cycle component contains no vertices from $X_1 \cup X_2$ since each vertex of the cycle is adjacent to an edge of colour 1 and an edge of colour 2. Similarly for paths, any internal vertices can not belong to $X_1 \cup X_2$. If we exchange colours 1 and 2 along a path component, then any endvertex which was in X_1 gets moved to X_2, and vice versa. Since $|X_1| > |X_2| + 2$ at least one component of H contains more vertices in X_1 than in X_2. This component must be a path P starting in X_1 and not ending in X_2. Exchanging colours 1 and 2 on P reduces $|X_1|$ by t for some $1 \leq t \leq 2$, and increases $|X_2|$ by the same amount t. Since $(|X_1|-t)^2+(|X_2|+t)^2 = |X_1|^2+|X_2|^2-2t(|X_1|-|X_2|-t) < |X_1|^2+|X_2|^2$ (where we used $t \geq 1 > 0$ and $|X_1| - |X_2| - t > 2 - t \geq 0$), this is a contradiction to our minimality assumption on the colouring. This proves (7).

Now by (6), we see that the average size of the sets $|X_i|$ is strictly smaller than 2, so there is a set of size 0 or 1. If there is no set of size 1, then it follows from (7) that each $|X_i$ is either 0 or 2, but this is impossible, as (6) shows that there sum must be odd. Hence, there is some set, say X_k, such that $|X_k| = 1$. That is, there is a unique neighbour u of v that misses the colour k.

Let G' be the graph obtained from G by deleting the edge $\{v, u\}$ and deleting all edges of colour k. So $G' \backslash \{v\}$ is $(k-1)$-edge-coloured. Moreover, in G', the vertex v and all its neighbours have degree at most $k-1$, and at most one neighbour has degree $k-1$ (since for every vertex we deleted at least one edge adjacent to it). So, by the induction hypothesis, G' is $(k-1)$-edge-colourable. Restoring the colour k edges and giving edge $\{v, u\}$ the colour k then yields a proper k-edge-colouring of G.[3] □

Remark 7.35. Vizing's Theorem shows that the chromatic index of any graph G takes one of two values: either $\Delta(G)$ or $\Delta(G) + 1$. However, it is an NP-Complete problem to determine whether or not $\chi'(G) = \Delta(G)$.

8 Extremal Graph Theory

Extremal problems are a wide class of problems that deal with questions of the following general form: how large can a structure be if it does not contain a forbidden substructure? Such questions arise in several different applications, and have led to the development of several different techniques with which to attack them. In this section, we will be introduced to two classic branches of extremal graph theory.

8.1 Ramsey Theory

Ramsey theory refers to a large body of deep results in mathematics concerning partitions of large collections. An informal way of describing this is Motzkin's statement that "Complete disorder is impossible". Although this does not always seem to be the case in the world around us, in mathematics this principle appears in great generality.

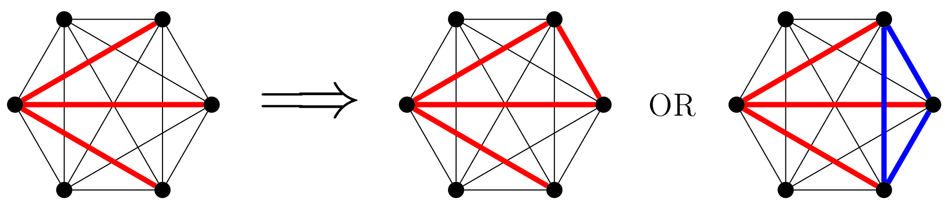
This general principle can be illustrated by the following story, which took place in Hungary in the 1950's. There was a sociologist named Szalai who was studying the social dynamics within a group of 20 schoolchildren. Every day, he would observe them during their breaks, making of note of who played with whom. At the end of his study, he noticed that every day he could find a group of 4 children where either every pair had played together, or no pair had played together.

