CHAPTER 28
Martingales and Measures

Practice Questions

28.1
The market price of risk for a variable that is not the price of an investment asset is the market price of risk of an investment asset whose price is instantaneously perfectly positively correlated with the variable.

28.2
If its market price of risk is zero, gold must, after storage costs have been paid, provide an expected return equal to the risk-free rate of interest. In this case, the expected return after storage costs must be 6% per annum. It follows that the expected growth rate in the price of gold must be 7% per annum.

28.3
The market price of risk is

$$\frac{\mu - r}{\sigma}$$

This is the same for both securities. From the first security, we know it must be

$$\frac{0.08 - 0.04}{0.15} = 0.26667$$

The volatility, σ, for the second security is given by

$$\frac{0.12 - 0.04}{\sigma} = 0.26667$$

The volatility is 30%.

28.4
It can be argued that the market price of risk for the second variable is zero. This is because the risk is unsystematic; that is, it is totally unrelated to other risks in the economy. To put this another way, there is no reason why investors should demand a higher return for bearing the risk since the risk can be totally diversified away.

28.5
Suppose that the price, f, of the derivative depends on the prices, S_1 and S_2, of two traded securities. Suppose further that:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dz_1$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dz_2$$

where dz_1 and dz_2 are Wiener processes with correlation ρ. From Ito's lemma

$$df = \left(\mu_1 S_1 \frac{\partial f}{\partial S_1} + \mu_2 S_2 \frac{\partial f}{\partial S_2} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} \right.$$
$$\left. + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} \right) dt + \sigma_1 S_1 \frac{\partial f}{\partial S_1} dz_1 + \sigma_2 S_2 \frac{\partial f}{\partial S_2} dz_2$$

To eliminate the dz_1 and dz_2 we choose a portfolio, Π, consisting of

$$\begin{array}{ll} -1: & \text{derivative} \\ +\dfrac{\partial f}{\partial S_1}: & \text{first traded security} \\ +\dfrac{\partial f}{\partial S_2}: & \text{second traded security} \end{array}$$

$$\Pi = -f + \frac{\partial f}{\partial S_1} S_1 + \frac{\partial f}{\partial S_2} S_2$$

$$d\Pi = -df + \frac{\partial f}{\partial S_1} dS_1 + \frac{\partial f}{\partial S_2} dS_2$$
$$= -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} \right) dt$$

Since the portfolio is instantaneously risk-free, it must instantaneously earn the risk-free rate of interest. Hence,

$$d\Pi = r\Pi dt$$

Combining the above equations,

$$-\left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} \right] dt$$

$$= r\left[-f + \frac{\partial f}{\partial S_1} S_1 + \frac{\partial f}{\partial S_2} S_2 \right] dt$$

so that:

$$\frac{\partial f}{\partial t} + rS_1 \frac{\partial f}{\partial S_1} + rS_2 \frac{\partial f}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} = rf$$

28.6

The process for x in a risk neutral world is from the end of Section 28.8,

$$dx = \left[a(x_0 - x) - \lambda c\sqrt{x} \right] dt + c\sqrt{x} dz$$

Hence, the drift rate should be reduced by $\lambda c\sqrt{x}$. In practice, λ is negative so that the drift rate increases.

28.7

As suggested in the hint, we form a new security f^* which is the same as f except that all income produced by f is reinvested in f. Assuming we start doing this at time zero, the relationship between f and f^* is

$$f^* = fe^{qt}$$

If μ^* and σ^* are the expected return and volatility of f^*, Ito's lemma shows that

$$\mu^* = \mu + q$$

$$\sigma^* = \sigma$$

From equation (28.9),

$$\mu^* - r = \lambda\sigma^*$$

It follows that

$$\mu + q - r = \lambda\sigma$$

28.8

As suggested in the hint, we form two new securities f^* and g^* which are the same as f and g at time zero, but are such that income from f is reinvested in f and income from g is reinvested in g. By construction f^* and g^* are non-income producing and their values at time t are related to f and g by

$$f^* = fe^{q_f t} \qquad\qquad g^* = ge^{q_g t}$$

From Ito's lemma, the securities g and g^* have the same volatility. We can apply the analysis given in Section 28.3 to f^* and g^* so that from equation (28.15)

