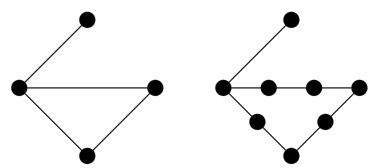
- This idea of a non-constructive existence proof has proved extremely effective, and has led to the development of probabilistic combinatorics.

9 Kuratowski's Theorem (non-examinable)

In Corollary 6.14 we saw that K_5 and $K_{3,3}$ are not planar. In this section we will prove a remarkable theorem of Kuratowski, roughly saying that K_5 and $K_{3,3}$ are in some sense the *only* obstructions to being planar. We will also simultaneously prove Fáry's theorem, that any planar graph can in fact be drawn using only straight lines.

Definition 9.1. A *subdivision* of a graph H is a graph obtained from H by replacing the edges of H by internally vertex disjoint paths of non-zero length with the same endpoints.

Example 9.2.



H Subdivision of H

Remark 9.3. If G contains a subdivision of H, it also contains an H-minor.

Definition 9.4. A *Kuratowski graph* is a graph which is a subdivision of K_5 or $K_{3,3}$. If G is a graph and H is a subgraph of G which is a Kuratowski graph then we say that H is a *Kuratowski subgraph* of G.

Theorem 9.5 (Kuratowski 1930)**.** *A graph is planar if and only if it has no Kuratowski subgraph.*

It is straightforwardly true that no Kuratowski graph is planar. Indeed, given a planar drawing (by polygonal curves) of a Kuratowski subgraph, we can consider the subdivided paths to be long polygonal curves in a planar drawing of K_5 or $K_{3,3}$, which is impossible. So, to prove Theorem 9.5 we need only prove that every graph with no Kuratowski subgraph has a plane drawing. As promised, we prove a stronger theorem, combining Kuratowski's Theorem with Fáry's Theorem on straightline drawings.

Definition 9.6. A *straightline drawing* of a planar graph G is a drawing in which every edge is a straight line.

Theorem 9.7. *If G is a graph with no Kuratowski subgraph then G has a straightline drawing in the plane.*

The proof of Theorem 9.7 is broken down into two parts. In the first part, we prove a version of the theorem for 3-connected graphs, then in the second part we show how the general case can be reduced to the 3-connected case.

9.1 Convex drawings of 3-connected graphs

Definition 9.8. A *convex drawing* of G is a straightline drawing in which every non-outer face of G is a convex polygon, and the outer face is the complement of a convex polygon. (That is, the boundary of each face is the boundary of a convex polygon).

Theorem 9.9 (Tutte 1960)**.** *If G is a 3-connected graph which has no Kuratowski subgraphs then G has a convex drawing in the plane with no three vertices on a line.*

We will need some lemmas to prove Theorem 9.9.

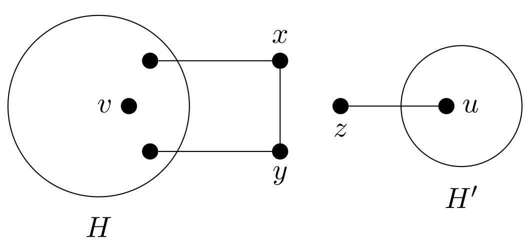
Lemma 9.10 (Thomassen 1980)**.** *Every 3-connected graph G with at least five vertices has an edge e such that G/e is 3-connected.*

Proof. We proceed by contradiction. Suppose G has no edge whose contraction yields a 3-connected graph. For every edge $(x, y) \in E(G)$, the graph G is 3-connected but $G/(x, y)$ is not 3-connected. A minimal (2 vertex) separating set in $G/(x, y)$ must therefore contain the newly contracted vertex, so there exists a vertex $z \in V(G) \setminus \{x, y\}$ such that $\{x, y, z\}$ is a separating set in G. We call such a vertex z a *mate* of the edge (x, y).

Choose an edge $(x, y) \in E(G)$ and a mate z and a connected component H in $G\backslash\{x, y, z\}$, in such a way that H has maximum possible size. Let H' be another connected component of $G\backslash\{x, y, z\}$. Since $\{x, y, z\}$ is a separating set of minimal size, each of x, y and z have a neighbour in both of H and H'. Let u be a neighbour of z in H' and let v be a mate of the edge (u, z).