[3] Giving $\{v, u\}$ the colour k does not violate the properness of the colouring: v has no other adjacent edges coloured k by construction, and since $u \in X_k$, u also has no other adjacent edges coloured k.

Perhaps as a result of being in Hungary, a country with a proud and strong combinatorial tradition, he suspected that there might be some mathematical magic at work. So before announcing a sociological breakthrough, he discussed the finding with some local mathematicians by the names of Erdős, Sós and Turán, who confirmed that indeed, Szalai's observations were not, in fact, about schoolchildren, but rather a fact about graphs. The following proposition demonstrates a simpler version of this.

Proposition 8.1. *Among six people it is possible to find three mutual acquaintances or three mutual non-acquaintances.*

Proof. Consider the complete graph whose vertices are the six people. Colour an edge between two people red if those people know each other, and blue otherwise. Single out some vertex u. Out of the five edges incident to u, at least three are blue or at least three are red. Without loss of generality say there are three red edges $\{u, v_1\}$, $\{u, v_2\}$, $\{u, v_3\}$. If there is a red edge $\{v_i, v_j\}$, then u, v_i and v_j are mutual acquaintances. Otherwise v_1, v_2 and v_3 are mutual non-acquaintances.



□

This is a particular case of a more general phenomenon — given any k, a red-/blue-colouring of the edges of a large enough complete graph must contain either an all-red or an all-blue copy of K_k. This was first proven by Ramsey in 1930, who needed the result for his work in Logic. When it came to the attention of combinatorialists, they were interested in a quantitative version: how large does "large enough" have to be?

Definition 8.2. Given integers $s, t \geq 2$, the *Ramsey number* $R(s,t)$ is the smallest n such every red-/blue-colouring of $E(K_n)$ contains either a red K_s or a blue K_t. In the *diagonal* case, when $s = t = k$, we write $R(k)$ for the Ramsey number $R(k,k)$.

It is obvious that

$$R(s,t) = R(t,s)$$

for every $s, t \geq 2$ and $R(s,2) = R(2,s) = s$, since in a red-/blue-colouring of K_s either there is a blue edge or else every edge is red. The following result, due to Erdős and Szekeres, states that $R(s,t)$ is finite for every s and t, and at the same time it gives an upper bound on $R(s,t)$. This bound is considerably better than the original bound given by Ramsey (who was primarily concerned with showing that $R(s,t)$ is finite).

Theorem 8.3. *The function $R(s,t)$ is finite for all $s, t \geq 2$. Quantitatively, if $s \geq 2$ and $t \geq 2$ then*

$$R(s,t) \leq R(s-1,t) + R(s,t-1) \tag{8}$$

and

$$R(s,t) \leq \binom{s+t-2}{s-1}. \tag{9}$$

Proof. When proving (8) we may assume that $R(s-1,t)$ and $R(s,t-1)$ are finite. Let $N = R(s-1,t) + R(s,t-1)$ and consider a colouring of the edges of K_N with red and blue. We have to show that in this colouring there is either a red K_s or a blue K_t. To this end, let x be a vertex of K_N. Since $d(x) = N-1$, either there are at least $N_1 = R(s-1,t)$ red edges incident with x or there are at least $N_2 = R(s,t-1)$ blue edges incident with x. Without loss of generality assume that the first case holds. Consider a subgraph K_{N_1} of K_N spanned by N_1 vertices joined to x by red edges. If K_{N_1} has a blue K_t, we are done. Otherwise, by the definition of $R(s-1,t)$, the graph K_{N_1} contains a red K_{s-1} which forms a red K_s with x.

Inequality (9) holds if $s = 2$ or $t = 2$ (in fact, we have equality). Assume that $s > 2$, $t > 2$, and that (9) holds for every pair (s', t') with $2 \leq s' + t' < s + t$. Then by (8) we have

$$\begin{aligned} R(s,t) &\leq R(s-1,t) + R(s,t-1) \\ &\leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}. \end{aligned}$$

□

Remark. Note that $R(s,t) \leq N$ means that every graph on N vertices has either $\omega(G) \geq s$ or $\alpha(G) \geq t$.

