Exponential and logarithmic functions (6)

Applications

Outline

- Population model (logistic)
- Cooling and carbon dating
- Gaussian distribution
- Logarithmic scale

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Population model (logistic)

Exponential Growth

- Growth rate proportional to own quantity:

Bacterial colony, spread of virus, debt with interest

- **Example:** model the population if the community had 10000 in 1980 and 200000 in 2000.

$$P(t) = P_0 \cdot e^{kt} \text{ with } P(0) = 1, P(20) = 20$$

Population model (logistic)

Logistic Models

- Given Carrying capacity K, $P(t) = \frac{K}{1+Ke^{-rt}}$, $\lim_{t\to\infty} P(t) = K$
- Standard logistic function: $f(x) = \frac{1}{1+e^{-x}}$
- **Example:** If $f(x)$ represents percentage to max capacity, when will we reach 80% ? 99.9%?

Cooling and carbon dating

- Newton's Law of Cooling

$$T(t) = K + (T(0) - K) \cdot e^{-rt}$$

Temperature at time *t* **Room temperature**

$$\lim_{t \to \infty} e^{-rt} = 0$$

$$\lim_{t \to \infty} T(t) = K$$

- Carbon Dating (C-14 half-life ~5730 years)

$$C(t) = C_0 \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

When $t = 5730$: $\frac{C(t)}{C_0} = \frac{1}{2}$

$$C(t) = 12.5\%\ C_0 \Rightarrow t = 3 \times 5730 = 17190$$

Gaussian Distribution

- Standard Normal Distribution: $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$
- Mean $= 0$, standard deviation$= 1$
- $68 - 95 - 99.7$ Rule
- Example: Intelligence Quotient has $\mu = 100, \sigma = 15$

  where μ = average(mean),σ = standard deviation

Logarithmic scale

- Richter magnitude, decibel, music, acidity
- Converts exponential growth to linear growth

Example

- $y = K \cdot e^{rx}, Y = \ln y = \ln K + r \cdot x$
- Order of magnitude
- Octave $\log_2(\frac{f_2}{f_1})$, compare frequencies
- pH: $\log(\frac{1}{a_{H^+}})$, hydrogen ion activity

Logarithmic scale

- Log Plot

- Log-log plots $y = f(x) \rightarrow Y = \ln f(e^x)$
- **Example: Cobb-Douglas production**
- Semi-log plots $y = f(x) \rightarrow Y = \ln f(x)$
- **Example: Progression of pandemic**

Review

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- What are some different population models?
- How does exponential decay show up in real life?
- What is a normal distribution?
- Where would you see logarithmic scale in action?