We claim now that the induced subgraph F of G with vertex set $V(H) \cup \{x, y\} \setminus v$ is connected. If that is the case then we reach a contradiction because $|F| > |H|$ and F is a connected subgraph of $G\backslash\{u, v, z\}$.

To prove the claim, we consider two cases: $v \notin V(H)$ or $v \in V(H)$. The first case is straightforward: since H is connected and since x and y each have a neighbour in H, the graph F must also be connected. Consider now the second case. The following picture gives an illustration of the situation.



Observe that $G\backslash\{v, z\}$ is connected since G is 3-connected. Thus, for any vertex $w \in V(H) \setminus \{v, z\}$ there exists a path in $G\backslash\{v, z\}$ from w to $\{x, y\}$. If we can find such a path for any $w \in V(H) \setminus \{v\}$ inside of F then it follows that F is connected.

We claim that we can just choose these paths to be the shortest paths (in $G\backslash\{v, z\}$) from each w to $\{x, y\}$. Indeed, since $\{x, y, z\}$ was the separating set with which we defined H, any path from w avoiding z must pass through x or y before it leaves H. □

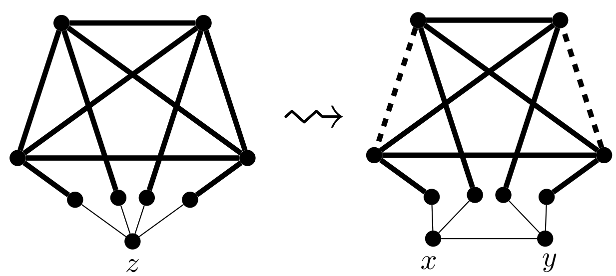
Lemma 9.11. *If G has no Kuratowski subgraphs, then G/e has no Kuratowski subgraph, for any edge $e \in E(G)$.*

Proof. We prove the contrapositive: if G/e contains a Kuratowski subgraph then so does G.

Let z be the vertex of G/e obtained by contracting $e = (x, y)$. Let H be a Kuratowski subgraph in G/e. If $z \notin V(H)$ then it is clear that H is also a subgraph of G. Otherwise, write $N_H(z) = A \cup B$, where the vertices in A were adjacent to x while the vertices in B were adjacent to y, before the contraction.

If $|A| = 0$ (respectively, $|B| = 0$) then we obtain a copy of H in G by replacing z with y (respectively x), in the copy of H in G/e. If $|A| = 1$ (respectively $|B| = 1$) we can replace z with y, add x and include the edge (x, y) and the edge from x to A, to obtain a copy of H with one of its edge subdivided (specifically, the edge from z to A, respectively from z to B).

Since H is a subdivision of $K_{3,3}$ or K_5, every vertex has degree 2,3 or 4. The only possibility that we have not accounted for is that $d_H(z) = |N_H(z)| = 4$, and $|A| = |B| = 2$. We can then obtain a subdivision of K_5 in G from H by replacing z with the edge (x, y) and deleting the inner vertices of two of the paths in H. This is illustrated below; the thicker lines indicate subdivided paths, and the four vertices directly above z are in A, A, B and B respectively.



□

Proof of Theorem 9.9. Let G be a 3-connected graph which has no Kuratowski subgraphs. We prove that G has a convex drawing in the plane with no three vertices on a line, by induction on $|G|$.

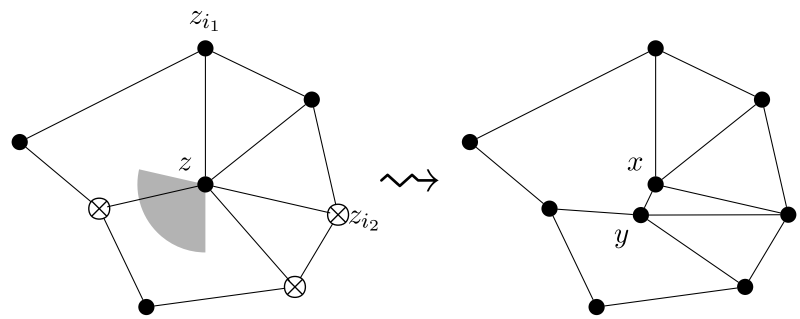
The base case is $|G| = 4$. In that case, since G is 3-connected, the only possibility is that $G = K_4$ and it is easy to find a suitable drawing (see for example Example 6.2).

