Graph Theory I

Math 7703

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Chapter 1: Fundamentals

1. Graphs
2. Graph isomorphism
3. The adjacency and incidence matrices
4. Degrees
5. Subgraphs
6. Special graphs
7. Walks, paths and cycles
8. Connectivity
9. Graph operations and parameters

§1.1: Graphs

1. Graphs
   - Definition
   - Terminology
2. Graph isomorphism
3. The adjacency and incidence matrices
4. Degrees
5. Subgraphs
6. Special graphs
7. Walks, paths and cycles
8. Connectivity
9. Graph operations and parameters

Defining graphs

Informal description

- Vertices: objects in the network
- Edges: pairs that are connected in the network

Formal definition

A (simple) graph is a pair (V, E) of sets, where

- V is a set of vertices
- $E \subseteq \binom{V}{2}$ is a set of edges

$\binom{V}{2} = \{\{u, v\} : u, v \in V, u \neq v\}$: set of unordered pairs of vertices

Interpretation

Graphs can represent symmetric pairwise relations between objects.

Terminology

Graph parameters

- order: $|V(G)|$ — number of vertices
- size: $e(G) = |E(G)|$ — number of edges

Vertex-edge relations

- Vertices u, v are adjacent if $\{u, v\} \in E(G)$
- Edge $e \in E(G)$ is incident to vertex $v \in V(G)$ if $v \in e$
- Edges $e, e' \in E(G)$ are incident if $e \cap e' \neq \emptyset$
- If $u, v \in V(G)$ are adjacent, then v is a neighbour of u

Typical notation

Graphs

- Capital letters, usually around G
- $G, H, F, \ldots$ (sometimes Γ)

Vertices

- Lower-case letters, usually around v
- $u, v, w, x, y, z, \ldots$

Edges

- Lower-case letters, usually around e
- $e, f, g, h, \ldots$

§1.2: Graph isomorphism

1. Graphs
2. Graph isomorphism
   - Example
   - Definition
   - Equivalence relation
3. The adjacency and incidence matrices
4. Degrees
5. Subgraphs
6. Special graphs
7. Walks, paths and cycles
8. Connectivity
9. Graph operations and parameters

Spot the difference

A motivating example

Are these graphs different?

A dictionary

Map between the vertex sets:

$$1 \leftrightarrow a,\ 2 \leftrightarrow c,\ 3 \leftrightarrow b,\ 4 \leftrightarrow d$$

Isomorphisms

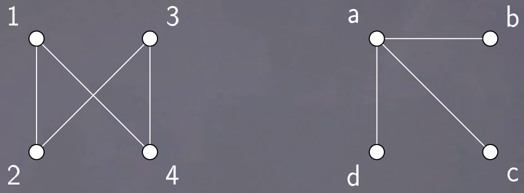
Definition

Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, an isomorphism is a map $\varphi : V_1 \rightarrow V_2$ is a bijection such that $\{u, v\} \in E_1$ if and only if $\{\varphi(u), \varphi(v)\} \in E_2$.
We say the graphs G_1, G_2 are isomorphic.

Explanation

- Relabel vertices of the graphs so that they have the same edges
- Only care about the structure of the edges, not vertex names

Non-isomorphic examples



Different number of edges

Only one graph has a triangle

Equivalence relation

Claim

Isomorphism defines an equivalence relation on the set of graphs.

In other words...

- Any graph G is isomorphic to itself
- If G_1 is isomorphic to G_2, then G_2 is isomorphic to G_1
- If G_1 is isomorphic to G_2, and G_2 is isomorphic to G_3, then G_1 is isomorphic to G_3

Unlabelled graphs

- Equivalence classes are unlabelled graphs
- We usually take the vertex set to be $[n] = \{1, 2, \ldots, n\}$

§1.3: Matrix representations

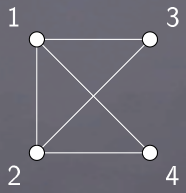
1. Graphs
2. Graph isomorphism
3. The adjacency and incidence matrices
   - Representing graphs
   - The adjacency matrix
   - The incidence matrix
4. Degrees
5. Subgraphs
6. Special graphs
7. Walks, paths and cycles
8. Connectivity
9. Graph operations and parameters

