29.10), $L = 10{,}000{,}000$, $s_K = 0.05$, $s_0 = 0.05$, $d_1 = 0.2\sqrt{4}/2 = 0.2$, $d_2 = -0.2$, and

$$A = \frac{1}{1.047^5} + \frac{1}{1.047^6} + \frac{1}{1.047^7} = 2.2790$$

The value of the swap option (in millions of dollars) is

$$10\times 2.2790[0.05N(0.2)-0.05N(-0.2)] = 0.181$$

This is also the value given by DerivaGem. (Note that the OIS rate is 4.593% with continuous compounding.)

29.15

The price of the bond at time t is $e^{-R(T-t)}$ where T is the time when the bond matures. Using Itô's lemma, the volatility of the bond price is

$$\sigma \frac{\partial}{\partial R} e^{-R(T-t)} = -\sigma(T-t)e^{-R(T-t)}$$

This tends to zero as t approaches T.

29.16

The cash price of the bond is

$$4e^{-0.05\times 0.50} + 4e^{-0.05\times 1.00} + \ldots + 4e^{-0.05\times 10} + 100e^{-0.05\times 10} = 122.82$$

As there is no accrued interest, this is also the quoted price of the bond. The interest paid during the life of the option has a present value of

$$4e^{-0.05\times 0.5} + 4e^{-0.05\times 1} + 4e^{-0.05\times 1.5} + 4e^{-0.05\times 2} = 15.04$$

The forward price of the bond is therefore

$$(122.82-15.04)e^{0.05\times 2.25} = 120.61$$

The yield with semiannual compounding is 5.0630%.
The duration of the bond at option maturity is

$$\frac{0.25\times 4e^{-0.05\times 0.25} + \ldots + 7.75\times 4e^{-0.05\times 7.75} + 7.75\times 100e^{-0.05\times 7.75}}{4e^{-0.05\times 0.25} + 4e^{-0.05\times 0.75} + \ldots + 4e^{-0.05\times 7.75} + 100e^{-0.05\times 7.75}}$$

or 5.994. The modified duration is 5.994/1.025315 = 5.846. The bond price volatility is therefore $5.846\times 0.050630\times 0.2 = 0.0592$. We can therefore value the bond option using Black's model with $F_B = 120.61$, $P(0, 2.25) = e^{-0.05\times 2.25} = 0.8936$, $\sigma_B = 5.92\%$, and $T = 2.25$. When the strike price is the cash price $K = 115$ and the value of the option is 1.74. When the strike price is the quoted price $K = 117$ and the value of the option is 2.36. This is in agreement with DerivaGem.

29.17

Choose the Caps and Swap Options worksheet of DerivaGem and choose Cap/Floor as the Underlying Type and Black–European as the pricing model.. Enter the OIS rates as 6.5%. (It is only necessary to enter this for one maturity as the rate for all maturities will then automatically be assumed to be 6.5%). The LIBOR forward rates are input as 6.7%. (Again this only needs to be done for one maturity.) Enter Semiannual for the Settlement Frequency, 100 for the Principal, 0 for the Start (Years), 5 for the End (Years), 8% for the Cap/Floor Rate, and $3 for the Price. Check the Cap button. Check the Imply Volatility box and hit *Calculate*. The implied volatility is 31.51%. Then uncheck Implied Volatility, select Floor, check Imply Breakeven Rate. The floor rate that is calculated is 5.9%. This is the floor rate for which the floor is worth $3. A collar when the floor rate is 5.9% and the cap rate is 8% has zero cost.

29.18

We prove this result by considering two portfolios. The first consists of the swap option to receive s_K; the second consists of the swap option to pay s_K and the forward swap. Suppose that the actual swap rate at the maturity of the options is greater than s_K. The swap option to pay s_K will be exercised and the swap option to receive s_K will not be exercised. Both portfolios are then worth zero since the swap option to pay s_K is neutralized by the forward swap. Suppose next that the actual swap rate at the maturity of the options is less than s_K. The swap option to receive s_K is exercised and the swap option to pay s_K is not exercised. Both portfolios are then equivalent to a swap where s_K is received and floating is paid. In all states of the world, the two portfolios are worth the same at time T_1. They must therefore be worth the same today. This proves the result. When s_K equals the current forward swap rate $f = 0$ and $V_1 = V_2$. A swap option to pay fixed is therefore worth the same as a similar swap option to receive fixed when the fixed rate in the swap option is the forward swap rate.

29.19

Choose the Caps and Swap Options worksheet of DerivaGem and choose Swap Option as the Underlying Type and Black–European as the pricing model. Enter 100 as the Principal, 1 as the Start (Years), 6 as the End (Years), 6% as the Swap Rate, and Semiannual as the Settlement Frequency. Enter 21% as the Volatility and check the Pay Fixed button. Do not check the Imply Breakeven Rate or Imply Volatility boxes. The value of the swap option is 3.75.