Consider now $|G| \geq 5$. The induction hypothesis says that every 3-connected graph on at most $|G| - 1$ vertices without Kuratowski subgraphs has a convex drawing in the plane with no three vertices on a line.

By Lemma 9.10 there exists an edge $e = (x, y) \in E(G)$ such that G/e is 3-connected. Moreover, by Lemma 9.11 G/e does not have any Kuratowski subgraphs. From the induction hypothesis it follows that G/e has a convex drawing in the plane with no three vertices on a line. Fix such a convex drawing and let z be the vertex of G/e obtained by contracting the edge e in G. Note that if we delete z from G/e we get a graph which is 2-connected and hence its faces are cycles. Let C be the cycle formed by the boundary of the face of $(G/e)\backslash z$ which contains z (this could be the outer face).

Observe that all the neighbours of z in G/e (and thus, also of x and y in G) lie in C. Moreover, since G is 3-connected, x and y each have at least 2 neighbours in C. Let $z_1, \ldots, z_k$ be the neighbours of z

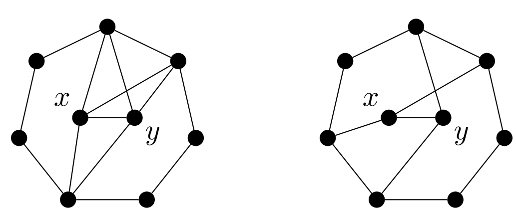
in G/e ordered cyclically (according to an orientation of C) and let $z_{i_1}, \ldots, z_{i_\ell}$ denote the neighbours of x in G (in the same order). If the neighbours of y all fall in between two neighbours of x (that is, all are in $\{z_{i_j}, z_{i_j+1}, \ldots, z_{i_{j+1}}\}$ for some $j \in \{1, \ldots, \ell\}$) then we can obtain a convex drawing of G with no three vertices on a line by putting x at z and putting y at a point close to z in the wedge formed by $\big(x, z_{i_j}\big)$ and $\big(x, z_{i_{j+1}}\big)$. Note that it is possible for the angle between $\big(x, z_{i_j}\big)$ and $\big(x, z_{i_{j+1}}\big)$ to be a reflex angle, as illustrated below (the vertices marked $\otimes$ correspond to neighbours of y). In this case we need to put our point in the region formed by continuing the lines z_{i_j}, z and $z_{i_{j+1}}, z$, shaded in the picture below. This is so that we can draw the triangles x, y, z_{i_j} and $x, y, z_{i_{j+1}}$ without intersecting existing lines, if necessary.



In general, the reason it is always possible to choose an appropriate position for y very close to z is that if we take a convex polygon with no three points on a line, and we perturb one of the vertices slightly, then the result is still a convex polygon. Note also that it is always possible to choose our new point such that still no three vertices are on a line, because the set of bad positions has measure zero.

We claim the neighbours of y are always situated in between two neighbours of x, so the above argument is always valid. Indeed, otherwise, one of the following two cases holds:

- y shares three neighbours z_{j_1}, z_{j_2} and z_{j_3} with x. In this case, the cycle C together with (x, y) and the edges from $\{x, y\}$ to $\{z_{j_1}, z_{j_2}, z_{j_3}\}$ forms a subdivision of K_5 in G (left image below).
- y has neighbours z_{j_1} and z_{j_3} which alternate with two neighbours z_{j_2} and z_{j_4} of x in the cycle C. In that case, the cycle C together with (x, y), (x, z_{j_2}), (x, z_{j_4}), (y, z_{j_1}) and (y, z_{j_3}) forms a subdivision of $K_{3,3}$ in G (right image below):

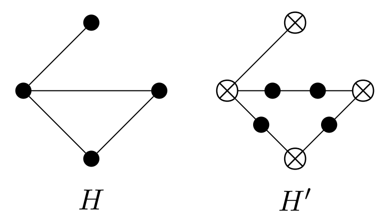


Since G has no Kuratowski subgraph, both of these cases are impossible. $\square$