Representing graphs

Drawing graphs

- Formal definition is not very convenient for presenting graphs
  e.g.: $G = (\{1,2,3,4\}, \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\}\})$
- Drawings can be much more readable

$G =$



Problems with drawings

- Larger graphs can be too dense to be read
- Drawings are not very computer-friendly

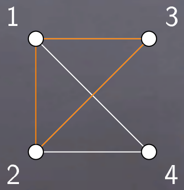
The adjacency matrix

Definition

Let $G = (V, E)$ be a graph, where $V = [n]$. The adjacency matrix $A = A(G)$ is an $n \times n$ matrix defined by

$$a_{i,j} = \begin{cases} 1 & \text{if } \{i,j\} \in E, \\ 0 & \text{if } \{i,j\} \notin E. \end{cases}$$

$G =$



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

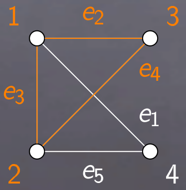
The incidence matrix

Definition

Let $G = (V, E)$ be a graph, with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. The incidence matrix $B = B(G)$ is an $n \times m$ matrix defined by

$$b_{i,j} = \begin{cases} 1 & \text{if } v_i \in e_j, \\ 0 & \text{if } v_i \notin e_j. \end{cases}$$

$G =$



$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

§1.4: Degrees

1. Graphs
2. Graph isomorphism
3. The adjacency and incidence matrices
4. Degrees
   - Definition
   - Extreme values
   - Handshake Lemma
5. Subgraphs
6. Special graphs
7. Walks, paths and cycles
8. Connectivity
9. Graph operations and parameters

Definition

Motivation: isomorphic graphs

- Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic $\Rightarrow$ they have the same number of edges
- More is true: if $\varphi : V_1 \to V_2$ is an isomorphism, $v \in V_1$ and $\varphi(v) \in V_2$ are incident to the same number of edges

Definitions

Given a graph $G = (V, E)$ and a vertex $v \in V$, the neighbourhood $N(v)$ of v is the set its neighbours: $N(v) = \{u \in V : \{u, v\} \in E\}$.

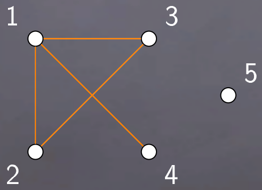
The degree $d(v)$ of v is the number of neighbours: $d(v) = |N(v)|$.

A vertex is isolated if it has no neighbours.

Connection to matrices

Observation

The degree $d(v)$ is the number of ones in the row corresponding to v in either the adjacency matrix $A(G)$ or the incidence matrix $B(G)$.



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$d(1) = 3, d(2) = 2, d(3) = 2, d(4) = 1, d(5) = 0$$

Extreme values

Definition

We say a graph $G = (V, E)$ is *d-regular* if $d(v) = d$ for all $v \in V$.

Notation

Let $G = (V, E)$ be an n-vertex graph.

- The *minimum degree* is $\delta(G) = \min\{d(v) : v \in V\}$
- The *maximum degree* is $\Delta(G) = \max\{d(v) : v \in V\}$
- The *average degree* is

$$\bar{d}(G) = \frac{1}{n} \sum_{v \in V} d(v)$$

Observation

Trivially, for all $v \in V$, $\delta(G) \leq d(v) \leq \Delta(G)$.
Thus, $\delta(G) \leq \bar{d}(G) \leq \Delta(G)$.

Handshake Lemma

Proposition

For every graph $G = (V, E)$, we have $\sum_{v \in V} d(v) = 2e(G)$.

Proof

- Summing over vertices vs summing over edges
  $\Rightarrow$ count vertex-edge incidences
- We count the number of pairs $(v, e) \in V \times E$, where $v \in e$
- Every vertex v is in $d(v)$ edges
  $\Rightarrow \sum_{v \in V} d(v)$ pairs
- Every edge e contains 2 vertices
  $\Rightarrow 2e(G)$ pairs

$\Rightarrow \sum_{v \in V} d(v) = 2e(G)$ □

Consequences

Proposition

For every graph $G = (V, E)$*, we have* $\sum_{v \in V} d(v) = 2e(G)$.