$$f_0^* = g_0^* E_g\left(\frac{f_T^*}{g_T^*}\right)$$

or

$$f_0 = g_0 E_g\left(\frac{f_T e^{q_f T}}{g_T e^{q_g T}}\right)$$

or

$$f_0 = g_0 e^{(q_f - q_g)T} E_g\left(\frac{f_T}{g_T}\right)$$

28.9

This statement implies that the interest rate has a negative market price of risk. Since bond prices and interest rates are negatively correlated, the statement implies that the market price of risk for a bond price is positive. The statement is reasonable. When interest rates increase, there is a tendency for the stock market to decrease. This implies that interest rates have negative systematic risk, or equivalently that bond prices have positive systematic risk.

28.10

(a) In the traditional risk-neutral world, the process followed by S is

$$dS = (r-q)S\,dt + \sigma_S S\,dz$$

where r is the instantaneous risk-free rate. The market price of dz-risk is zero.

(b) In the traditional risk-neutral world for currency B, the process is

$$dS = (r-q+\rho_{QS}\sigma_S\sigma_Q)S\,dt + \sigma_S S\,dz$$

where Q is the exchange rate (units of A per unit of B), σ_Q is the volatility of Q and ρ_{QS} is the coefficient of correlation between Q and S. The market price of dz-risk is $\rho_{QS}\sigma_Q$.

(c) In a world defined by numeraire equal to a zero-coupon bond in currency A maturing at time T

$$dS = (r-q+\rho_{SP}\sigma_S\sigma_P)Sdt + \sigma_S Sdz$$

where σ_P is the bond price volatility and ρ_{SP} is the correlation between the stock and bond. The market price of dz-risk is $\rho_{SP}\sigma_P$.

(d) In a world defined by a numeraire equal to a zero-coupon bond in currency B maturing at time T

$$dS = (r-q+\rho_{SP}\sigma_S\sigma_P + \rho_{FS}\sigma_S\sigma_F)Sdt + \sigma_S Sdz$$

where F is the forward exchange rate, σ_F is the volatility of F (units of A per unit of B, and ρ_{FS} is the correlation between F and S. The market price of dz-risk is $\rho_{SP}\sigma_P + \rho_{FS}\sigma_F$.

28.11

The forward value of a stock price, commodity price, or exchange rate is the delivery price in a forward contract that causes the value of the forward contract to be zero. A forward bond price is calculated in this way. However, a forward interest rate is the interest rate implied by the forward bond price.

28.12

$$d\ln f = \left[r + \sum_{i=1}^{n} (\lambda_i \sigma_{f,i} - \sigma_{f,i}^2 / 2) \right] dt + \sum_{i=1}^{n} \sigma_{f,i} dz_i$$

$$d\ln g = \left[r + \sum_{i=1}^{n} (\lambda_i \sigma_{g,i} - \sigma_{g,i}^2 / 2) \right] dt + \sum_{i=1}^{n} \sigma_{g,i} dz_i$$

so that

$$d\ln \frac{f}{g} = d(\ln f - \ln g) = \left[\sum_{i=1}^{n} (\lambda_i \sigma_{f,i} - \lambda_i \sigma_{g,i} - \sigma_{f,i}^2 / 2 + \sigma_{g,i}^2 / 2) \right] dt + \sum_{i=1}^{n} (\sigma_{f,i} - \sigma_{g,i}) dz_i$$

Applying Ito's lemma again

$$d\frac{f}{g} = \frac{f}{g} \left[\sum_{i=1}^{n} (\lambda_i \sigma_{f,i} - \lambda_i \sigma_{g,i} - \sigma_{f,i}^2 / 2 + \sigma_{g,i}^2 / 2) + (\sigma_{f,i} - \sigma_{g,i})^2 / 2 \right] dt + \frac{f}{g} \sum_{i=1}^{n} (\sigma_{f,i} - \sigma_{g,i}) dz_i$$

When $\lambda_i = \sigma_{g,i}$ the coefficient of dt is zero and f/g is a martingale.