9.2 Reducing the general case to the 3-connected case

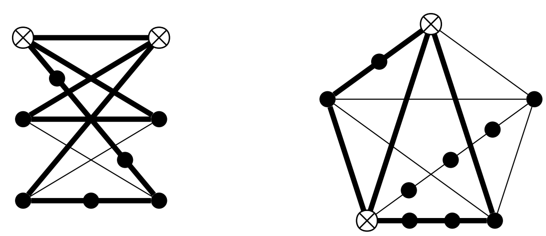
Definition 9.12. Given a subdivision H' of H, we call the vertices of the original graph *branch vertices.*

Example 9.13. In the following picture, the branch vertices are marked with an $\otimes$.



Fact. *We make three observations. See the picture below.*

1. *In a Kuratowski subgraph, there are three internally vertex-disjoint paths connecting any two branch vertices. For K_5-subdivisions, we even have four such paths.*
2. *In a Kuratowski subgraph, there are four internally vertex-disjoint paths between any two pairs of branch vertices.*
3. *Any cycle in a subdivision contains at least three branch vertices.*



Proposition 9.14. *Let G be a graph with at least 4 vertices which has no Kuratowski subgraph, and suppose that adding an edge joining any pair of non-adjacent vertices creates a Kuratowski subgraph. Then G is 3-connected.*

Theorem 9.7 is a corollary of Proposition 9.14, as follows.

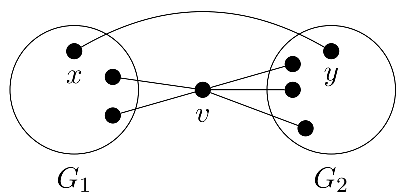
Proof of Theorem 9.7. Start with a graph G which has no Kuratowski subgraph, and add edges as long as possible without creating a Kuratowski subgraph. Let the final graph be $G' \supseteq G$. Then G' is 3-connected so by Theorem 9.9 it has a planar convex drawing. Therefore G has a planar straightline drawing. □

Proof of Proposition 9.14. The proof is by induction on $|G|$. If $|G| = 4$ then G is complete so we are done. So assume $|G| \geq 5$.

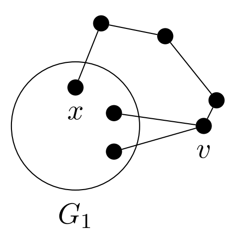
First, note that G is connected. Otherwise, we could add an edge between connected components, and this cannot create a Kuratowski subgraph because Kuratowski graphs have no bridges.

Next, we prove that G is 2-connected. Suppose otherwise, that we can write $G = G_1 \cup G_2$ in such a way that $G_1 \cap G_2 = v$ for some vertex v, each G_i contains more vertices than just v, and there are no edges from $G_1 \backslash v$ to $G_2 \backslash v$. We claim that the addition of any edge to each G_i creates a Kuratowski subgraph, so we can apply the induction hypothesis to each. Indeed, consider $G_i \cup \{e\}$ for some edge e between vertices of G_i. By assumption, $G \cup \{e\}$ contains a Kuratowski subgraph, which is 2-connected. Therefore deleting v (which separates G_1 from G_2) cannot disconnect the Kuratowski subgraph, so this subgraph must have been entirely contained in G_i.

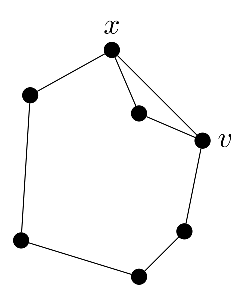
By the induction hypothesis and Theorem 9.9, each G_i has a planar convex drawing (if either has fewer than 4 vertices, then it trivially has such a drawing). Now, consider any $x \in G_1 \backslash v$, $y \in G_2 \backslash v$. The addition of $e = (x, y)$ to G creates a Kuratowski subgraph.



By the first observation above, a branch vertex cannot be disconnected from the other branch vertices by deleting an edge and a vertex. Hence, the branch vertices of this subdivision are either all in G_1 or all in G_2 (say G_1). Therefore the only part of the Kuratowski subgraph in $G \cup \{e\}$ which is not contained in G_1, is a path between v and x.



Hence, if we start with G_1 and add any path from v to any x, then the resulting graph also has a Kuratowski subgraph. But if we choose x to be a neighbour of v, then we can add such a path to a plane drawing of G_1 while preserving planarity.