It is not surprising that the diagonal Ramsey numbers are of greatest interest, but despite decades of effort, we know precisely few values precisely. We have shown in Proposition 8.1 that $R(3) \leq 6$, and the equality comes from considering a colouring of K_5 were we colour the edges of a 5-cycle red, and the remaining edges (which also form a 5-cycle) blue. Aside from this, it is known that $R(4) = 18$. Those are the only exact values that are known!

These Ramsey numbers have also proven very hard to estimate — for $R(5)$, we only know $43 \leq R(5) \leq 48$, while for $R(6)$ the range already widens considerably: $102 \leq R(6) \leq 165$. We will thus give up on finding more values exactly, and instead try to establish some general bounds.

We see from Theorem 8.3 that

$$R(k) \leq \binom{2k-2}{k-1} \leq \frac{2^{2k-2}}{\sqrt{k}}. \tag{10}$$

Although the proof above is very simple, the bound (10) has hardly been improved over the last 70 years.

8.2 Turán's Thoerem

To prove $R(k) > N$, we need to give a red-/blue-colouring of $E(K_N)$ with no monochromatic K_k. Equivalent, by focussing on the red edges, we need to show there is an N-vertex graph with neither a clique nor independent set on k vertices. However, controlling the independence number is not so easy. Heuristically, we might expect that the more edges a graph has, the smaller its independent sets will be. Therefore, rather than looking for a graph with no large clique or independent set, we might instead look for a graph with no large clique and many edges.

Definition 8.4. $\mathrm{ex}(n, H)$ is the maximal value of $e(G)$ over all n-vertex graphs G that do not contain H as a subgraph.

Example 8.5. Consider the case where H is a triangle. Recall that bipartite graphs contain no triangles. So $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ gives a triangle-free graph with $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

As a generalization Example 8.5, notice that dense graphs not having K_{r+1} as a subgraph can be obtained by dividing the vertex set V into r pairwise disjoint subsets $V = V_1 \cup \ldots \cup V_r$, $|V_i| = n_i$, $n = n_1 + \ldots + n_r$, joining two vertices if and only if they lie in distinct sets V_i, V_j. We denote the resulting graph by $K_{n_1,\ldots,n_r}$. To see that this is K_{r+1}-free, observe that any set of $r+1$ vertices must contain two vertices from the same part, and these vertices will not be adjacent.

The graph $e(K_{n_1,n_2,\ldots,n_r})$ has $\sum_{i<j} n_i n_j$ edges. Assuming n is fixed, we get the maximal number of edges among such graphs when we divide the numbers n_i as evenly as possible, that is $|n_i - n_j| \leq 1$ for all i, j. Indeed, suppose $n_1 \geq n_2 + 2$. By shifting one vertex from V_1 to V_2, we obtain $K_{n_1-1,n_2+1,\ldots,n_r}$, and that contains $(n_1-1)(n_2+1) - n_1 n_2 = n_1 - n_2 - 1 \geq 1$ more edges than $K_{n_1,\ldots,n_r}$. In particular, if r divides n, then we may choose $n_i = \frac{n}{r}$ for all i, obtaining

$$\binom{r}{2}\left(\frac{n}{r}\right)^2 = \left(1 - \frac{1}{r}\right)\frac{n^2}{2}$$

edges. Turán's theorem states that this number is an upper bound for the number of edges in *any* graph on n vertices without an $(r+1)$-clique.

Definition 8.6. We call the graph $K_{n_1,\ldots,n_r}$ with $|n_i - n_j| \leq 1$ the *Turán graph*, denoted by $T_{n,r}$.