29.20

(a) To calculate flat volatilities from spot volatilities, we choose a strike rate and use the spot volatilities to calculate caplet prices. We then sum the caplet prices to obtain cap prices and imply flat volatilities from Black's model. The answer is typically slightly dependent on the strike price chosen. This procedure ignores any volatility smile in cap pricing.

(b) To calculate spot volatilities from flat volatilities, the first step is usually to interpolate between the flat volatilities so that we have a flat volatility for each caplet payment date. We choose a strike price and use the flat volatilities to calculate cap prices. By subtracting successive cap prices, we obtain caplet prices from which we can imply spot volatilities. The answer is typically slightly dependent on the strike price chosen. This procedure also ignores any volatility smile in caplet pricing.

29.21

The present value of the coupon payment is

$$35e^{-0.08\times 0.25} = 34.31$$

Equation (29.2) can therefore be used with $F_B = (910-34.31)e^{0.08\times 8/12} = 923.66$, $r = 0.08$, $\sigma_B = 0.10$ and $T = 0.6667$. When the strike price is a cash price, $K = 900$ and

$$d_1 = \frac{\ln(923.66/900) + 0.005\times 0.6667}{0.1\sqrt{0.6667}} = 0.3587$$

$$d_2 = d_1 - 0.1\sqrt{0.6667} = 0.2770$$

The option price is therefore

$$900e^{-0.08\times0.6667}N(-0.2770)-875.69N(-0.3587)=18.34$$

or \$18.34.
When the strike price is a quoted price, 5 months of accrued interest must be added to 900 to get the cash strike price. The cash strike price is $900+35\times0.8333=929.17$. In this case,

$$d_1=\frac{\ln(923.66/929.17)+0.005\times0.6667}{0.1\sqrt{0.6667}}=-0.0319$$

$$d_2=d_1-0.1\sqrt{0.6667}=-0.1136$$

and the option price is

$$929.17e^{-0.08\times0.6667}N(0.1136)-875.69N(0.0319)=31.22$$

or \$31.22.

29.22
The payoff from the swaption is a series of five cash flows equal to $\max[0.076-R,0]$ in millions of dollars where R is the five-year swap rate in four years. The value of an annuity that provides \$1 per year at the end of years 5, 6, 7, 8, and 9 is

$$\sum_{i=5}^{9}e^{-0.078i}=2.914$$

The value of the swaption in millions of dollars is therefore,

$$2.914[0.076N(-d_2)\text{-}0.08N(\text{-}d_1)$$

where

$$d_1=\frac{\ln(0.08/0.076)+0.25^2\times4/2}{0.25\sqrt{4}}=0.3526$$

and

$$d_2=\frac{\ln(0.08/0.076)-0.25^2\times4/2}{0.25\sqrt{4}}=-0.1474$$

The value of the swaption is

$2.914[0.076N(0.1474)-0.08N(-0.3526)]=0.03927$

or \$39,273. This is the same answer as that given by DerivaGem.

29.23
Use the Caps and Swap Options worksheet of DerivaGem. To set the OIS zero curve as flat at 5.8% with continuous compounding, you need only enter 5.8% for one maturity. Similarly, you only need to enter the LIBOR forward rate as 6% for one maturity. To value the cap, select Cap/Floor as the Underlying Type, select Black–European as the pricing model, enter Quarterly for the Settlement Frequency, 100 for the Principal, 0 for the Start (Years), 5 for the End (Years), 7% for the Cap/Floor Rate, and 20% for the Volatility. Check the Cap button. Do not check the Imply Breakeven Rate, and Imply Volatility boxes. Clicking on Calculate gives the value of the cap as 1.514. To value the floor change the Cap/Floor Rate to 5% and check the Floor button rather than the Cap button. Clicking on Calculate gives the

value as 1.116. The collar is a long position in the cap and a short position in the floor. The value of the collar is therefore,

$$1.514 - 1.116 = 0.398$$

29.24

Choose the Cap and Swaptions worksheet of DerivaGem, choose Swap Option as the Underlying Type, and Black–European as the pricing model. Enter 100 as the Principal, 2 as the Start (Years), 7 as the End (Years), 6% as the Swap Rate, and Semiannual as the Settlement Frequency. Enter the zero curve information and enter a forward rate curve that is flat at 7%. Enter 15% as the Volatility and check the Pay Fixed button. Do not check the Imply Breakeven Rate and Imply Volatility boxes. The value of the swaption is 4.384. For a company that expects to borrow at LIBOR plus 50 basis points in two years and then enter into a swap to convert to five-year fixed-rate borrowings, the swaption guarantees that its effective fixed rate will not be more than 6.5%. The swaption is the same as an option to sell a five-year 6% coupon bond for par in two years.