28.13

$$d\ln h = \ldots + \sum_{i=1}^{n} \sigma_{h,i} dz_i$$

$$d\ln g = \ldots + \sum_{i=1}^{n} \sigma_{g,i} dz_i$$

so that

$$d\ln \frac{h}{g} = \ldots + \sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i}) dz_i$$

Applying Ito's lemma again

$$d\frac{h}{g} = \ldots + \frac{h}{g} \sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i}) dz_i$$

This proves the result.

28.14

If the expected value of a variable at time *t* was expected to be greater (less) than its expected value at time zero, the expected value at time zero would be wrong. It would be too low (too high). A general result is that the expected value of a variable today is the expectation of its expected value at a future time.

28.15

(a) The no-arbitrage arguments in Chapter 5 show that

$$F(t) = \frac{f(t)}{P(t,T)}$$

(b) From Ito's lemma:

$$d\ln P = (\mu_P - \sigma_P^2 / 2)\, dt + \sigma_P\, dz$$

$$d \ln f = (\mu_f - \sigma_f^2 / 2)\, dt + \sigma_f \, dz$$

Therefore,

$$d \ln \frac{f}{P} = d(\ln f - \ln P) = (\mu_f - \sigma_f^2 / 2 - \mu_P + \sigma_P^2 / 2)\, dt + (\sigma_f - \sigma_P)\, dz$$

so that

$$d \frac{f}{P} = (\mu_f - \mu_P + \sigma_P^2 - \sigma_f \sigma_P) \frac{f}{P} dt + (\sigma_f - \sigma_P) \frac{f}{P} dz$$

or

$$dF = (\mu_f - \mu_P + \sigma_P^2 - \sigma_f \sigma_P) F\, dt + (\sigma_f - \sigma_P) F\, dz$$

In a world defined by numeraire $P(t,T)$, F has zero drift. The process for F is

$$dF = (\sigma_f - \sigma_P) F\, dz$$

(c) In the traditional risk-neutral world, $\mu_f = \mu_P = r$ where r is the short-term risk-free rate and

$$dF = (\sigma_P^2 - \sigma_f \sigma_P) F\, dt + (\sigma_f - \sigma_P) F\, dz$$

Note that the answers to parts (b) and (c) are consistent with the market price of risk being zero in (c) and σ_P in (b). When the market price of risk is σ_P, $\mu_f = r + \sigma_f \sigma_P$ and $\mu_P = r + \sigma_P^2$.

(d) In a world defined by a numeraire equal to a bond maturing at time T^*, $\mu_P = r + \sigma_P^* \sigma_P$ and $\mu_f = r + \sigma_P^* \sigma_f$ so that

$$dF = [\sigma_P^2 - \sigma_f \sigma_P + \sigma_P^* (\sigma_f - \sigma_P)] F\, dt + (\sigma_f - \sigma_P) F\, dz$$

or

$$dF = (\sigma_f - \sigma_P)(\sigma_P^* - \sigma_P) F\, dt + (\sigma_f - \sigma_P) F\, dz$$

28.16

(a) The futures price is a martingale in the traditional risk-neutral world.

(b) The forward price for a contract maturing at time T is a martingale is a world defined by numeraire $P(t,T)$.

(c) Define σ_P as the volatility of $P(t,T)$ and σ_F as the volatility of the forward price. The forward rate has zero drift in a world defined by numeraire $P(t,T)$. When we move from the traditional world to a world defined by numeraire $P(t,T)$, the volatility of the numeraire ratio is σ_P and the drift increases by $\rho_{PF} \sigma_P \sigma_F$ where ρ_{PF} is the correlation between $P(t,T)$ and the forward price. It follows that the drift of the forward price in the traditional risk neutral world is $-\rho_{PF} \sigma_P \sigma_F$. The drift of the futures price is zero in the traditional risk neutral world. It follows that the excess of the drift of the futures price over the forward price is $\rho_{PF} \sigma_P \sigma_F$.

(d) P is inversely correlated with interest rates. It follows that when the correlation between interest rates and F is positive the futures price has a lower drift than the forward price. The futures and forward prices are the same at maturity. It follows that the futures price is above the forward price prior to maturity. This is consistent with Section 5.8. Similarly, when the correlation between interest rates and F is negative, the future price is below the forward price prior to maturity.