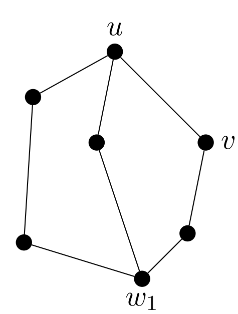
This is impossible because no planar graph has a Kuratowski subgraph. It follows that G is 2-connected.

The final step is to show that G is 3-connected. Suppose otherwise, so that we can write $G = G_1 \cup G_2$ with $G_1 \cap G_2 = \{u, v\}$, where each G_i contains more than just the vertices u, v, and there are no edges from $G_1 \backslash \{u, v\}$ to $G_2 \backslash \{u, v\}$. We know that G is 2-connected so we can assume each G_i is connected (each of u and v must have a neighbour in each component of each $G_i \backslash \{u, v\}$, otherwise deleting one of u and v would disconnect G).

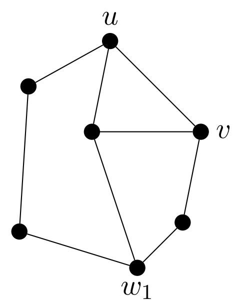
Now, we claim that $(u, v) \in E(G)$. If not, then $G \cup \{(u, v)\}$ contains a Kuratowski subgraph. The branch vertices of this subgraph must be entirely in G_1 or in G_2 (say G_1) because, by the first observation, deleting 2 vertices from a Kuratowski graph cannot disconnect a pair of branch vertices. Thus $G_2 \cup \{(u, v)\}$ contains at most 2 branch vertices, so by the third observation, the intersection of the Kuratowski subgraph with $G_2 \cup \{(u, v)\}$is cycle-free, i.e. it is just the edge (u, v). On the other hand, G_2 is connected, so it contains a u-v path (disjoint from the Kuratowski subgraph). We can view that path as a subdivision of (u, v) showing that G contains a subdivision of $G_1 \cup \{(u, v)\}$, whence G contains a Kuratowski subgraph as well.

Next, we claim that adding any edge to each G_i creates a Kuratowski subgraph, so we can apply the induction hypothesis. Indeed, consider $G_1 \cup \{e\}$ for some edge e between vertices of G_1. By assumption, $G \cup \{e\}$ contains a Kuratowski subgraph, but deleting u and v disconnects $G \cup \{e\}$. Therefore $G_2 \backslash \{u, v\}$ contains no branch vertices, and since $(u, v) \in E(G)$, a Kuratowski subgraph is also contained in $G_1 \cup \{e\}$.

By the induction hypothesis and Theorem 9.9 (and by considering separately the case $|G_i| = 3$), each G_i has a convex drawing. For each i choose $w_i \in V(G_i)$ such that u, v and w_i belong to the boundary of some convex face of G_i. By hypothesis the graph $G \cup \{(w_1, w_2)\}$ contains a Kuratowski subgraph K. Note that, by the second observation, deleting two vertices and an edge cannot disconnect these pairs from one another, and it follows that all but possibly one of the branch vertices are contained in some G_i (say G_1). Note that we can use the same reasoning as in the proof of 2-connectedness to see that in fact $G_2 \backslash \{u, v\}$ has exactly one branch vertex x. Indeed, otherwise G_1 contains all the branch vertices, so the other vertices of K can only be on a path from w_1 to u or from w_1 to v. But we can add such a path to our convex drawing of G_1 (as illustrated below), which contradicts the nonplanarity of K.



Note that x is separated from the other branch vertices of K by the 3-element set $\{u, v, w_1\}$; by the first observation, K must be a subdivision of $K_{3,3}$. It also follows that if in $G \cup \{(w_1, w_2)\}$ we identify all vertices of $V(G_2) \backslash V(G_1)$ into a single vertex, the new graph also contains a subdivision of $K_{3,3}$. But we can construct a planar drawing of this graph, which is a contradiction! Indeed, u, v and w_1 are on the boundary of some convex face of G_1. Put the new vertex anywhere inside this face and connect it to u, v and w_1 with straight lines, as illustrated below.



We conclude that G is 3-connected, as required. $\square$