Theorem 8.7 (Turán, 1941)**.** *Among all the n-vertex simple graphs with no $(r+1)$-clique, $T_{n,r}$ is the unique graph having the maximum number of edges.*

Proof. Let $G = (V, E)$ have the maximum number of edges among graphs containing no $(r+1)$-cliques. Let $v_m \in V$ be a vertex of maximal degree $\Delta(G)$. Denote the set of neighbours of v_m by S, $|S| = d_m$, and set $T = V \setminus S$. As G contains no $(r+1)$-clique and v_m is adjacent to all vertices of S, we note that S contains no r-clique.

We now construct the following graph H on V. H corresponds to G on S and contains all edges between S and T, but no edges within T. In other words, T is an independent set in H, and it follows that H again has no $(r+1)$-cliques. If $v \in S$, then we certainly have $d_H(v) \geq d_G(v)$ by the construction of H, and for $v \in T$ we see $d_H(v) = |S| = \Delta(G) \geq d_G(v)$ by the choice of v_m. We infer $|E(H)| \geq |E(G)|$ and find that among all graphs with a maximal number of edges, there must be one of the form of H. By induction, the graph induced by S has at most as many edges as a suitable graph $K_{n_1,\ldots,n_{r-1}}$ on S. So $|E(G)| \leq |E(H)| \leq E(K_{n_1,\ldots,n_r})$ with $n_r = |T|$. We have established that $E(K_{n_1,\ldots,n_r})$ is maximized for $|n_i - n_j| \leq 1$, which implies that $T_{n,r}$ has the maximum possible number of edges.

To prove uniqueness, note that equality can only hold in our previous bound if S induces the complete $(r-1)$-partite graph $K_{n_1,\ldots,n_{r-1}}$ and T touches exactly Δn_r edges in G. But the latter can only happen if T is an independent set in G. Indeed, the sum of the degrees of the vertices in T counts each edge spanned by T twice, and each edge connecting T and S once. As Δ is the maximum degree in G, the sum of degrees is at most Δn_r, so T can only touch this many edges if it spans none of them. But then G is r-partite and since it has the maximum number of edges, $G = T_{n,r}$. □

8.3 Lower bounds on Ramsey numbers

Turán's Theorem solves an extremal problem about the largest graphs without a copy of K_{r+1}. However, our original motivation was to obtain lower bounds on the Ramsey number $R(k)$. If we set $k = r + 1$, or $r = k - 1$, then we find that $T_{n,k-1}$ is a graph with many edges that does not contain K_k. For which parameters does it also not contain an independent set of order k?

In the Turán graph, any independent set must be fully contained within the vertex parts, the largest of which has size $\left\lceil \frac{n}{k-1} \right\rceil$. Thus, for there to be no independent set of k vertices, we must have $\left\lceil \frac{n}{k-1} \right\rceil \leq k-1$, which implies $n \leq (k-1)^2$. That is, the Turán graphs show that $R(k) > (k-1)^2$.

This quadratic lower bound is far from the exponential upper bound we proved earlier. However, it is very hard to find better constructions, and indeed, Turán tried to show that the upper bound can be reduced to quadratic instead. However, Erdős put a dramatic stop to those efforts by proving the following result.

Theorem 8.8 (Erdős, 1947)**.** *For $k \geq 3$, we have $R(k) \geq \frac{1}{\sqrt{2}} 2^{k/2}$.*

Proof. Consider the set of all graphs with vertex set $[n]$, of which there are $2^{\binom{n}{2}}$, as there are $\binom{n}{2}$ potential edges, each of which can be present or not.

A given k-clique is present in $2^{\binom{n}{2}-\binom{k}{2}}$ of these graphs, since we specify the state of $\binom{k}{2}$ edges. Similarly, a given set of k vertices forms an independent set in $2^{\binom{n}{2}-\binom{k}{2}}$ of these graphs.