Corollary

Every graph G *has an even number of vertices of odd degree.*

Can use this to rule out the existence of certain graphs

Corollary

There is no 3-regular graph on 9 vertices.

§1.5: Subgraphs

Types of subgraphs

It is often necessary to focus on a smaller part of a graph

Subgraph

A graph $H = (U, F)$ is a subgraph of a graph $G = (V, E)$ if $U \subseteq V$ and $F \subseteq \binom{U}{2} \cap E$.

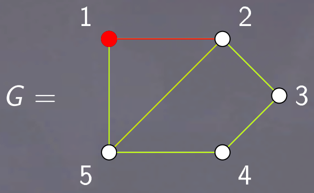
Spanning

A subgraph $H = (U, F)$ of a graph $G = (V, E)$ is spanning if $U = V$.

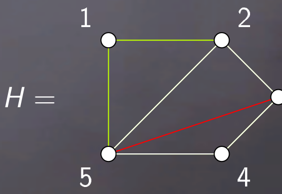
Induced

A subgraph $H = (U, F)$ of a graph $G = (V, E)$ is induced if $F = \binom{U}{2} \cap E$. We write $H = G[U]$.

Examples



the original graph



Hello worldSubgraph: not induced, not spanninginduced subgraph, not sp

§1.6: Special graphs

Graph classes

The following types of graphs are of particular importance

Complete graphs (cliques)

$K_n = \left([n], \binom{[n]}{2}\right)$ — every pair of vertices is an edge

Empty graphs

$G = ([n], \emptyset)$ — no pair of vertices forms an edge

Bipartite graphs

We can partition the vertices into parts $V(G) = V_1 \cup V_2$ so that every edge intersects V_1 and V_2; i.e., $G[V_1]$ and $G[V_2]$ are empty

Complete bipartite graphs

$K_{m,n}$ is a bipartite graph with parts of size m and n, and all mn possible edges

§1.7: Walks, paths and cycles

Walks & Co.

Definitions

A walk in a graph is a sequence of vertices $v_0, v_1, \ldots, v_k$ and a sequence of edges $\{v_i, v_{i+1}\}, 0 \leq i \leq k-1$.

A path is a walk where all the vertices are distinct.

If, for a path with $k \geq 2$, $\{v_0, v_k\} \in E(G)$, then $v_0, v_1, \ldots, v_k, v_0$ is a cycle.

The length of a walk/path/cycle is the number of edges it contains.

v_5, v_1

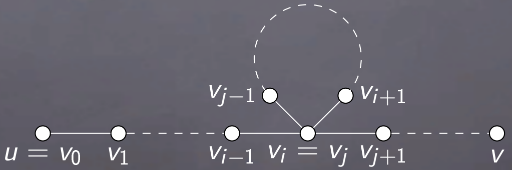
Walks ⇒ paths

Proposition

Every walk from u to v in G contains a path between u and v.

Proof

Induction on the length ℓ of the walk $u = v_0, v_1, \ldots, v_\ell = v$.
If $\ell = 1$, the walk is a path.
For $\ell \geq 2$, if the walk is not a path, there is some $i < j$ with $v_i = v_j$.



Then $u = v_0, \ldots, v_i, v_{j+1}, \ldots, v_\ell = v$ is a shorter walk from u to v.
Induction ⇒ contains a path from u to v. □

Finding long cycles

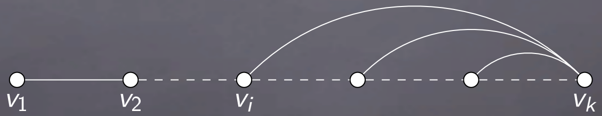
Can we guarantee the existence of cycles?

Proposition

Every graph G with $\delta(G) = \delta \geq 2$ contains a path of length δ and a cycle of length at least $\delta + 1$.

