Polynomials (10)

Signs of Polynomials

Outline

- Signs of Polynomials

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Signs of Polynomials

After factorization, we can determine intervals on which a polynomial is positive or negative.

- Roots of a polynomials divide the real line into several intervals.
- Factors of the polynomial may change signs at these roots but they remain same signs on each of the intervals.

By listing signs of each factor we can determine whether the polynomial is positive or negative.

Signs of Polynomials

In particular, a polynomial $f(x)$ can be factorized into a product of linear factors and irreducible quadratic factors.

We could further make leading coefficients of every factors positive before discussing their signs.

1. If $ax + b$ is a linear factor with $a > 0$, then

$$ax + b = a\left(x + \frac{b}{a}\right)\begin{cases} < 0, & \text{for } x < -\frac{b}{a} \\ > 0, & \text{for } x > -\frac{b}{a} \end{cases}$$

Signs of Polynomials

2. If $ax^2 + bx + c$ is an irreducible factor (i.e. $b^2 - 4ac < 0$) and $a > 0$, then

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] > 0 \quad \text{for all } x.$$

Since the irreducible factor $ax^2 + bx + c$ is always positive, it doesn't affect signs of $f(x)$ and we can ignore it.

In conclusion, signs of each factor are determined and we can know where $f(x)$ is positive or negative.

Signs of Polynomials

Example

Find the intervals on which

$$f(x) = 2x^3 + 7x^2 - 15x$$

is positive or negative.

Example

Find the intervals on which

$$f(x) = 1 - x^6$$

is positive or negative.

Signs of Polynomials

1. A linear factor $ax + b$ with $a > 0$ may repeat n times.
   - If n is even, then $(ax+b)^n > 0$ for all $x \neq -b/a$. Note that in this case, $(ax+b)^n$ does **NOT** change signs at $-b/a$.
   - If n is odd, then

$$(ax+b)^n \begin{cases} < 0, & \text{for } x < -\frac{b}{a} \\ > 0, & \text{for } x > -\frac{b}{a} \end{cases}$$

2. An irreducible factor $ax^2 + bx + c$ with $a > 0$ is always positive. Hence $(ax^2+bx+c)^m$ is always positive no matter m is even or odd.

Signs of Polynomials

Example

Find the intervals on which

$$f(x) = x^5 + 4x^4 + 4x^3$$

is positive or negative.

Example

Find the intervals on which

$$f(x) = (-x+4)^2(-2x+3)^3(x^2+x+1)^5$$

is positive or negative.

Review

- How do the linear and quadratic factors affects the sign of a polynomial function?
- How do the multiplicity of the linear and quadratic factors affects the sign of a polynomial function?

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