If we discard all these graphs, ranging over all choices of k-sets of vertices, we are left with graphs having no k-clique or independent k-set. Since there are $\binom{n}{k}$ ways to choose k vertices, the inequality $2^{\binom{n}{2}} > 2\binom{n}{k}2^{\binom{n}{2}-\binom{k}{2}}$, which is equivalent to $1 > \binom{n}{k}2^{1-\binom{k}{2}}$, would imply that there are such graphs left over, and so $R(k) > n$.

For some rough approximations, we can take $\binom{n}{k} \leq \frac{n^k}{2}$, so that we see $\binom{n}{k}2^{1-\binom{k}{2}} \leq \left(n2^{-\frac{k-1}{2}}\right)^k$. This will be smaller than 1 if $n < 2^{\frac{k-1}{2}} = \frac{1}{\sqrt{2}}2^{k/2}$. □

Remark 8.9. Although the proof is very simple, it took a relatively long time to find because it relies on the crucial idea that we can show that the desired graph exists without actually constructing the graph itself. Note that although we have proven that there is a graph on $\approx 2^{k/2}$ vertices without a clique or independent set of size k, the proof tells us nothing about what the graph looks like. Some further comments:

- Although this bound is, like the upper bound, exponential, the two bounds remain exponentially far apart.
- Despite considerable effort, this original lower bound of Erdős has only been improved by a *constant factor* in the past 70+ years.
- If we run through the calculations, the proof actually shows that *almost all* graphs on $\approx 2^{k/2}$ vertices have no clique or independent set on k vertices. However, we do not know how to actually find one. In fact, it is an open problem to construct such a graph even on 1.00000000001^k vertices!

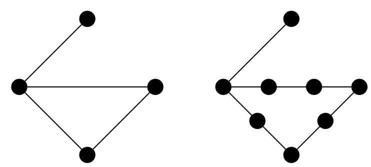
- This idea of a non-constructive existence proof has proved extremely effective, and has led to the development of probabilistic combinatorics.

9 Kuratowski's Theorem (non-examinable)

In Corollary 6.14 we saw that K_5 and $K_{3,3}$ are not planar. In this section we will prove a remarkable theorem of Kuratowski, roughly saying that K_5 and $K_{3,3}$ are in some sense the *only* obstructions to being planar. We will also simultaneously prove Fáry's theorem, that any planar graph can in fact be drawn using only straight lines.

Definition 9.1. A *subdivision* of a graph H is a graph obtained from H by replacing the edges of H by internally vertex disjoint paths of non-zero length with the same endpoints.

Example 9.2.



H Subdivision of H

Remark 9.3. If G contains a subdivision of H, it also contains an H-minor.

Definition 9.4. A *Kuratowski graph* is a graph which is a subdivision of K_5 or $K_{3,3}$. If G is a graph and H is a subgraph of G which is a Kuratowski graph then we say that H is a *Kuratowski subgraph* of G.

Theorem 9.5 (Kuratowski 1930)**.** *A graph is planar if and only if it has no Kuratowski subgraph.*

It is straightforwardly true that no Kuratowski graph is planar. Indeed, given a planar drawing (by polygonal curves) of a Kuratowski subgraph, we can consider the subdivided paths to be long polygonal curves in a planar drawing of K_5 or $K_{3,3}$, which is impossible. So, to prove Theorem 9.5 we need only prove that every graph with no Kuratowski subgraph has a plane drawing. As promised, we prove a stronger theorem, combining Kuratowski's Theorem with Fáry's Theorem on straightline drawings.

Definition 9.6. A *straightline drawing* of a planar graph G is a drawing in which every edge is a straight line.

Theorem 9.7. *If G is a graph with no Kuratowski subgraph then G has a straightline drawing in the plane.*

The proof of Theorem 9.7 is broken down into two parts. In the first part, we prove a version of the theorem for 3-connected graphs, then in the second part we show how the general case can be reduced to the 3-connected case.