Proof

Let $v_1, v_2, \ldots, v_k$ be a longest path in G



All neighbours of v_k must be on the path

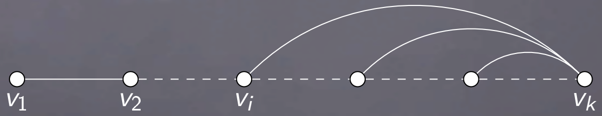
- otherwise we would have a longer path

Finding long cycles

Proposition

Every graph G with $\delta(G) = \delta \geq 2$ contains a path of length δ and a cycle of length at least $\delta + 1$.

Proof



Let v_i be the first neighbour of v_k on the path

$\Rightarrow$ all neighbours of v_k are in $\{v_i, v_{i+1}, \ldots, v_{k-1}\} \Rightarrow k - i \geq \delta$

The path is of length $\geq \delta$

The cycle $v_i, v_{i+1}, \ldots, v_k, v_i$ is of length $\geq \delta + 1$ $\square$

Can we improve this?

Are these the longest paths/cycles we can hope to find?

Observation

The complete graph $K_{\delta+1}$ has:

- minimum degree $\delta(K_{\delta+1}) = \delta$
- no path of length $\delta + 1$
- no cycle of length $\delta + 2$

So the previous result is best possible

§1.8: Connectivity

1. Graphs
2. Graph isomorphism
3. The adjacency and incidence matrices
4. Degrees
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6. Special graphs
7. Walks, paths and cycles
8. Connectivity
   - Definition
   - Many components
9. Graph operations and parameters

Connectivity

It is often critical to be able to go from one vertex to another

Definition

A graph G is connected if for all pairs of vertices $u, v \in V(G)$, there is a path in G from u to v.

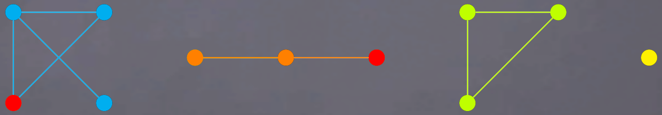
Remark

By our earlier proposition, it is enough to have a walk from u to v.

Components

Definition

A (connected) component of a graph G is a maximal connected subgraph of G.



[illegible]

Observation

G is connected if and only if it has one connected component.

Graphs with few edges

Proposition

A graph G with n vertices and m edges has at least $n - m$ connected components.

Proof

Induction on m

If $m = 0$, all vertices are isolated and their own component

$\Rightarrow$ $n = n - 0$ components

If $m \geq 1$, let $e = \{u, v\} \in E(G)$, and let $G' = G - e$

- Induction $\Rightarrow$ G' has at least $n - m + 1$ components
- Form G by adding e back to G'
  - Components of u and v are merged
  - Other components left unchanged

    $\Rightarrow$ at least $n - m$ components

□

§1.9: Operations and parameters

1. Graphs
2. Graph isomorphism
3. The adjacency and incidence matrices
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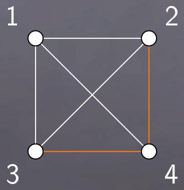
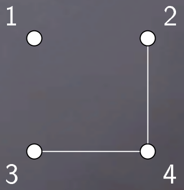
6. Special graphs
7. Walks, paths and cycles
8. Connectivity
9. Graph operations and parameters
   - Complements
   - Clique and independence numbers

Complements

Definition

Given a graph $G = (V, E)$, the complement is $\bar{G} = \left(V, \binom{V}{2} \setminus E\right)$.
That is, $\{u, v\} \in E(\bar{G})$ if and only if $\{u, v\} \notin E(G)$.

$G =$ 

$\overline{G} =$ 

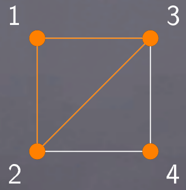
Cliques and independent sets

Definitions

A clique in a graph G is a complete subgraph.

An independent set in a graph G is an empty induced subgraph.

$G =$



An independent set

A clique

Graph parameters

Cliques

The clique number $\omega(G)$ of a graph G is the maximum order of a clique in G.

Independence number

The independence number $\alpha(G)$ of a graph G is the maximum order of an independent set in G.

Observation

Clique in $G \leftrightarrow$ independent set in $\bar{G}$

Corollary

$\omega(G) = \alpha(\bar{G})$ *and* $\alpha(G) = \omega(\bar{G})$

Thank you for listening!

Any questions?